Multigrid Semismooth Newton Methods for Elastic Contact Problems

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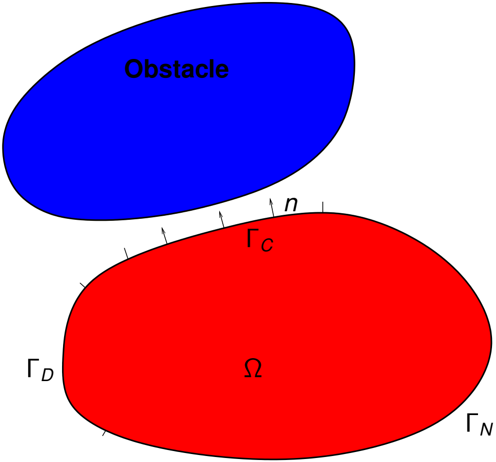
Outline

- Contact problem in 3D elasticity
- Regularized dual problem and error estimates
- Application of semismooth Newton methods
- Multigrid method for discrete semismooth Newton system
- Convergence result and condition number estimate
- Numerical results

Elastic 3D Contact Problem (Signorini Problem)



Elastic 3D Contact Problem (Signorini Problem)

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Elastic 3D Contact Problem as Optimization Problem (P):

$$\min_{u\in V} \; J(u) := \int_\Omega \Big(\mu\epsilon(u) : \epsilon(u) + \frac{\lambda}{2}\mathrm{div}(u)^2 - f_V^T u\Big)\,dx - \int_{\Gamma_N} f_S^T u \, dS(x)$$
$$\text{s. t.} \quad u^T n \le g \quad \text{on } \Gamma_C$$

$\Omega \subset \mathbb{R}^3$	reference domain of an elastic body,
$\Gamma_D, \Gamma_N \subset \partial\Omega$	Dirichlet boundary, Neumann boundary,
$\Gamma_C \subset \partial\Omega$	possible contact boundary on Ω,
$u \in V$	displacement, $V = \left\{u \in H^1(\Omega)^3 \, ; \; u\vert_{\Gamma_D} = 0\right\}$
$\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$	linearized strain,
λ, μ	Lamé material constants,
$u^T n$	normal displacement on Γ_C,
$g \in H^{1/2}(\Gamma_C)$	normal distance of the body to the obstacle,
$f_V \in L^2(\Omega)^3$, $f_S \in L^2(\Gamma_N)^3$	volume / surface forces.

Related Work

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- **Semismooth Newton methods for contact problems:**
  Christensen, Hoppe, Hüeber, Ito, Kunisch, Pang, Stadler, M. Ulbrich, S. U., Wohlmuth, ...
- **Multilevel methods for contact problems:**
  Dostal, Hüeber, Kornhuber, Krause, Schöberl, Stadler, Wohlmuth, ...
- **Abstract multilevel theory (only the references we build on):**
  Bornemann, Yserentant (... and many more)
- **Multilevel trust region methods:**
  Gratton, von Loesch, Toint, ...
- **Regularization of obstacle and state constrained problems:**
  Hintermüller, Ito, Kunisch, Meyer, Prüfert, Rösch, Schiela, Tröltzsch, Weiser, ...

A Class of Nonlinear Elastic 3D Contact Problems

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$$\min_{u \in V} \; J(u) := \int_\Omega \Big(\Phi(x, \epsilon(u) : \epsilon(u)) + \frac{1}{2}\Psi(x, \mathrm{div}(u)^2) - f_V^T u\Big)\, dx - \int_{\Gamma_N} f_S^T u \, dS(x)$$
$$\text{s. t.} \quad u^T n \le g \quad \text{on } \Gamma_C$$

cf. Necas, Hlavacek 81; Axelsson, Padiy 00; Blaheta 97

$\Phi(x, s) = \mu s$, $\Psi(x, s) = \lambda s$ recovers the linear case.

Assumptions:

- $0 < \mu_0 \le \Phi'(s) \le \mu_1$
- $0 < \lambda_0 \le \Psi'(s) \le \lambda_1$
- $0 < \mu_0' \le \frac{\partial}{\partial s}(\Phi'(s^2)s) \le \mu_1'$
- $0 < \mu_0' \le \frac{\partial}{\partial s}(\Psi'(s^2)s) \le \mu_1'$

Several results of the talk can be extended to this case (current work).

