ON THE PARAMETRIC LCP: A HISTORICAL PERSPECTIVE

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The year 2014

A few important centennials

The year 2014

A few important centennials

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The year 2014

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- ____________________
- The birth of GEORGE B. DANTZIG

This talk is meant to review

1. the origins of the PLCP

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2. some PLCP algorithms

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1. the origins of the PLCP
2. some PLCP algorithms
3. the equivalence of some PLCP algorithms

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WM. ORCHARD-HAYS,
A composite simplex algorithm—II. RM-1275.
The RAND Corp. 1954.

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G.B. DANTZIG,
Linear Programming and Extensions,
Princeton University Press, 1963, pp. 245–247.

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Equilibrium points of bimatrix games,
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Bimatrix equilibrium points and mathematical programming,
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Developments in the late 60s

G.B. DANTZIG AND R.W. COTTLE
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Developments in the early 70s

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On the parametric complementarity problem, Engineering Summer Conference Notes, University of Michigan. August, 1971.

A. Ravindran
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C. van de Panne
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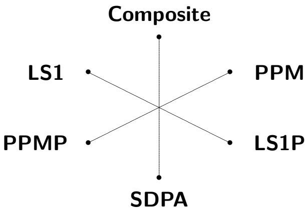
LCP algorithms



Primal LP as in SDPA

Dantzig (1963) considered the primal LP:

$$(\mathrm{P}) \qquad \begin{array}{ll} \text{minimize} & 0^T v + c^T x \\ \text{subject to} & Iv + Ax = b \\ & x \geq 0,\ v \geq 0 \end{array} \qquad A \in R^{m \times n}$$

The identity matrix I is the initial basis in $[\, I\ A \,]$.

Neither b nor c is assumed to be nonnegative.

The dual of (P)

Let $\tilde{A} = [\, I \; A \,]$ and $\tilde{c}^T = [\, 0^T \; c^T \,]$.

Then the dual of (P) would be

$$
\text{(D)} \qquad \begin{array}{ll} \text{maximize} & \tilde{y}^T b \\ \text{subject to} & \tilde{y}^T \tilde{A} \leq \tilde{c}^T \\ & \tilde{y} \text{ free} \end{array}
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A feasible $\tilde{y}$ must be *nonpositive*. The dual can be rewritten in terms of $y = -\tilde{y}$ as

$$\text{(D)} \qquad \begin{array}{ll} \text{maximize} & -y^T b \\ \text{subject to} & -y^T A \le c^T \\ & y \ge 0 \end{array}$$

Optimality conditions for (P)

The optimality conditions for (P) are:

$$
\begin{array}{rcccccc}
u & = & c & & & + & A^T y \\
v & = & b & - & Ax & &
\end{array}
$$

$$u,\ v,\ x,\ y \geq 0$$

$$x^T u + y^T v = 0$$

This can be viewed as a *composite problem*. It is also a special LCP.

Dantzig's Primal LP as in SDPA

Dantzig (1963) formulated the self-dual parametric LP:

$$
\text{(P)} \qquad \begin{array}{ll} \text{minimize} & 0^T v + (c + d'\lambda)^T x \\ \text{subject to} & Iv + Ax = b + d''\lambda \\ & x \ge 0, \ v \ge 0 \end{array} \qquad A \in R^{m \times n}
$$

The identity matrix I is the initial basis in $[\, I \; A \,]$.

Neither b nor c is assumed to be nonnegative.

Parameter $\lambda \in [0, \bar{\lambda}]$, $\quad \bar{\lambda} = \min\left\{\min\{b_i\}, \min\{c_j\}\right\}$.

$$
d' \ge 0, \ c + d'\bar{\lambda} \ge 0 \quad \text{and} \quad d'' \ge 0, \ b + d''\bar{\lambda} \ge 0
$$

Tableau for SDPA

In tableau form, Dantzig's (P) would be given as

$$
\begin{array}{ccc||c}
-\zeta & v & x & 1 \\
\hline
0 & I & A & b \\
\hline
1 & 0^T & c^T & 0 \\
\hline
\end{array}
$$

where ζ denotes the value of the objective function.

Tableau for the parametric algorithm

In tableau form, the parametric version of (P) would be

$$\begin{array}{cc} \begin{array}{cc||c|c} v & x & 1 & \lambda \\ \hline I & A & b & d'' \\ \hline 0^T & c^T & 0 & 0 \\ \hline 0^T & {d'}^T & 0 & 0 \\ \hline \end{array} & \begin{array}{c} \\ \\ 1 \\ \lambda \end{array} \end{array}$$

The ζ-column has been omitted.

