On QPCCs, QCQPs, and Completely Positive Programs[1]

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1. Introduction

2. Relationship between QCQPs and QPCCs

3. Completely Positive Relaxation

4. Conclusions

Outline

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Introduction

Linear complementarity constraints are quadratic constraints, so a conic quadratic program with complementarity constraints (QPCC) is a conic quadratically constrained quadratic program (QCQP).

- We show a converse: any conic QCQP can be represented as an equivalent conic QPCC.
- We show any conic QCQP has an equivalent convex completely positive reformulation.
- Thus we have equivalent convex formulations for several classes of nonconvex optimization problems defined over convex cones, including rank-constrained semidefinite programs and quadratically constrained quadratic programs (QCQPs).
- Our results make no boundedness assumptions on the feasible regions of the various problems considered.

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Conic Quadratic Programs with Complementarity Constraints

$$\begin{array}{lll} \underset{x \triangleq (x^0, x^1, x^2)}{\text{minimize}} & c^T x + \frac{1}{2} x^T Q x & \\ \text{subject to} & Ax = b & (QPCC) \\ \text{and} & x^0 \in \mathcal{K}^0, \ \mathcal{K}^1 \ni x^1 \perp x^2 \in \mathcal{K}^{1*} & \end{array}$$

where $\mathcal{K}^0$ and $\mathcal{K}^1$ are closed convex cones,
and $\mathcal{K}^{1*}$ is the dual cone to $\mathcal{K}^1$.

Eg:

- $\mathcal{K}^1$ is the nonnegative orthant: get standard QPCC.
- $\mathcal{K}^1$ is the semidefinite cone: get SDP-MPCC as considered by Defeng Sun et al.

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Conic Complementarity

Since $x^1 \in \mathcal{K}^1$ and $x^2 \in \mathcal{K}^2$, we have $(x^1)^T x^2 \geq 0$.
So complementarity condition can be expressed $(x^1)^T x^2 \leq 0$.

If $\mathcal{K}^1$ is a polyhedral cone then we get a combinatorial structure, and can express the problem equivalently as a finite number of (possibly nonconvex) quadratic programs.

We construct an equivalent convex reformulation for any conic QPCC. Solving the convex problem would give a globally optimal solution to the conic complementarily problem.

Conic Quadratically Constrained Quadratic Programs

$$
\begin{array}{lll}
\underset{\tilde{x} \in \tilde{\mathcal{K}}}{\text{minimize}} & f_0(\tilde{x}) \triangleq (g^0)^T \tilde{x} + \frac{1}{2} \tilde{x}^T M^0 \tilde{x} & \\
\text{subject to} & H\tilde{x} = p & (QCQP) \\
\text{and} & f_i(\tilde{x}) \triangleq \nu_i + (g^i)^T \tilde{x} + \frac{1}{2} \tilde{x}^T M^i \tilde{x} \leq 0, \quad i = 1, \cdots, I, &
\end{array}
$$

for some closed convex cone $\tilde{\mathcal{K}} \subseteq \mathbb{R}^{\tilde{n}}$ and for some positive integer I, $\nu_i \in \mathbb{R}$, $g^i \in \mathbb{R}^{\tilde{n}}$, and $M^i \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ symmetric for $i = 0, 1, \cdots, I$.

A conic QPCC is a conic QCQP.

“no Slater point”

A conic QPCC is a conic QCQP with just one quadratic constraint:

$$(x^1)^T x^2 \leq 0$$

Further, every point in $\{x \in \mathcal{K} : Ax = b\}$ satisfies $(x^1)^T x^2 \geq 0$.

So this is a QCQP with no Slater point.

Binary quadratic programs also lead to QCQPs with no Slater point:

$$x_j \in \{0, 1\} \iff x_j(1 - x_j) \leq 0$$

provided the conic and linear constraints imply $0 \leq x_j \leq 1 \; \forall j \in B$.

Nonconvex constraints of this type can be aggregated. Eg:

$$\sum_{j \in B} x_j(1 - x_j) \leq 0.$$

QCQPs with no Slater point

$$\begin{array}{lll} \underset{x\in\mathcal{K}}{\text{minimize}} & q_0(x) \triangleq (c^0)^T x + \frac{1}{2} x^T Q^0 x & \\ \text{subject to} & Ax = b & \\ & q(x) \triangleq \mathbf{h} + \mathbf{g}^T x + \frac{1}{2} x^T \mathbf{Q} x \leq 0 & \\ \text{and} & q_j(x) \triangleq h_j + (c^j)^T x + \frac{1}{2} x^T Q^j x \leq 0, & j = 1, \cdots, J, \end{array}$$

where $q(x) \geq 0 \; \forall x \in \mathcal{K} \cap \mathcal{M}$ with $\mathcal{M} := \{x \in \mathbb{R}^n : Ax = b\}$
and where the constraints $q_j(x) \leq 0$, $j = 1 \ldots, J$, are convex.