Elastic 3D Contact Problem

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Elastic 3D Contact Problem as Optimization Problem (P):

$$\min_{u \in V} \; J(u) := \int_\Omega \left(\mu \epsilon(u) : \epsilon(u) + \frac{\lambda}{2} \mathrm{div}(u)^2 - f_V^T u \right) dx - \int_{\Gamma_N} f_S^T u \, dS(x)$$
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KKT-System of the Elastic Contact Problem

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We define $a : V \times V \to \mathbb{R}$, $A \in \mathcal{L}(V, V^*)$, $N \in \mathcal{L}(V, H^{1/2}(\Gamma_C))$, $f \in V^*$ by

$$a(v, w) = \langle v, Aw \rangle_{V,V^*} = \int_\Omega \big(2\mu\epsilon(v) : \epsilon(w) + \lambda \mathrm{div}(v)\mathrm{div}(w)\big)\, dx,$$

$$Nu = u^T n|_{\Gamma_C}, \qquad \langle f, u \rangle_{V^*,V} = \int_\Omega f_V^T u\, dx + \int_{\Gamma_N} f_S^T u\, dS(x).$$

Contact problem (P) in abstract form:

$$\min_{u \in V} \ \frac{1}{2} a(u, u) - \langle f, u \rangle_{V^*,V} \quad \text{s. t.} \quad Nu \leq g.$$

The problem is uniformly convex and quadratic.

KKT-System of the Elastic Contact Problem

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$$\min_{u \in V} \ \frac{1}{2} a(u,u) - \langle f, u \rangle_{V^*,V} \quad \text{s. t.} \quad Nu \le g.$$

The problem is uniformly convex and quadratic.

Optimality conditions:

$u \in V$ solves (P) if and only if there exists $z \in H^{1/2}(\Gamma_C)^*$ such that

$$Au - f + N^* z = 0$$

$$z \ge 0, \quad Nu - g \le 0, \quad \langle z, Nu - g \rangle_{(H^{1/2})^*, H^{1/2}} = 0.$$

Here, $z \ge 0$ means $\quad \langle z, v \rangle_{(H^{1/2})^*, H^{1/2}} \ge 0 \quad \forall\, v \in H^{1/2}(\Gamma_C),\ v \ge 0.$

Dual Problem

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Applying Lagrange duality yields the

Equivalent dual problem (D):

$$\max_{z \in H^{1/2}(\Gamma_C)^*} \quad -\frac{1}{2}\langle z, NA^{-1}N^* z\rangle_{(H^{1/2})^*, H^{1/2}} + \langle z, NA^{-1}f - g\rangle_{(H^{1/2})^*, H^{1/2}}$$
$$\text{s. t.} \quad z \geq 0.$$

In the following, we assume sufficient regularity of the problem data and the solution u of (P) to ensure the following:

Assumption: The optimal solution of (D) satisfies $z \in L^2(\Gamma_C)$ (Necas).

Idea: Replace the numerically inconvenient space $H^{1/2}(\Gamma_C)^*$ by $L^2(\Gamma_C)$.

But: Objective function of (D) is coercive in $H^{1/2}(\Gamma_C)^*$ but not in $L^2(\Gamma_C)$.

Remedy: We introduce an L^2-regularization.

Regularization of the Dual Problem

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Dual problem (D):

$$\max_{z \in H^{1/2}(\Gamma_C)^*} -\frac{1}{2}\langle z, NA^{-1}N^* z\rangle_{(H^{1/2})^*,H^{1/2}} + \langle z, NA^{-1}f - g\rangle_{(H^{1/2})^*,H^{1/2}}$$
$$\text{s. t.} \quad z \geq 0.$$

We add an L^2-regularization and obtain the following

Regularized dual problem (D$_\gamma$):

$$\max_{z \in L^2(\Gamma_C)} -\frac{1}{2}(z, NA^{-1}N^* z)_{L^2} - \frac{\gamma}{2}\|z - z^r\|_{L^2}^2 + (z, NA^{-1}f - g)_{L^2}$$
$$\text{s. t.} \quad z \geq 0.$$

Here, $\gamma > 0$ and $z^r \in L^2(\Gamma_C)$ are suitably chosen.
Problem is uniformly concave and quadratic (variant of normal compliance reg.).

Error Estimates

Dual problem (D):

$$\max_{z\in L^2(\Gamma_C)^*} -\frac{1}{2}\left(z, NA^{-1}N^*z\right)_{L^2} + \left(z, NA^{-1}f - g\right)_{L^2} \quad \text{s. t.} \quad z \geq 0.$$

Regularized dual problem (D_γ):

$$\max_{z\in L^2(\Gamma_C)} -\frac{1}{2}(z, (NA^{-1}N^*z)_{L^2} - \frac{\gamma}{2}\|z - z^r\|_{L^2}^2 + (z, NA^{-1}f - g)_{L^2} \ \text{s. t.} \ z \geq 0.$$

Let z^* and z_γ be solutions of (D) and (D_γ), with displacements $u^*, u_\gamma \in V$, i.e.,

$$Au^* - f + N^*z^* = 0, \quad Au_\gamma - f + N^*z_\gamma = 0.$$

Then: $\|z_\gamma - z^*\|_{(H^{1/2})^*} = o(\gamma^{1/2}), \quad \|u_\gamma - u^*\|_{H^1} = o(\gamma^{1/2}).$