Assumption: For all $\lambda \in [0, \bar{\lambda}]$ the vector $(c + d'\lambda,\ b + d''\lambda)$ has at most one zero component (and exactly one when $\lambda = \bar{\lambda}$).

The algorithmic concept

- If $b \geq 0$ and $c \geq 0$, $x = 0$ and $y = 0$ are optimal solutions of (P) and (D). Stop. Else $\bar{\lambda} > 0$.
- If $c_r + d_r'\bar{\lambda} = 0$, attempt to perform (primal) a simplex method pivot step in the column of x_r.
- If $b_r + d_r''\bar{\lambda} = 0$, attempt to perform a dual simplex method pivot step in the row of v_r.
- If neither pivot is possible, stop. (P) has no optimal solution.
- Reduce λ to 0 or the next critical value. Repeat.

Primal simplex pivots

If decreasing λ makes $\bar{c}_r + \bar{d}_r'\lambda = 0$, we have $\bar{d}_r' > 0$.

The method acts as though $\bar{c}_r + \bar{d}_r'\lambda < 0$ (as it would be for slightly smaller λ).

This is followed by a minimum ratio test. The variable x_r is increased as much as possible so as to maintain primal feasibility:

$$x_r \leq \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{ir}} : \bar{a}_{ir} > 0 \right\}$$

If $\bar{a}_{ir} \leq 0$ for all i, then (D) is infeasible. Stop.

Dual simplex pivots

If decreasing λ makes $\bar{b}_r + \bar{d}''_r \lambda = 0$, we have $\bar{d}''_r > 0$.

The method acts as though $\bar{b}_r + \bar{d}''_r \lambda < 0$ (as it would be for slightly smaller λ).

This is followed by a minimum ratio test. The variable y_r is increased as much as possible so as to maintain dual feasibility:

$$y_r \leq \min_j \left\{ \frac{\bar{c}_j}{\bar{a}_{rj}} : \bar{a}_{rj} > 0 \right\}$$

If $\bar{a}_{rj} \geq 0$ for all j, then (P) is infeasible. Stop.

The LCP (q, M) in dictionary form

The system equation $w = q + Mz$ is represented in dictionary form as

$$
\begin{array}{c|c|cccc|}
 & 1 & z_1 & z_2 & \cdots & z_n \\
\hline
w_1 & q_1 & m_{11} & m_{12} & \cdots & m_{1n} \\
w_2 & q_2 & m_{21} & m_{22} & \cdots & m_{2n} \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
w_n & q_2 & m_{n1} & m_{n2} & \cdots & m_{nn} \\
\hline
\end{array}
$$

The (Symmetric) Principal Pivoting Method (PPM): Generalities

General features:

- Limited to “sufficient” matrices
- Not parametric
- Works with a sequence of major cycles, each of which is associated with a distinguished basic variable
- Each major cycle ends with the distinguished variable becoming nonbasic or an unblocked driving (increasing nonbasic) variable
- Preserves complementarity of bases
- Monotonically reduces the number of negative variables

Principal pivoting

For $M \in R^{n \times n}$ with $\det(M_{\alpha\alpha}) \neq 0$, the corresponding *principal pivotal transformation* of M is the matrix $\wp_\alpha(M) = \bar{M}$ such that

$$\begin{bmatrix} \bar{M}_{\alpha\alpha} & \bar{M}_{\alpha\bar{\alpha}} \\ \bar{M}_{\bar{\alpha}\alpha} & \bar{M}_{\bar{\alpha}\bar{\alpha}} \end{bmatrix} = \begin{bmatrix} M_{\alpha\alpha}^{-1} & -M_{\alpha\alpha}^{-1} M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} & M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\bar{\alpha}} \end{bmatrix}$$

The idea behind the PPM

$$w = q + Mz$$

- Initially $m_{ij} = \partial w_i / \partial z_j$ and $m_{ii} \geq 0$ for all i.
- If $q_i < 0$ and $m_{ii} > 0$, increasing z_i makes w_i increase.
- If $q_k \geq 0$ and $m_{ki} < 0$, increase of z_i could make w_k decrease to 0 (before w_i increases to 0).
- If M is sufficient, $m_{ii} = 0$, and $m_{ki} < 0$, then $m_{ik} > 0$, so the increase of z_k would make w_i increase to zero.

