Revisiting Preprocessing for Generalized Nash Games

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Outline

- Generalized Nash game formulation
- Parametric variational inequalities
- Preprocessing techniques

Generalized Nash Games

Problem Formulation

- Non-cooperative game played by n individuals
  - Each player selects a strategy to optimize their objective
  - Strategies for the other players are fixed
- Equilibrium reached when no improvement is possible

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- Characterization of two player equilibrium (x^*, y^*)

$$x^* \in \begin{cases} \displaystyle\operatorname*{arg\,min}_{x \geq 0} & f_1(x, y^*) \\ \text{subject to} & c_1(x, y^*) \leq 0 \end{cases}$$
$$y^* \in \begin{cases} \displaystyle\operatorname*{arg\,min}_{y \geq 0} & f_2(x^*, y) \\ \text{subject to} & c_2(x^*, y) \leq 0 \end{cases}$$

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- Many applications

Generalized Nash Games

Complementarity Formulation

- Assume each optimization problem is convex
  - $f_1(\cdot, y)$ and $c_1(\cdot, y)$ are convex for each y
  - $f_2(x, \cdot)$ and $c_2(x, \cdot)$ are convex for each x
  - $c_1(\cdot, y)$ and $c_2(x, \cdot)$ satisfy constraint qualification
- Then the first-order conditions are necessary and sufficient

$$\begin{array}{ll} \min\limits_{x \geq 0} & f_1(x, y^*) \\ \text{subject to} & c_1(x, y^*) \leq 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{lcl} 0 \leq x & \perp & \nabla_x f_1(x, y^*) + \lambda_1^T \nabla_x c_1(x, y^*) \geq 0 \\ 0 \leq \lambda_1 & \perp & -c_1(x, y^*) \geq 0 \end{array}$$

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$$

- Nonlinear complementarity problem
  - Each solution is an equilibrium for the Nash game
  - Other (better) formulations can be constructed
  - Recommend using the MOPEC machinery

Generalized Nash Games

A Simple Example

- First player

$$\begin{array}{ll} \min_{x\geq 0} & \frac{1}{2}x^2 - x \\ \text{subject to} & x + y \geq 0 \end{array}$$

- Second player

$$\min_{y\geq 0} \quad \frac{1}{2}y^2 - xy$$

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- Complementarity problem

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- Notice that the first player constraint is redundant
  - What structure do we need to identify to eliminate?
  - What other operations can be done with this structure?

Parametric Variational Inequalities

Definition

- Let $F : \Re^n \to \Re^n$ be continuously differentiable
- Let $X(p) \subseteq \Re^n$ be a closed convex set for each p
- Given $p \in \Re^n$, find find $x \in X(p)$ such that

$$\langle F(x), y - x \rangle \geq 0 \quad \forall\, y \in X(p)$$

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- Equivalent formulation is the generalized equation

$$0 \in F(x) + N_{X(p)}(x)$$

where the normal cone to $X(p)$ at $x \in X(p)$ is

$$N_{X(p)}(x) := \{\bar{x} \mid \langle \bar{x}, y - x \rangle \leq 0\ \forall\, y \in X(p)\}$$

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- Special cases include
  - Nonlinear equations when $X = \Re^n$
  - Nonlinear complementarity when $X = \Re^n_+$
  - Mixed complementarity when $X = [l, u]^n$

Parametric Variational Inequalities

Simplified Theory (based on Robinson)

- Let $F : \Re^n \to \Re^n$, $A \in \Re^{m \times n}$, $B \in \Re^{m \times n}$, and $b \in \Re^m$.

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- If x and λ solve

$$\begin{array}{rcl} 0 \leq x & \perp & F(x) - A^T \lambda \geq 0 \\ \lambda & & Ax + Bp + b = 0 \end{array} \tag{1}$$

then

$$0 \in F(x) + N_{X(p)}(x) \tag{2}$$

where $X = \{x \mid x \geq 0 \text{ and } Ax + Bp + b = 0\}$.

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Note: $X(p)$ and $N_{X(p)}(\cdot)$ are geometric objects and we are free to choose among equivalent algebraic representations.