(M. Ulbrich, S.U., Bratzke 13; see also Chouly, Hild 12)

Nonsmooth Reformulation

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Optimality conditions of the regularized dual problem (D_γ):

$u_\gamma \in V$ and $z_\gamma \in L^2(\Gamma_C)$ satisfy

$$Au_\gamma - f + N^* z_\gamma = 0$$
$$z_\gamma \geq 0, \quad Nu_\gamma - \gamma(z_\gamma - z^r) - g \leq 0, \quad z_\gamma\,(Nu_\gamma - \gamma(z_\gamma - z^r) - g) = 0.$$

Using the NCP-Function $\min(a, \gamma^{-1} b) = a - \max(0, a - \gamma^{-1} b)$ this can be rewritten as follows:

Nonsmooth reformulation (R_γ):

$$Au_\gamma - f + N^* z_\gamma = 0$$
$$z_\gamma - \max(0, \gamma^{-1}(Nu_\gamma - g) + z^r) = 0.$$

This system is a semismooth equation.

Semismooth Operators

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Let be given a continuous operator $H : X \to Y$ between Banach spaces and a setvalued **generalized differential** $\partial H : X \rightrightarrows \mathcal{L}(X, Y)$.

The operator H is called **∂H-semismooth** at $x \in X$ if

$$\sup_{M \in \partial H(x+s)} \|H(x+s) - H(x) - Ms\|_Y = o(\|s\|_X) \quad (\|s\|_X \to 0).$$

(Kummer; Hintermüller, Ito Kunisch; M. Ulbrich)

- If H is semismooth and all $M \in \partial H(x)$ are uniformly bounded invertible near the solution then Newton's method converges locally q-superlinearly.

Semismoothness of the Nonsmooth Reformulation

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We use the following fact (Hintermüller, Ito, Kunisch; M. Ulbrich):

For all $p \in (2, \infty]$ and all $b \in L^2(\Gamma_C)$, the operator

$$S : L^p(\Gamma_C) \to L^2(\Gamma_C), \quad S(w) = \max(0, w + b)$$

is ∂S-semismooth with $\partial S(w)$ consisting of all operators

$$D \in \mathcal{L}(L^p(\Gamma_C), L^2(\Gamma_C)), \;\; Dv = d \cdot v, \;\; d \begin{cases} = 1 & \text{on } \{w + b > 0\}, \\ = 0 & \text{on } \{w + b < 0\}, \\ \in [0, 1] & \text{on } \{w + b = 0\}. \end{cases}$$

Let $p > 2$ be such that the embedding $H^{1/2}(\Gamma_C) \subset L^p(\Gamma_C)$ is continuous. Then

$$u \in V \mapsto \gamma^{-1} N u \in L^p(\Gamma_C)$$

is linear and continuous.

Semismoothness of the Nonsmooth Reformulation

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From the above considerations, we conclude:
The operator

$$H(u,z) = \begin{pmatrix} Au - f + N^* z \\ z - \max(0, \gamma^{-1}(Nu - g) + z^r) \end{pmatrix}$$

is ∂H-semismooth and ∂H contains the operator

$$M \in \mathcal{L}(V \times L^2(\Gamma_C), V^* \times L^2(\Gamma_C)), \quad M = \begin{pmatrix} A & N^* \\ -\gamma^{-1} DN & I \end{pmatrix}$$

with $Dv = \mathbf{1}_{\{\gamma^{-1}(Nu-g)+z^r \geq 0\}} v$.

Superlinear Convergence of Semismooth Newton's Method

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In each iteration the following linear operator equation has to be solved:

Semismooth Newton system:

$$\begin{pmatrix} A & N^* \\ -\gamma^{-1}DN & I \end{pmatrix} \begin{pmatrix} s_u \\ s_z \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

with D = multiplication operator.

Since A is uniformly elliptic and N is onto:
For all $u \in V$ and $z \in L^2(\Gamma_C)$, all $M \in \partial H(u, z)$ are uniformly bounded invertible.

Together with the semismoothness of H, we obtain:

The semismooth Newton method, applied to (R_γ), converges locally q-superlinearly.

Appropriate Form for Multigrid Method

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Semismooth Newton system:

$$\begin{pmatrix} A & N^* \\ -\gamma^{-1} DN & I \end{pmatrix} \begin{pmatrix} s_u \\ s_z \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}.$$

Block elimination yields

$$\begin{pmatrix} A + \gamma^{-1} N^* DN & 0 \\ -\gamma^{-1} DN & I \end{pmatrix} \begin{pmatrix} s_u \\ s_z \end{pmatrix} = \begin{pmatrix} r_1 - N^* r_2 \\ r_2 \end{pmatrix}.$$

The upper left block $A_\gamma = A + \gamma^{-1} N^* DN$ is an elliptic operator.