Steps of the (Symmetric) PPM (simple case)

Step 0. *Input data, choose* $\mu < \min q_i < 0$

Step 1. *Determine the distinguished variable,* w_r^ν

Step 2. *Determine the blocking variable,* w_s^ν

Step 3. *Pivot* $\langle w_s, z_r \rangle$*, return to Step 1*

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Many details have been omitted. The important features are:

- If $s = r$ and $m_{rr}^\nu > 0$, pivot $\langle w_r^\nu, z_r^\nu \rangle$
- If $s \neq r$ and $m_{ss}^\nu > 0$, pivot $\langle w_s^\nu, z_s^\nu \rangle$
- If $s \neq r$ and $m_{ss}^\nu = 0$, pivot $\{\langle w_s^\nu, z_r^\nu \rangle, \langle w_r^\nu, z_s^\nu \rangle\}$

Bisymmetry

A matrix $M \in R^{n \times n}$ is *bisymmetric* if there exists an $n \times n$ permutation matrix P such that

$$PMP^T = \begin{bmatrix} G & -A^T \\ A & H \end{bmatrix}$$

where G and H are symmetric.

Symmetry ($M - M^T = 0$) and *skew-symmetry* ($M + M^T = 0$) are special cases of *bisymmetry*.

Invariance under principal pivoting

If all principal pivotal transforms $\wp_\alpha$ of all members M of a class $\mathcal{C}$ of squares matrices belong to that class, the class is said to be *invariant under principal pivoting.*

$$M \in \mathcal{C} \implies \wp_\alpha(M) \in \mathcal{C}$$

Examples of such classes are the positive semidefinite matrices and the bisymmetric matrices.

The LP case

The matrix M of an LCP (q, M) associated with a linear program in nonnegative variables is

- Skew-symmetric
- Positive semidefinite

On $K(M)$ and its convexity

Given $M \in R^{n\times n}$, the union of the corresponding complementary cones is the closed cone

$$K(M) = \bigcup_{\alpha} \operatorname{pos} B_\alpha, \qquad B_\alpha := [I_{\cdot\bar{\alpha}} \ - M_{\cdot\alpha}]$$

$$= \{q \in R^n : (q, M) \text{ has a solution } \}$$

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The cone $K(M)$ is *convex* if and only if for all $q \in R^n$,

$$\text{FEA}\,(q, M) \neq \emptyset \Longrightarrow \text{SOL}\,(q, M) \neq \emptyset$$

When this holds, we say $M \in \mathcal{Q}_0$.

On the monotonicity of the parameter

Suppose the LCP (q, M) represents the optimality conditions of an LP for which $\min q_i < 0$.

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In these parametric methods (and LS1), the parameter λ will first be increased to the value $\bar{\lambda} > 0$.

Several authors have established that λ *does not increase during the solution process.*

Why should this be the case? It has to do with the positive semi-definiteness of M and the invariance of positive semi-definiteness under principal pivoting.

Why the parameter does not increase in the LP case

Let (q, M) represent an LP and $\min q_i < 0$ for some i.

Let $\underline{\lambda} = \min\{\lambda \geq 0 : \text{FEA}\,(q + d\lambda, M) \neq \emptyset\}$.

- M is skew-symmetric
- All skew-symmetric matrices are positive semidefinite
- All positive semidefinite matrices belong to $\mathcal{P}_0 \cap \mathcal{Q}_0$
- $K(M)$ is convex for all $M \in \mathcal{Q}_0$
- $q + d\lambda \in K(M)$ for all $\lambda \in [\underline{\lambda}, \bar{\lambda}]$
- The positive semi-definiteness is invariant under principal pivoting.

Lemke's Algorithm, Scheme 1 (LS1)

Step 0. *Initialization.* Input (q, d, M) with $q + dz_0 \geq 0$ for all $z_0 \geq \bar{z}_0 = \max\{\frac{-q_i}{d_i} : q_i < 0\}$. Stop if $q \geq 0$. Else some w_r blocks z_0. Pivot $\langle w_r, z_0 \rangle$ where $=$ arg max $\{\frac{-q_i}{d_i} : q_i < 0\}$.

Step 1. *Finding blocking variable (if any).* The new driving variable is the complement of the last blocking variable. Increase the driving variable. Stop if it is unblocked. (Interpret this outcome, if possible.)

Step 2. *Pivoting.* If z_0 blocks the driving variable, pivot

$$\langle z_0, \textit{driving variable} \rangle.$$

A solution has been found. Otherwise pivot

$$\langle \textit{blocking variable}, \textit{driving variable} \rangle$$

and return to Step 1 with the complement of the blocking variable as the new driving variable.