We call this problem an nSp-QCQP.
If $J = 0$, it is denoted as an nSp0-QCQP.

Can always represent a convex quadratic constraint as a second order cone constraint, so an nSp-QCQP is equivalent to an nSp0-QCQP.

nSp0-QCQP is a broad class

Theorem

A conic QCQP with variables $\tilde{x} \in \mathbb{R}^n$ is equivalent to an nSp-QCQP with variables $x \in \mathbb{R}^{n+2}$ with a convex objective.

Main ideas of proof

Introduce variables r, s satisfying $s + r = 1$, and two constraints:

- $\tilde{x}^T\tilde{x} + r^2 - s^2 \leq 0$: represent as a second order cone constraint.
- $q(\tilde{x}, r, s) := -\tilde{x}^T\tilde{x} - r^2 + s^2 \leq 0$, our nonconvex constraint; trivially, $q(\tilde{x}, r, s) \geq 0 \ \forall(\tilde{x}, r, s)$ satisfying $\tilde{x}^T\tilde{x} + r^2 - s^2 \leq 0$.

Note that $s - r = s^2 - r^2 = \tilde{x}^T\tilde{x}$.

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Note that $s - r = s^2 - r^2 = \tilde{x}^T\tilde{x}$.
If M^i not positive semidefinite, replace quadratic constraint by

$$\nu_i + (g^i)^T\tilde{x} - \frac{1}{2}\lambda_i(s - r) + \frac{1}{2}\tilde{x}^T(M^i + \lambda_i I)\tilde{x} \leq 0$$

for appropriately chosen λ_i. Similarly modify the objective.

nSp0-QCQP is a broad class (continued)

Corollary

A conic QCQP with variables $x \in \mathbb{R}^n$ and l quadratic constraints is equivalent to an nSp0-QCQP with variables $\tilde{x} \in \mathbb{R}^{n+2+2l}$.

Since:

Any convex quadratic constraint can be replaced by an equivalent second order cone constraint, after the addition of two variables:

$$h + c^T x + \tfrac{1}{2} x^T L L^T x \;\leq\; 0$$

$$\Longleftrightarrow$$

$$||L^T x||_2^2 + u^2 \leq \; v^2, \qquad \text{(second order cone)}$$

with $u = \frac{1}{2} + h + c^T x$ and $v = \frac{1}{2} - h - c^T x$.

A conic QCQP is equivalent to a conic QPCC

Theorem

Any conic QCQP can be reformulated as an equivalent conic QPCC, by first constructing an equivalent nSp0-QCQP.

Main ideas of proof

For the nSp0-QCQP, $\{\, x \in \mathcal{K} \cap \mathcal{M} \mid q(x) \leq 0\} = \underset{x \in \mathcal{K} \cap \mathcal{M}}{\operatorname{argmin}}\, q(x)$.

For $-\nabla q(x)$ be in the normal cone to $\mathcal{K} \cap \mathcal{M}$ at $x \in \mathcal{K} \cap \mathcal{M}$ need:

$$\begin{aligned} \mathcal{K} \ni x &\perp \mathbf{g} + \mathbf{Q}x + A^T \lambda \in \mathcal{K}^* \\ 0 &= Ax - b. \end{aligned}$$

Thus, for local minimizers of $q(x)$, have $q(x) = \frac{1}{2}\left(\mathbf{g}^T x - b^T \lambda\right) + \mathbf{h}$, so add linear constraint $\frac{1}{2}\left(\mathbf{g}^T x - b^T \lambda\right) + \mathbf{h} = 0$.

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Lifting the nSp0-QCQP

We have a QCQP with a single quadratic constraint:

$q(x) := \mathbf{h} + \mathbf{g}^T x + \frac{1}{2} x^T \mathbf{Q} x \leq 0,$

such that $Ax = b, x \in \mathcal{K}$ implies $q(x) \geq 0$.

Can be lifted to a completely positive program in a well-known manner:

$$\begin{array}{lll} \underset{x,X}{\text{minimize}} & (c^0)^T x + \frac{1}{2} \langle Q^0, X \rangle & \\ \text{subject to} & Ax = b \quad \text{and} \quad A_i X A_i^T = b_i^2, \quad i = 1, \cdots, k, & \\ & \mathbf{h} + \mathbf{g}^T x + \frac{1}{2} \langle \mathbf{Q}, X \rangle = 0 & \\ \text{and} & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{CP}_{1+n}(\mathcal{K}) & \left(\begin{array}{l} \text{cone of completely} \\ \text{positive matrices over } \mathcal{K} \end{array} \right). \end{array}$$

In general, this **convex** problem is a relaxation of a QCQP.