We will show how a **multigrid method** can be derived for the solution of

$$A_\gamma s_u = r_1 - N^* r_2.$$

Challenges for multigrid methods:

- $\gamma^{-1} N^* DN$ is a large perturbation of A (unresolved on coarse grids)
- we do not want to require H^2-regularity for A_γ

Finite Element Space for the Displacements

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For simplicity, we assume that $\Omega \subset \mathbb{R}^3$ is a polyhedral domain.

Multigrid hierarchy:

Let $\mathcal{T}_0$ be a conforming simplicial triangulation of Ω such that Γ_D, Γ_N and Γ_C are composed of faces of simplices in $\mathcal{T}_0$.

Let $\mathcal{T}_1$, ..., $\mathcal{T}_J$ be simplicial triangulations obtained by successively refining $\mathcal{T}_0$ according to standard rules (Bornemann, Yserentant).

Finite element space hierarchy:

We define the spaces

$$\mathcal{S}_k = \left\{ \mathbf{v} \in C(\bar{\Omega})\,;\ \mathbf{v} \text{ piecewise linear on } \mathcal{T}_k,\ \mathbf{v}|_{\Gamma_D} = 0 \right\}$$

Then: $\mathcal{S}_k \subset \mathcal{S}_l, \quad k \leq l.$

We set $\mathcal{S}_k^3 = \mathcal{S}_k \times \mathcal{S}_k \times \mathcal{S}_k$, $\mathcal{S} = \mathcal{S}_J$, and $\mathcal{S}^3 = \mathcal{S}_J^3$.

Discretization of the Non-Penetration Condition

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Finite element space for multiplier z:

Let $\overline{Z} = \overline{Z}^* \subset L^2(\Gamma_C)$ be a finite element space for the multipliers (usually derived from $\mathcal{S}_J$, e.g. by biorthogonality, see Wohlmuth).
Let $\{\phi_i\}_{1 \le i \le K}$ be a positive basis of $\overline{Z}^*$ such that with $0 < \kappa_1 \le \kappa_2$:

$$\kappa_1 \|\mathbf{v}|_{\Gamma_C}\|_{L^2(\Gamma_C)} \le \left(\sum_{i=1}^{K} (\phi_i, \mathbf{v})^2_{L^2(\Gamma_C)} \right)^{1/2} \le \kappa_2 \|\mathbf{v}|_{\Gamma_C}\|_{L^2(\Gamma_C)} \quad \forall\, \mathbf{v} \in \mathcal{S}.$$

Let $\tau^n : \mathcal{S}^3 \to \overline{Z}$ be a discrete version of the normal trace operator
$N : u \in V \to u^T n|_{\Gamma_C} \in H^{1/2}(\Gamma_C)$.

Discretized non-penetration condition:

As discretization of the constraint $Nu - g \le 0$ we choose

$$\mathbf{N}\mathbf{u} \le \mathbf{g}$$

with $\quad \mathbf{N} : \mathcal{S}^3 \mapsto \mathbb{R}^K, \quad (\mathbf{N}\mathbf{u})_i = (\tau^n(\mathbf{u}), \phi_i)_{L^2(\Gamma_C)}, \quad (\mathbf{g})_i = (g, \phi_i)_{L^2(\Gamma_C)}.$

Discretized Elastic Contact Problem

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Discretized elastic contact problem (P)

$$\min_{\mathbf{u}\in\mathcal{S}^3} \quad a(\mathbf{u},\mathbf{u}) - \langle f, \mathbf{u}\rangle_{V^*,V} \quad \text{s. t.} \quad \mathbf{N}\mathbf{u} \le \mathbf{g}.$$

Operator formulation:

We introduce the L^2-like norm, the operator $\mathbf{A} : \mathcal{S}^3 \to \mathcal{S}^3$ and $\mathbf{f} \in \mathcal{S}^3$ by

$$(\mathbf{v},\mathbf{w})_0 = \sum_{T\in\mathcal{T}_0} \frac{1}{\operatorname{diam}(T)^2} \int_T \mathbf{v}^T\mathbf{w}\,dx,$$

$$(\mathbf{A}\mathbf{v},\mathbf{w})_0 = a(\mathbf{v},\mathbf{w}) \quad \forall\, \mathbf{v},\mathbf{w} \in \mathcal{S}^3, \quad (\mathbf{f},\mathbf{v})_0 = \langle f,\mathbf{v}\rangle_{V^*,V} \quad \forall\, \mathbf{v} \in \mathcal{S}^3,$$