Lemke's Algorithm (Scheme 1) in parametric form (LS1P)

Step 0. Input (q, d, M) where $d \geq 0$ covers q. If $q \geq 0$, stop: $z = 0$ solves (q, M). Otherwise, let $\bar{z}_0 = \max_i\{-q_i/d_i\}$ and let $r = \arg\max_i\{-q_i/d_i\}$. Set $\nu = 0$ and define

$$(q^\nu, d^\nu, M^\nu) = (q, d, M), \quad (w^\nu, z^\nu) = (w, z), \quad z_0^\nu = \bar{z}_0.$$

Step 1. Define w_r^ν as the distinguished variable and its complement z_r^ν as the driving variable. Apply LS1 on (q^ν, d^ν, M^ν). This will yield either a solution of (q^ν, d^ν, M^ν) or termination on a ray (i.e., with an unblocked driving variable).

Step 2. If $q^\nu \geq 0$, decrease z_0 to zero. Stop: a solution of (q, M) has been found. If $d^\nu \geq 0$, stop. The algorithm terminates unsuccessfully (on a z_0-ray). Otherwise $\min_i d_i^\nu < 0 < \max_i d_i^\nu$. Let $z_0^{\nu-1}$ denote the current value of z_0; it is either $\max\{-q_i^\nu/d_i^\nu : d_i^\nu > 0\}$ or $\min\{-q_i^\nu/d_i^\nu : d_i^\nu < 0\}$.

- If $z_0^{\nu-1} = \max\{-\frac{q_i^\nu}{d_i^\nu} : d_i^\nu > 0\}$, set $z_0^\nu = \min\{-\frac{q_i^\nu}{d_i^\nu} : d_i^\nu < 0\}$;
- if $z_0^{\nu-1} = \min\{-\frac{q_i^\nu}{d_i^\nu} : d_i^\nu < 0\}$, set $z_0^\nu = \max\{-\frac{q_i^\nu}{d_i^\nu} : d_i^\nu > 0\}$.

(Put $w^\nu = q^\nu + d^\nu z_0^\nu$.) Return to Step1 with the unique $r = \arg\min_i w_i^\nu$.

Dantzig's claim

- Dantzig (1963)

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- 1970 Ravindran's paper, McCammon's thesis

Ravindran's paper

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- was published in the same year as McCammon's Ph.D. thesis (to be discussed shortly)
- compares Dantzig's SDPA with Lemke's Scheme 1 on LPs
- gives a *parametric, structure-preserving* variant of Lemke's Scheme 1 on LPs
- does not establish the general equivalence of Lemke's Scheme 1 and its parametric form
- establishes that the parameter z_0 decreases monotonically

Ravindran's statement

Those who are familiar with the complementary pivot theory, can easily see that in our algorithm we are essentially applying Lemke's method but to a reduced tableau, taking advantage of M-structure.

Dantzig has claimed that the Lemke's complementary pivot method to solve linear programs is identical with respect to the pivot steps to his self-dual parametric algorithm. Though this fact is not very obvious to see, using our algorithm we can see that the pivot steps are identical to those in the self-dual parametric algorithm though Dantzig uses the idea of primal and dual simplex method while we make use of complementarity between the variables. Since our method is a condensed form of Lemke's method in some sense, we have shown that Dantzig's claim may be valid.

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- discusses a few standard matrix classes

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- develops the parametric version of Lemke's algorithm (replacing the "pseudo-variable" z_0 by a parameter θ)

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- discusses a few standard matrix classes
- develops the parametric version of Lemke's algorithm (replacing the "pseudo-variable" z_0 by a parameter θ)
- shows that the parametric pivoting algorithm and Lemke's algorithm are equivalent
- discusses the principal pivoting method

McCammon's thesis (continued)

- notes the strict decrease of z_0 in the case where M is a P-matrix
- alludes to the fact that the choice of covering vector can influence the number of iterations and the solution found
- discusses the Dantzig's SDPA for LP and the restoration of skew symmetry after a principal block pivot of order 2
- gives a direct numerical comparison of the SDPA and the parametric pivoting method (LS1P)
- the above comparison reaches a faulty conclusion which will be illustrated here

McCammon's statement

At the time of initiating the study which culminated in this dissertation, it was not clear what the relationship between the principal pivoting method and Lemke's method was, and it was felt that an investigation would clarify the similarities and distinctions. This dissertation, indeed, sheds light on this relationship. A related study was generated by two remarks of Dantzig made to Professor Lemke: in a letter to Lemke in 1965, Professor Dantzig expressed the feeling that the use of one pseudo-vector, as in Lemke's method, was not the most efficient way to handle the problem. This suggested the use of n pseudo-variables, and is considered in this thesis. Professor Dantzig also remarked in a conversation with Lemke in 1966 that the Lemke method when applied to the linear programming problem was equivalent to the 'self-dual parametric algorithm'. This suggested the attempt to view Lemke's method as a 'Parametric method', and gave rise to the Parametric Pivoting Method developed by the author.

van de Panne's paper

C. VAN DE PANNE,
A complementary variant of Lemke's method for the linear complementarity problem, *Mathematical Programming* 7 (1974), 283–310.