Completely Positive Matrices

Cone $\mathcal{CP}_{1+n}(\mathcal{K})$ of completely positive matrices over $\mathcal{K}$:

$$\mathcal{CP}_{1+n}(\mathcal{K}) \triangleq \text{conv}\left\{ M \in \mathcal{S}^{1+n} \mid M = xx^T, \, x \in \mathbb{R}_+ \times \mathcal{K} \right\},$$

Dual cone $\mathcal{COP}_{1+n}(\mathcal{K})$ of copositive matrices over $\mathcal{K}$:

$$\mathcal{COP}_{1+n}(\mathcal{K}) \triangleq \left\{ M \in \mathcal{S}^{1+n} \mid x^T M x \geq 0, \, \forall \, x \in \mathbb{R}_+ \times \mathcal{K} \right\}.$$

We also use the notation $\mathcal{CP}_{1+n} := \mathcal{CP}_{1+n}(\mathbb{R}^n_+)$ and $\mathcal{COP}_{1+n} := \mathcal{COP}_{1+n}(\mathbb{R}^n_+)$.

Even with $\mathcal{K} = \mathbb{R}^n$, determining membership in $\mathcal{CP}_{1+n}(\mathcal{K})$ or $\mathcal{COP}_{1+n}(\mathcal{K})$ is NP-Complete.

May be able to approximate $\mathcal{CP}_{1+n}(\mathcal{K})$ or $\mathcal{COP}_{1+n}(\mathcal{K})$.
Eg, see work of Dür et al, or Dickinson.

Burer's result

Theorem (Burer, Math Progg, 2009)

If $\mathcal{K} = \mathbb{R}^n_+$ and $q(x) = \sum_{i \in B} x_i(1 - x_i)$ and if $Ax = b, x \geq 0$ implies $0 \leq x_i \leq 1 \, \forall i \in B$ then the QCQP and its completely positive relaxation are equivalent.

Note that Burer imposes no convexity assumption on the objective function Q^0.

Burer extended his results to LPCCs and later to problems defined over convex cones. See also Dickinson, Eichfelder, and Povh for results on more general sets.
All these results require a bounded feasible region.

An example where the relaxation of QPCC is not tight

The QPCC has an optimal value of 0, but the completely positive lifting is unbounded below.

$$\begin{array}{lrcl} \underset{x\geq 0}{\text{minimize}} & -x_2^2 & & \\ \text{subject to} & x_1 + x_2 - x_3 & = & 3 \\ & x_1 - x_4 & = & 2 \\ \text{and} & x_1 & \perp & x_2. \end{array}$$

Optimal value is 0,
since $x_1 \geq 2$, so $x_2 = 0$.

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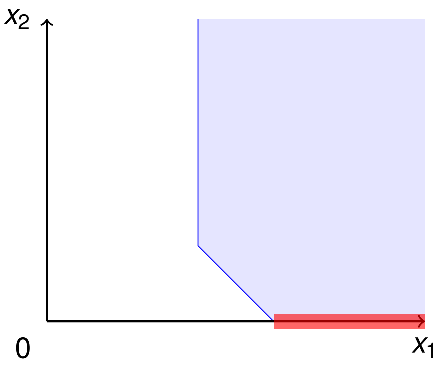

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$\bar{d} \triangleq (0,1,1,0)^T$ has $A\bar{d} = 0$.

Not valid direction in QPCC since must have $x_1 > 0$, so must have $d_2 = 0$.

$$\begin{array}{ll} \underset{x, X}{\text{minimize}} & -X_{2,2} \\ \text{subject to} & x_1 + x_2 - x_3 = 3 \\ & x_1 - x_4 = 2 \\ & X_{1,1} + 2X_{1,2} + X_{2,2} - 2X_{1,3} \\ & \quad -2X_{2,3} + X_{3,3} = 9 \\ & X_{1,1} + X_{4,4} - 2X_{1,4} = 4 \\ & X_{1,2} = 0 \\ & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{CP}_5. \end{array}$$

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$\bar{d} \triangleq (0,1,1,0)^T$ has $A\bar{d} = 0$.

Not valid direction in QPCC since must have $x_1 > 0$, so must have $d_2 = 0$.

But $\begin{pmatrix} 0 & 0 \\ 0 & \bar{d}\,\bar{d}^T \end{pmatrix}$ is a feasible ray of the completely positive program, so it is unbounded below.