Then we can write (**P**) as follows:

$$\min_{\mathbf{u}\in\mathcal{S}^3} \quad \tfrac{1}{2}(\mathbf{u},\mathbf{A}\mathbf{u})_0 - (\mathbf{f},\mathbf{u})_0 \quad \text{s. t.} \quad \mathbf{N}\mathbf{u} \le \mathbf{g}.$$

Regularization of the Optimality Conditions

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Optimality conditions:

$\mathbf{u} \in \mathcal{S}$ solves ($\mathbf{P}$) if and only if there exists $\mathbf{z} \in \mathbb{R}^K$ such that

$$\mathbf{A}\mathbf{u} - \mathbf{f} + \mathbf{N}^T\mathbf{z} = 0$$

$$\mathbf{z} \geq 0, \quad \mathbf{N}\mathbf{u} - \mathbf{g} \leq 0, \quad \mathbf{z}^T(\mathbf{N}\mathbf{u} - \mathbf{g}) = 0.$$

As in the infinite dimensional setting, we introduce a regularization:

Regularized optimality conditions:

$\mathbf{u}_\gamma \in \mathcal{S}$ and $\mathbf{z}_\gamma \in \mathbb{R}^K$ satisfy

$$\mathbf{A}\mathbf{u}_\gamma - \mathbf{f} + \mathbf{N}^T\mathbf{z}_\gamma = 0$$

$$\mathbf{z}_\gamma \geq 0, \quad \mathbf{N}\mathbf{u}_\gamma - \gamma(\mathbf{z}_\gamma - \mathbf{z}^r) - \mathbf{g} \leq 0, \quad \mathbf{z}_\gamma^T(\mathbf{N}\mathbf{u}_\gamma - \gamma(\mathbf{z}_\gamma - \mathbf{z}^r) - \mathbf{g}) = 0.$$

Discrete Semismooth Newton System

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Reformulated regularized optimality conditions $(\mathbf{R}_\gamma)$

$$\mathbf{H}(\mathbf{u}, \mathbf{z}) := \begin{pmatrix} \mathbf{A}\mathbf{u} - \mathbf{f} + \mathbf{N}^T\mathbf{z} \\ \mathbf{z} - \max(0, \gamma^{-1}(\mathbf{N}\mathbf{u} - \mathbf{g}) + \mathbf{z}^r) \end{pmatrix} = 0,$$

where the max is applied componentwise.

Semismooth Newton system:

$$\begin{pmatrix} \mathbf{A} & \mathbf{N}^T \\ -\gamma^{-1}\mathbf{D}\mathbf{N} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{s}_u \\ \mathbf{s}_z \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix}$$

with $\mathbf{D} = \mathrm{diag}(\mathbf{d}), \quad \mathbf{d}_i = \begin{cases} 1 & \text{if } \gamma^{-1}(\mathbf{N}\mathbf{u} - \mathbf{g})_i + \mathbf{z}^r_i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$

Discrete Semismooth Newton System (2)

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Semismooth Newton system:

$$\begin{pmatrix} \mathbf{A} & \mathbf{N}^T \\ -\gamma^{-1}\mathbf{D}\mathbf{N} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{s}_u \\ \mathbf{s}_z \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix}$$

with $\mathbf{D} = \mathrm{diag}(\mathbf{d}), \quad \mathbf{d}_i = \begin{cases} 1 & \text{if } \gamma^{-1}(\mathbf{N}\mathbf{u} - \mathbf{g})_i + \mathbf{z}_i^r \geq 0, \\ 0 & \text{otherwise.} \end{cases}$

Block elimination yields

$$\begin{pmatrix} \mathbf{A} + \gamma^{-1}\mathbf{N}^T\mathbf{D}\mathbf{N} & 0 \\ -\gamma^{-1}\mathbf{D}\mathbf{N} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{s}_u \\ \mathbf{s}_z \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 - \mathbf{N}^T\mathbf{r}_2 \\ \mathbf{r}_2 \end{pmatrix}$$

Discrete Semismooth Newton System (3)

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Semismooth Newton system after block elimination:

$$\begin{pmatrix} \mathbf{A} + \gamma^{-1}\mathbf{N}^T\mathbf{D}\mathbf{N} & 0 \\ -\gamma^{-1}\mathbf{D}\mathbf{N} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{s}_u \\ \mathbf{s}_z \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 - \mathbf{N}^T\mathbf{r}_2 \\ \mathbf{r}_2 \end{pmatrix}$$

Our aim is to solve the hard part of the system,

$$\mathbf{A}_\gamma \mathbf{s}_u = \mathbf{b}$$

with

$$\mathbf{A}_\gamma = \mathbf{A} + \gamma^{-1}\mathbf{N}^T\mathbf{D}\mathbf{N}, \quad \mathbf{b} = \mathbf{r}_1 - \mathbf{N}^T\mathbf{r}_2$$

by a multigrid method.