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- cites McCammon but not Ravindran
- gives an algorithm that is related to McCammon's (i.e., LS1P) but is *multi-parametric*

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- cites McCammon but not Ravindran
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- supposedly has the advantage of allowing for infeasibility tests

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- cites McCammon but not Ravindran
- gives an algorithm that is related to McCammon's (i.e., LS1P) but is *multi-parametric*
- supposedly has the advantage of allowing for infeasibility tests
- asserts that the "complementary variant" and Lemke's method (LS1) are equivalent

van de Panne's statement

The complementary variant is related to the parametric pivot method proposed by McCammon. . . . In the latter method only λ is varied as a parameter, while in the complementary variant also a number of nonbasic variables are varied as parameters.

Both McCammon's method and the complementary variant are equivalent to Lemke's method in the sense that the same successive solutions are generated . McCammon's method and the complementary variant are identical as long as no fixed variables appear in the latter method.

McCammon's statement . . . that this parametric pivoting method and Dantzig's self-dual parametric method for linear programming are not necessarily equivalent is based on errors in his example; Lemke's method and consequently McCammon's method and the complementary variant applied to linear programming problems are equivalent to Dantzig's method.

Lustig's thesis

I.J. LUSTIG
Comparisons of Composite Simplex Algorithms,
Ph.D. thesis, Stanford University, 1987.

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- compared the SDPA with LS1 (but not LS1P) applied to LP
- noted strict decrease of the artificial variable z_0 in LS1
- commented on McCammon's conclusion on Dantzig's claim

Lustig's comments

McCammon (1970), in his Ph.D. thesis at Rensselaer Polytechnic Institute, considered Dantzig's claim as well. He presented his own parametric pivoting method for general linear complementarity problems . . .
McCammon proved that his algorithm is equivalent to Lemke's algorithm applied to the LCP (q, M). McCammon, however, believed that the solution path of his algorithm did not correspond to that of Dantzig's self-dual algorithm. In the last section of Appendix II of his thesis, he applied his algorithm and Dantzig's algorithm to a numerical example in order to show that the two solution paths need not correspond. A careful examination of his calculations indicates that he applied the two algorithms to two different linear programs.

A short description of McCammon's algorithm (without proof) is given by Lemke (1970), pp. 359–361. I find McCammon's proof and the statement of his algorithm somewhat unclear. It is possible that his algorithm is equivalent to Cottle's (1972) parametric pivoting algorithm. Cottle's algorithm is well-defined when M has positive principal minors or M is positive semi-definite. If they are equivalent, then it is not clear whether McCammon realized the necessity of having assumptions on the properties of M in order to make his algorithm well-defined.

Lustig's result (paraphrased)

Theorem

Assume that Dantzig's self-dual parametric algorithm is executed on a linear program (in inequality form) and that Lemke's algorithm is executed on the corresponding LCP and an optimal solution is found by the self-dual method in ℓ iterations. Then the pivots of iteration t of the self-dual algorithm correspond in a precise way to the pivots of iterations $2t-1,\ 2t,$ and $2t+1$ of Lemke's algorithm for $t = 1, 2, \ldots, \ell$.

McCammon's LP formulated and solved by SDPA

7. Dantzig's self-dual parametric algorithm

Consider the problem

$$\text{minimize } 4x_1 - 3x_2 \quad \text{subject to } \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \quad \begin{matrix} x_1 \geq 0, \\ x_2 \geq 0. \end{matrix}$$

The self-dual parametric method considers the problem:

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$$\text{minimize } 4x_1 + (\theta - 3x_2) \text{ subject to } \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \theta - 2 \\ \theta - 1 \end{pmatrix},$$

$x_1 \geq 0$, $x_2 \geq 0$.

McCammon's LP: solved Dantzig-style in tableaux

1	2
3	4

y_1	y_2	x_1	x_2	1	θ	
1	0	-1	1	-2	1	row1
0	1	1	-2	-1	1	row2
0	0	4	-3	0	0	1
0	0	0	1	0	0	θ

$\theta \geq 3,\ \langle y_1, x_2 \rangle_{\text{(P)}}$

y_1	y_2	x_1	x_2	1	θ	
1	0	-1	1	-2	1	row1
2	1	-1	0	-5	3	row2
3	0	1	0	-6	3	1
-1	0	1	0	2	-1	θ

$\theta \in [2, 3],\ \langle x_2,\ x_1 \rangle_{\text{(D)}}$

y_1	y_2	x_1	x_2	1	θ	
-1	0	1	-1	2	-1	row1
1	1	0	-1	-3	2	row2
4	0	0	1	-8	4	1
0	0	0	1	0	0	θ