Our assumptions

1. Our QCQP is an nSp0-QCQP.

2. Let $L \triangleq \{ d \in \mathcal{K} \mid A\,d = 0 \text{ and } d^T \mathbf{Q} d = 0 \}$.
   Assume objective function matrix Q^0 is copositive on L.

* **Note:** No boundedness assumption on any of the variables.
* **Note:** Assumptions all hold if $q(x) = \sum_{i \in B} x_i(1 - x_i)$ and $0 \leq x_i \leq 1 \; \forall i \in B$. (Burer: third assumption holds provided optimal value of BQP is finite.)

Completely positive relaxation is tight

Theorem

Under our two assumptions, the nSp0-QCQP and its completely positive relaxation are equivalent in the sense that

1. *The nSp0-QCQP is feasible if and only if the completely positive program is feasible.*
2. *Either the optimal values of the nSp0-QCQP and the completely positive program are finite and equal, or both of them are unbounded below.*
3. *Assume both the nSp0-QCQP and the completely positive program are bounded below, and $(\bar{x}, \bar{X})$ is optimal for the completely positive program, then $\bar{x}$ is in the convex hull of the optimal solutions of the nSp0-QCQP.*
4. *The optimal value of the nSp0-QCQP is attained if and only if the same holds for the completely positive program.*

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Completely positive representations of QPCCs

Corollary

The QPCC

$$\begin{array}{lll} \underset{x \triangleq (x^0, x^1, x^2)}{\text{minimize}} & (c^0)^T x + \frac{1}{2} x^T Q^0 x & \\ \text{subject to} & Ax = b & (QPCC) \\ \text{and} & x^0 \in \mathcal{K}^0, \ \mathcal{K}^1 \ni x^1 \perp x^2 \in \mathcal{K}^{1*} & \end{array}$$

is equivalent to its convex completely positive lifting, provided Q^0 is copositive on an appropriate subset of the recession cone.

Note that we impose no boundedness assumption on the complementary variables.

Our second assumption

Recall our second assumption:

* Let $L \triangleq \{\, d \in \mathcal{K} \mid A\,d = 0 \text{ and } d^T\mathbf{Q}d = 0 \,\}$.

  Assume objective function matrix Q^0 is copositive on L.

This assumption can be **removed**:

replace the quadratic objective function by a linear objective function $\min (c^0)^T x + z$, and add the constraint $-z + \frac{1}{2}x^T Q^0 x \leq 0$.
Then construct the corresponding nSp0-QCQP.

Hence we have the following theorem:

Theorem

Any QCQP is equivalent to a convex completely positive program.

Rank-constrained SDPs

$$\begin{array}{llcll} \min_X & \langle C, X\rangle & & & \\ \text{subject to} & \langle A_i, X\rangle & = & b_i & \forall i \\ & \text{rank}(X) & \leq & p & \\ & X & \in & S^n_+ & \end{array}$$

where S^n_+ is set of $n \times n$ symmetric psd matrices, C, A_i symmetric $n \times n$ matrices.

Equivalent to (Sun et al.):

$$\begin{array}{llcll} \min_{X,W} & \langle C, X\rangle & & & \\ \text{subject to} & \langle A_i, X\rangle & = & b_i & \forall i \\ & \text{trace}(W) & = & p & \\ & S^n_+ \ni X & \perp & I - W \in S^n_+ & \\ & W & \in & S^n_+. & \end{array}$$

Objective function is linear, this problem is equivalent to its convex completely positive lifting. The lifting has $O(n^4)$ variables.

Rank-constrained SDPs

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\text{subject to} & \langle A_i, X\rangle & = & b_i & \forall i \\
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\end{array}
$$

where S^n_+ is set of $n \times n$ symmetric psd matrices, C, A_i symmetric $n \times n$ matrices.

Equivalent to (Sun et al.):

$$
\begin{array}{llcll}
\min_{X,W} & \langle C, X\rangle & & & \\
\text{subject to} & \langle A_i, X\rangle & = & b_i \quad \forall i & \\
& \text{trace}(W) & = & p & \\
& S^n_+ \ni X & \perp & I - W \in S^n_+ & \\
& W & \in & S^n_+. &
\end{array}
$$

Objective function is linear, this problem is equivalent to its convex completely positive lifting. The lifting has $O(n^4)$ variables.

Outline

1 Introduction

2 Relationship between QCQPs and QPCCs

3 Completely Positive Relaxation

4 Conclusions

Conclusions

A conic quadratically constrained quadratic program is equivalent to a conic quadratic program with complementarity constraints.

Conic quadratically constrained quadratic programs are equivalent to convex conic programs, even if the variables are unbounded and even if the constraints and/or objective are nonconvex.