Multigrid Cycle: Subspace Decomposition

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We will propose and analyze a multigrid cycle applied to

$$\mathbf{A}_\gamma \mathbf{u} = \mathbf{b}, \qquad \mathbf{A}_\gamma = \mathbf{A} + \gamma^{-1}\mathbf{N}^T\mathbf{D}\mathbf{N}.$$

Subspace decomposition

$$\mathcal{S}^3 = \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_J, \ \text{ where } \mathcal{W}_J \subset \mathcal{S}^3,$$

$$\mathcal{W}_k \subset (\mathcal{S}^3)' := \left\{\mathbf{w} \in \mathcal{S}^3\,;\ \mathbf{DNw} = 0\right\}, \ \ \mathcal{W}_k \text{ has nodal basis w.r.t. } \mathcal{T}_k, \ \ 0 \leq k < J.$$

Example for subspace decomposition

Choose linear, local operators (easy to implement)

$$P_k(\mathbf{D}) : \mathcal{S}_k^3 \to (\mathcal{S}^3)' := \left\{\mathbf{w} \in \mathcal{S}^3\,;\ \mathbf{DNw} = 0\right\}, \quad 0 \leq k < J.$$

Now set $\mathcal{W}_J := \mathcal{S}^3, \quad \mathcal{W}_k := P_k(\mathbf{D})\,\mathcal{S}_k^3, \ \ 0 \leq k < J.$

Multigrid Cycle

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Subspace corrections:

Define the operators

$$\mathbf{r}_k \mapsto \mathbf{d}_k = \mathbf{B}_k^{-1}\mathbf{r}_k \quad \text{with} \quad \mathbf{B}_k \in \mathcal{L}(\mathcal{W}_k, \mathcal{W}_k),$$

$$\mathbf{d}_k = \mathbf{B}_k^{-1}\mathbf{r}_k \in \mathcal{W}_k : \left\{ \begin{array}{l} k = 0 : \text{exact solution of} \\ k \geq 1 : \ell \text{ sym. Gauss-Seidel} \\ \qquad\quad \text{steps for} \end{array} \right\} a_\gamma(\mathbf{d}_k, \mathbf{w}_k) = (\mathbf{r}_k, \mathbf{w}_k) \; \forall\, \mathbf{w}_k \in \mathcal{W}_k$$

Multigrid cycle:

$$\text{For } k = 0, \dots, J: \quad \mathbf{v} \leftarrow \mathbf{v} + \mathbf{B}_k^{-1}\mathbf{Q}_k(\mathbf{b} - \hat{\mathbf{A}}\mathbf{v}).$$

Here, the L^2-like projections $\mathbf{Q}_k \in \mathcal{L}(\mathcal{S}^3, \mathcal{W}_k)$ are defined by

L^2-like projections:

$$(\mathbf{Q}_k\mathbf{v}, \mathbf{w}_k)_0 = (\mathbf{v}, \mathbf{w}_k)_0 \quad \forall\, \mathbf{v} \in \mathcal{S}^3, \; \mathbf{w}_k \in \mathcal{W}_k.$$

Multigrid Cycle

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Denote by $\mathbf{v}^*$ the solution of $\mathbf{A}_\gamma \mathbf{v} = \mathbf{b}$.

The result $\mathbf{v}^+$ of a multigrid cycle with input $\mathbf{v}$ satisfies

$$\mathbf{v}^+ - \mathbf{v}^* = \mathbf{E}(\mathbf{v} - \mathbf{v}^*)$$

with

$$\mathbf{E} = (I - \mathbf{T}_J) \cdots (I - \mathbf{T}_0), \qquad \mathbf{T}_k = \mathbf{B}_k^{-1} \mathbf{Q}_k \mathbf{A}_\gamma.$$

We show that (M. Ulbrich, S.U., Bratzke 12)

$$\|\mathbf{E}\mathbf{v}\| \leq \eta < 1 \quad \forall\, \mathbf{v} \in \mathcal{S}^3$$

with the energy norm

$$\|\mathbf{v}\| = a_\gamma(\mathbf{v}, \mathbf{v})^{1/2}.$$

Consequences:

- Multigrid method converges with linear rate $\leq \eta$ and is a good preconditioner
- η is independent of grid levels J

Abstract Multilevel Theory

$$\|\mathbf{E}\mathbf{v}\|^2 \le \left(1 - \frac{2-\omega}{K_1(1+K_2)^2}\right)\|\mathbf{v}\|^2 \quad \forall\, \mathbf{v} \in \mathcal{S}^3$$

holds if the following assumptions are satisfied (e.g., Yserentant):