$\theta \in [\frac{3}{2}, 2],\ \langle y_2,\ x_2 \rangle_{\text{(D)}}$

y_1	y_2	x_1	x_2	1	θ	
-2	-1	1	0	5	-3	row1
-1	-1	0	1	3	-2	row2
5	1	0	0	-11	6	1
1	1	0	0	-3	2	θ

$\theta \in [0,\ \frac{3}{2}],\ \hat{\zeta} = 11$

LS1P on McCammon's LP: formulation of the model

McCammon formulated the system equation of a parametric LCP for solving his LP as follows:

$$w = \begin{pmatrix} 4 \\ -3 \\ -2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \theta + \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{pmatrix} z$$

LS1P on McCammon's LP: as he "solved" it in tableaux

	1	θ	z_1	z_2	z_3	z_4
w_1	4	0	0	0	1	-1
w_2	-3	1	0	0	-1	2
w_3	-2	1	-1	1	0	0
w_4	-1	1	1	-2	0	0

$\theta \geq 3; \quad r = 2; \quad \langle w_4,\ z_2 \rangle$

	1	θ	z_1	w_4	z_3	z_4
w_1	4	0	0	0	1	-1
w_2	-3	1	0	0	-1	2
w_3	$-5/2$	$3/2$	$-1/2$	$-1/2$	0	0
z_2	$-1/2$	$1/2$	$1/2$	$-1/2$	0	0

$\theta \geq 3; \quad r = 2; \quad \langle w_2,\ z_4 \rangle$

	1	θ	z_1	w_4	z_3	w_2
w_1	$5/2$	$1/2$	0	0	$1/2$	$-1/2$
z_4	$3/2$	$-1/2$	0	0	$1/2$	$1/2$
w_3	$-5/2$	$3/2$	-1	1	0	0
z_2	$-1/2$	$1/2$	1	-2	0	0

$\theta \in [\frac{5}{3}],\ 3]; \quad r = 3; \quad z_3 \uparrow \infty$

LS1P on McCammon's LP: as he "solved" it in tableaux

1	2
3	

	1	θ	z_1	z_2	z_3	z_4
w_1	4	0	0	0	1	-1
w_2	-3	1	0	0	-1	2
w_3	-2	1	-1	1	0	0
w_4	-1	1	1	-2	0	0

$\theta \geq 3; \quad r = 2; \quad \langle w_4,\, z_2 \rangle$

	1	θ	z_1	w_4	z_3	z_4
w_1	4	0	0	0	1	-1
w_2	-3	1	0	0	-1	2
w_3	$-5/2$	$3/2$	$-1/2$	$-1/2$	0	0
z_2	$-1/2$	$1/2$	$1/2$	$-1/2$	0	0

$\theta \geq 3; \quad r = 2; \quad \langle w_2,\, z_4 \rangle$

	1	θ	z_1	w_4	z_3	w_2
w_1	$5/2$	$1/2$	0	0	$1/2$	$-1/2$
z_4	$3/2$	$-1/2$	0	0	$1/2$	$1/2$
w_3	$-5/2$	$3/2$	-1	1	0	0
z_2	$-1/2$	$1/2$	1	-2	0	0

$\theta \in [\frac{5}{3}],\, 3]; \quad r = 3; \quad z_3 \uparrow \infty$

OOPS!

LS1P on McCammon's LP: as he "solved" it in tableaux

$$
\begin{array}{c|c|c|cc|cc|}
 & 1 & \theta & z_1 & z_2 & z_3 & z_4 \\
\hline
w_1 & 4 & 0 & 0 & 0 & 1 & -1 \\
w_2 & -3 & 1 & 0 & 0 & -1 & 2 \\
\hline
w_3 & -2 & 1 & -1 & 1 & 0 & 0 \\
w_4 & -1 & 1 & 1 & -2 & 0 & 0 \\
\hline
\end{array}
$$

$\theta \geq 3; \quad r = 2; \quad \langle w_4,\ z_2 \rangle$

$$
\begin{array}{c|c|c|cc|cc|}
 & 1 & \theta & z_1 & w_4 & z_3 & z_4 \\
\hline
w_1 & 4 & 0 & 0 & 0 & 1 & -1 \\
w_2 & -3 & 1 & 0 & 0 & -1 & 2 \\
\hline
w_3 & -5/2 & 3/2 & -1/2 & -1/2 & 0 & 0 \\
z_2 & -1/2 & 1/2 & 1/2 & -1/2 & 0 & 0 \\
\hline
\end{array}
$$

$\theta \geq 3; \quad r = 2; \quad \langle w_2,\ z_4 \rangle$

$$
\begin{array}{c|c|c|cc|cc|}
 & 1 & \theta & z_1 & w_4 & z_3 & w_2 \\
\hline
w_1 & 5/2 & 1/2 & 0 & 0 & 1/2 & -1/2 \\
z_4 & 3/2 & -1/2 & 0 & 0 & 1/2 & 1/2 \\
\hline
w_3 & -5/2 & 3/2 & -1 & 1 & 0 & 0 \\
z_2 & -1/2 & 1/2 & 1 & -2 & 0 & 0 \\
\hline
\end{array}
$$

$\theta \in [\frac{5}{3}],\ 3]; \quad r = 3; \quad z_3 \uparrow \infty$

OOPS!

How can this be?

LS1P on McCammon's LP: formulation of the model

Correct formulation of McCammon's system equation of a parametric LCP for solving his LP:

$$w = \begin{pmatrix} 4 \\ -3 \\ -2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \theta + \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \end{pmatrix} z$$

LS1P on McCammon's LP: correctly solved in 7 tableaux

	1	θ	z_1	z_2	z_3	z_4
w_1	4	0	0	0	-1	1
$\star w_2$	-3	1	0	0	1	-2
w_3	-2	1	1	-1	0	0
w_4	-1	1	-1	2	0	0

$\theta \geq 3; \quad r = 2; \quad \langle w_3, z_2 \rangle$

	1	θ	z_1	w_3	z_3	z_4
w_1	4	0	0	0	-1	1
$\star w_2$	-3	1	0	0	1	-2
z_2	-2	1	1	-1	0	0
w_4	-5	3	1	-2	0	0

$\theta \geq 3; \quad r = 2; \quad \langle w_2, z_3 \rangle$

	1	θ	z_1	w_3	w_2	z_4
w_1	1	1	0	0	-1	-1
z_3	3	-1	0	0	1	2
$\star z_2$	-2	1	1	-1	0	0
w_4	-5	3	1	-2	0	0

$\theta \in [2, 3]; \quad r = 3; \quad \langle w_1, w_2 \rangle$

	1	θ	z_1	w_3	w_1	z_4
w_2	1	1	0	0	-1	-1
z_3	4	0	0	0	-1	1
$\star z_2$	-2	1	1	-1	0	0
w_4	-5	3	1	-2	0	0

$\theta \in [2, 3]; \quad r = 3; \quad \langle z_2, z_1 \rangle$

	1	θ	z_2	w_3	w_1	z_4
w_2	1	1	0	0	-1	-1
z_3	4	0	0	0	-1	1
z_1	2	-1	1	1	0	0
$\star w_4$	-3	2	1	-1	0	0

$\theta \in [3/2, 2]; \quad r = 4; \quad \langle w_2, z_4 \rangle$

	1	θ	z_2	w_3	w_1	w_2
z_4	1	1	0	0	-1	-1
z_3	5	1	0	0	-2	-1
z_1	2	-1	1	1	0	0
$\star w_4$	-3	2	1	-1	0	0

$\theta \in [3/2, 2]; \quad r = 4; \quad \langle w_4, z_2 \rangle$

LS1P on McCammon's LP: reformulated and solved

$$
\begin{array}{c|c|c|cc|cc|}
 & 1 & \theta & w_4 & w_3 & w_1 & w_2 \\
\hline
z_4 & 1 & 1 & 0 & 0 & -1 & 1 \\
z_3 & 5 & 1 & 0 & 0 & 1 & -2 \\
\hline
z_1 & 5 & -3 & 1 & 2 & 0 & 0 \\
z_2 & 3 & -2 & 1 & 1 & 0 & 0 \\
\hline
\end{array}
$$

$\theta \in [0,\ 3/2]$; solution: $(z_1, z_2) = (5, 3)$

PPPM on LP is SDPA

- Dantzig's LP in tableau form

$$
\begin{array}{c}
\begin{array}{cccc} -\zeta & v & x & 1 \end{array} \\
\begin{array}{|ccc||c|}
\hline
0 & I & A & b \\
\hline
1 & 0^T & c^T & 0 \\
\hline
\end{array}
\end{array}
$$

- In dictionary form, the LCP for Dantzig's LP is

$$
\begin{array}{c|c|cc|}
 & 1 & x & y \\
\cline{2-4}
u & c & 0 & A^T \\
v & b & -A & 0 \\
\cline{2-4}
\end{array}
$$