There exist spaces $\mathcal{V}_k \subset \mathcal{W}_k$ such that $\mathcal{S}^3 = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_J$ and

A_1 The decomposition is stable, i.e., there exists a constant $K_1 > 0$ with

$$\sum_{k=0}^{J} (\mathbf{B}_k \mathbf{v}_k, \mathbf{v}_k)_0 \le K_1 \left\| \sum_{k=0}^{J} \mathbf{v}_k \right\|^2 \quad \forall \mathbf{v}_k \in \mathcal{V}_k,\ 0 \le k \le J.$$

A_2 There are $c_{kl} = c_{lk}$ with Spectral Radius$((c_{kl})_{0\le k,l\le J}) \le K_2$ such that

$$a_\gamma(\mathbf{w}_k, \mathbf{v}_l) \le c_{kl} (\mathbf{B}_k \mathbf{w}_k, \mathbf{w}_k)_0^{1/2} (\mathbf{B}_l \mathbf{v}_l, \mathbf{v}_l)_0^{1/2}, \quad \forall\, \mathbf{w}_k \in \mathcal{W}_k,\ \mathbf{v}_l \in \mathcal{V}_l,\ 0 \le k \le l \le J.$$

A_3 There exists $0 < \omega < 2$ such that $\quad a_\gamma(\mathbf{w}_k, \mathbf{w}_k) \le \omega (\mathbf{B}_k \mathbf{w}_k, \mathbf{w}_k)_0 \quad \forall\, \mathbf{w}_k \in \mathcal{W}_k.$

Verification of Assumptions A_1–A_3

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We choose for the analysis the auxiliary spaces

$$\mathcal{V}_0 = \mathcal{W}_0, \quad \mathcal{V}_k = P_k(\mathbf{D}) \left\{ \mathbf{Q}_k \mathbf{v} - \mathbf{Q}_{k-1}\mathbf{v} \, ; \, \mathbf{v} \in \mathcal{S}^3 \right\}, \; 1 \leq k \leq J \text{ with } P_J(\mathbf{D}) := \text{id}.$$

Then under reasonable assumptions, we can prove:

A_1 holds with

$$K_1 = C\left(1 + \max_{T_0 \in \mathcal{T}_0^*} \max_{\mathcal{T}_J^* \ni T \subset T_0} \frac{\gamma \, \text{diam}(T)}{(\text{diam}(T_0)/2^J)^2}\right),$$

where $\mathcal{T}_k^* = \{T \in \mathcal{T}_k \, ; \; T \cap \Gamma_C$ contains an interior point$\}$ and C depends only on the regularity of the initial mesh.

A_2 holds with

$$c_{kl} = C\left(\frac{1}{\sqrt{2}}\right)^{l-k} \left(1 + \delta_{lJ}(1 - \delta_{kJ}) \max_{T_0 \in \mathcal{T}_0^*} \max_{\mathcal{T}_J^* \ni T \subset T_0} \frac{\sqrt{\gamma \, \text{diam}(T)}}{\text{diam}(T_0)/2^J}\right),$$

where C depends only on the regularity of the initial mesh.

A_3 holds with $\omega = 1$.

See M. Ulbrich, S.U., Bratzke 13.

Remarks

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We obtain

$$\|\mathbf{E}\mathbf{v}\|^2 \leq \left(1 - \frac{1}{K_1(1+K_2)^2}\right)\|\mathbf{v}\|^2 \quad \forall\, \mathbf{v} \in \mathcal{S}^3$$

- K_1 and K_2 are independent of the number of grid levels
- Bounds for K_1 and K_2 are independent of the regularization parameter γ as long as

  $$\gamma \leq C\,\mathrm{diam}(T_0)/2^J \;\; (= C\,\mathrm{diam}(T) \;\text{ for uniform refinement along } \Gamma_C)$$

  for coarse grid elements T_0 (fine grid elements T) at the contact boundary
- The proof is based on the following stability estimate for the space decomposition $\mathcal{V}_k$, $0 \leq k \leq J$

$$\|Q_0\mathbf{v}\|_{H^1}^2 + \sum_{k=1}^{J} 4^k \|Q_k\mathbf{v} - Q_{k-1}\mathbf{v}\|_0^2 \leq C\|\mathbf{v}\|_{H^1}^2 \quad \forall \mathbf{v} \in \mathcal{S}_J{}^3.$$

Extension to Nonlinear Elastic Contact Problems

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$$\min_{u\in V} \; J(u) := \int_\Omega \Big(\Phi(x,\epsilon(u):\epsilon(u)) + \frac{1}{2}\Psi(x,\operatorname{div}(u)^2) - f_V^T u\Big)\,dx - \int_{\Gamma_N} f_S^T u\, dS(x)$$
$$\text{s. t.} \quad u^T n \le g \quad \text{on } \Gamma_C$$

cf. Necas, Hlavacek 81; Axelsson, Padiy 00; Blaheta 97

Assumptions on $\Phi(x,s), \Psi(x,s)$ as above.