Primal pivot $\langle v_r, x_s \rangle$ in Dantzig's tableau

After primal pivot $\langle v_r, x_s \rangle$, the Dantzig-style subtableau

$$
\begin{array}{ccc|c}
v_r & x_j & x_s & 1 \\
\hline
0 & a_{ij} & a_{is} & b_i \\
1 & a_{rj} & a_{rs} & b_r \\
\hline
0 & c_j & c_s & 0 \\
\hline
\end{array}
$$

becomes

$$
\begin{array}{ccc|c}
v_r & x_j & x_s & 1 \\
\hline
-a_{is} & a_{ij}a_{rs} - a_{rj}a_{is} & 0 & b_i a_{rs} - b_r a_{is} \\
1 & a_{rj} & a_{rs} & b_r \\
\hline
-c_s & c_j a_{rs} - c_s a_{rj} & 0 & -b_r c_s \\
\hline
\end{array}
\times \frac{1}{a_{rs}}
$$

Pivot $\langle v_r, x_s \rangle$ in LCP tableau

In the LCP subtableau

$$
\begin{array}{c|c|cc|}
 & 1 & x_j & x_s \\
\hline
v_i & b_i & -a_{ij} & -a_{is} \\
v_r & b_r & -a_{rj} & -a_{rs} \\
\hline
\end{array}
$$

the pivot $\langle v_r, x_s \rangle$ produces

$$
\begin{array}{c|c|cc|}
 & 1 & x_j & v_r \\
\hline
v_i & b_i a_{rs} - b_r a_{is} & -a_{ij} a_{rs} + a_{rj} a_{is} & a_{is} \\
x_s & b_r & -a_{rj} & -1 \\
\hline
\end{array}
\times \frac{1}{a_{rs}}
$$

This is the first half of the 2×2 block pivot $\langle v_r, x_s \rangle, \ \langle u_s, y_r \rangle$.

Pivot $\langle u_s, y_r \rangle$ in LCP tableau

In the LCP subtableau

$$\begin{array}{c|c|cc|} & 1 & y_i & y_r \\ \cline{2-4} u_j & c_j & a'_{ji} & a'_{jr} \\ u_s & c_s & a'_{si} & a'_{sr} \\ \cline{2-4} \end{array} \qquad\qquad A' = A^T \quad (a'_{ji} = a_{ij})$$

the pivot $\langle u_s, y_r \rangle$ produces

$$\begin{array}{c|c|cc|} & 1 & y_i & u_s \\ \cline{2-4} u_j & c_j a'_{sr} - c_s a'_{jr} & a'_{ji} a'_{sr} - a'_{jr} a'_{si} & a'_{jr} \\ y_r & -c_s & -a'_{si} & 1 \\ \cline{2-4} \end{array} \times \frac{1}{a'_{sr}}$$

Since $a'_{ji} = a_{ij}$ for all i, j, the matrix resulting from $\langle u_s, y_r \rangle$ is the negative transpose of the one resulting from $\langle v_r, x_s \rangle$

Comparison of SDPA and PPPM

SDPA

$$\begin{array}{cccc|c} v_r & x_j & x_s & & 1 \\ \end{array}$$

$$\left[\begin{array}{ccc||c} -a_{is} & a_{ij}a_{rs} - a_{rj}a_{is} & 0 & b_i a_{rs} - b_r a_{is} \\ 1 & a_{rj} & a_{rs} & b_r \\ \hline -c_s & c_j a_{rs} - c_s a_{rj} & 0 & -b_r c_s \end{array}\right] \times \frac{1}{a_{rs}}$$

PPPM

$$\begin{array}{c|c|cc|} & 1 & x_j & v_r \\ \hline v_i & b_i a_{rs} - b_r a_{is} & -a_{ij}a_{rs} + a_{rj}a_{is} & a_{is} \\ x_s & b_r & -a_{rj} & -1 \\ \hline \end{array} \times \frac{1}{a_{rs}}$$

$$\begin{array}{c|c|cc|} & 1 & y_i & u_s \\ \hline u_j & c_j a'_{sr} - c_s a'_{jr} & a'_{ji}a'_{sr} - a'_{jr}a'_{si} & a'_{jr} \\ y_r & -c_s & -a'_{si} & 1 \\ \hline \end{array} \times \frac{1}{a'_{sr}}$$

Further remarks

- The conclusion is the same if the first pivot is of the Dual Simplex type.
- As applied to Linear Programming, the SDPA and the PPPM are equivalent in terms of iterates, space, and computational effort.
- Question:

Further remarks

- The conclusion is the same if the first pivot is of the Dual Simplex type.
- As applied to Linear Programming, the SDPA and the PPPM are equivalent in terms of iterates, space, and computational effort.
- Question: Did Dantzig realize this too?