Results (work in progress):

- Error etimates w.r.t. γ and convergence of the semismooth Newton scheme after discretization still hold
- Analysis of the multigrid method for the semismooth Newton system similar as in the linear case, since $A(u) = J''(u)$ uniformly V-coercive.
- Convergence of the semismooth Newton scheme in function space leads to norm gap and requires a smoothing step

Numerical Results

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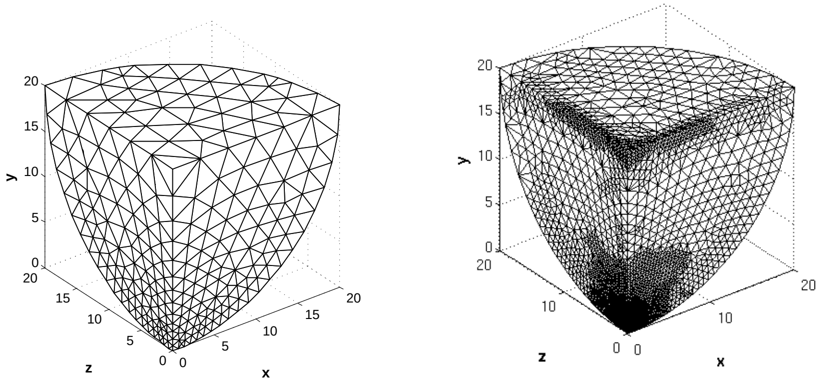
3D Hertzian contact problem (steel ball)



FE discretization, (l) coarse mesh - 3993 elem., (r) finest mesh - 1302330 elem.

Numerical Results

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Iteration history

Level l	n_l	num contact nodes	iterations Newton	avg. iterations pcg
0	922	69	3	1.00
1	1793	71	5	2.75
2	3117	244	4	3.00
3	6980	934	5	3.50
4	19851	3584	4	5.33
5	68682	14055	3	3.50
6	252377	55592	5	4.00

- Regularization parameter $\gamma = 10^{-8}$ (similar results for other values)
- pcg with the proposed multigrid preconditioner
- 2 symmetric Gauss-Seidel iterations as smoother
- V-cycle to get symmetric preconditioner (coarse $\rightarrow$ fine $\rightarrow$ coarse)

Numerical Results

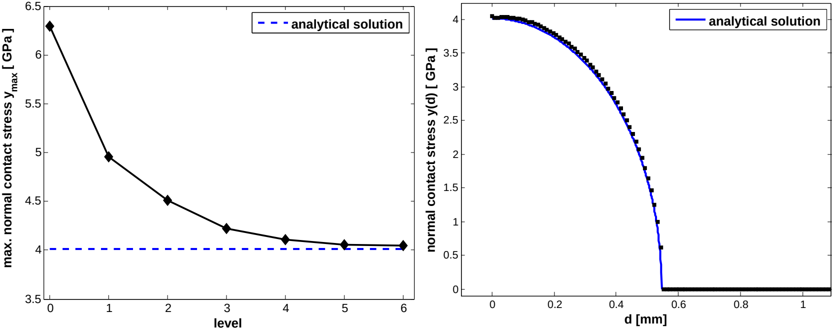

Figure : (l): Maximal contact normal stresses on level 0,. . . , 6, (r): Normal contact stress distribution in the x-y plane

Numerical Results

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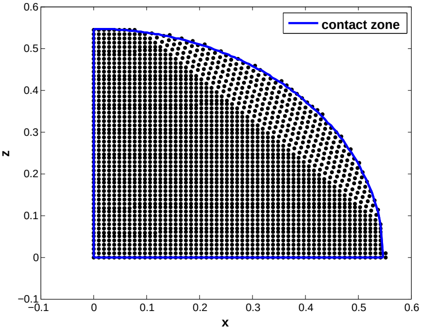

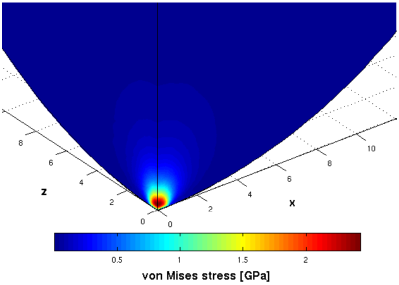

(l): contact zone, (r): von Mises stress distribution

Conclusions

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- Convergence of semismooth Newton methods for regularized elastic contact problems in function space
- Error estimate with respect to regularization parameter
- Multigrid solver/preconditioner for semismooth Newton system
- Convergence rate independent of number of grid levels and size of regularization parameters
- Extension to nonlinear contact problems possible