Preprocessing

Basic Methodology

- Given an arbitrary complementarity problem

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- Convert the problem into parametric variational inequality
- Choose representation of the polyhedral constraint set
  - Reduce model size and complexity
  - Improve algorithm performance
  - Detect unsolvable models

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  - Improve algorithm performance
  - Detect unsolvable models
- Recover reduced complementarity problem

Preprocessing

Discovering Structure

The structure we need to identify is

$$\begin{array}{lcl} 0 \le x & \perp & F(x,y) - A^T\lambda \ge 0 \\ 0 \le y & \perp & G(x,y) \ge 0 \\ 0 \le \lambda & \perp & Ax + By + b \ge 0 \end{array}$$

- Provided with a list of linear rows and columns for the problem
- For each linear column we perform the following
  - Check that the row is linear
  - Reject rows having a diagonal entry
  - Ensure elements in common have opposite signs

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- For each linear column we perform the following
  - Check that the row is linear
  - Reject rows having a diagonal entry
  - Ensure elements in common have opposite signs
    - Can negate rows corresponding to equalities

$$\begin{array}{rcl} 0 \le x & \perp & x - \lambda + 1 \ge 0 \\ 0 \le y & \perp & x + y + 4 \ge 0 \\ \lambda & & -2x - 4y + 6 = 0 \end{array}$$

is equivalent to

$$\begin{array}{rcl} 0 \le x & \perp & x - \lambda + 1 \ge 0 \\ 0 \le y & \perp & x + y + 4 \ge 0 \\ \lambda & & x + 2y - 3 = 0 \end{array}$$

Preprocessing

Assembling Sets

- Given eligible rows and columns
- Reject those requiring scaling or sign changes (can be relaxed)
- Use a greedy heuristic to assemble maximal polyhedral set
    - Choose a remaining column
    - Add next column sharing some nonzero entries if possible
    - Continue adding columns until no more can be added
    - Repeat to identify multiple polyhedral sets

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- Structure can be conveyed to capable solvers
  - A primal/dual structure recovered for optimization problems
  - Parametric polyhedral variational inequalities for other problems

Preprocessing

Possible Reductions

- Reductions on a single constraint include
  - Singleton rows
  - Doubleton rows with a column singleton
  - Forcing conditions
- Reductions using polyhedral sets include
  - Duplicate rows
  - Implied variable bounds

Example

Singleton Reduction

1. Complementarity problem

$$
\begin{aligned}
0 \leq x \quad &\perp \quad x - z - 1 \geq 0 \\
0 \leq y \quad &\perp \quad -x + y \geq 0 \\
0 \leq z \quad &\perp \quad x + y \geq 0
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2. Form equivalent polyhedral problem

$$\begin{array}{l} 0 \in x - 1 + N_{X(y)}(x) \\ 0 \in -x + y + N_{\Re_+}(y) \end{array}$$

where $X = \{x \mid x \ge 0 \text{ and } x + y \ge 0\}$ with $y \ge 0$

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3. Remove redundant inequality
4. Recover reduced complementarity problem

$$\begin{array}{lcl} 0 \le x & \perp & x - 1 \ge 0 \\ 0 \le y & \perp & -x + y \ge 0 \end{array}$$

   with the solution $x = 1$ and $y = 1$

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5. Compute multiplier $z = 0$

Example

Forcing Condition

1. Complementarity problem

$$
\begin{array}{lcl}
0 \le x \le 1 & \perp & x - z + 2 \\
0 \le y \le 1 & \perp & -2x + y \\
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\end{array}
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$$\begin{array}{l} 0 \in x + 2 + N_{X(y)}(x) \\ 0 \in -2x + y + N_Y(y) \end{array}$$

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3. Eliminate $x = 1$ but need to leave remaining side constraint
4. Recover reduced complementarity problem

$$\begin{array}{lcl} 0 \leq y \leq 1 & \perp & y - 2 \\ 0 \leq z & \perp & y \geq 1 \end{array}$$

   which has solution $y = 1$, $z \geq 0$

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5. Recover multiplier $z \geq 3$

Preprocessing

Block Structure

- Exploitation requires structural identification

x	x	0	0	0	0	0
x	x	0	0	0	0	0
x	0	x	x	x	0	0
x	0	x	x	x	0	0
x	0	x	x	x	0	0
0	0	0	0	0	x	x
x	0	x	x	x	x	x

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- Focus on small block sizes (at most 3×3)
  - Start from a single row
  - Add constraints for variables
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  - Equations removed via Schur complement when possible

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  - Preblocks use uniqueness arguments
  - Postblocks use existence arguments
- Matrix classes form the foundation for these methods

Preprocessing

Embedded Blocks and Forcing Conditions

- Determine subproblems with unique solution
  - Find a small index set α such that $M_{\alpha,\alpha}$ is strictly semimonotone
  - Determine possible right-hand sides Q_α
  - If $Q_\alpha \geq 0$, then fix $x_\alpha = 0$ and eliminate

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- All strictly semimonotone 2×2 matrices
  - Positive diagonal for singletons
  - Positive diagonals with one positive off diagonal for doubletons
  - Positive diagonals with positive determinant for doubletons
- Identification based the Jacobian structure

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- Identification based the Jacobian structure
- Currently implemented only for singleton subproblems

Summary and Status

- Parametric variational inequality structure is found in applications
- Uncovering the structure means that it can be exploited
- Modifications to standard preprocessor required
  - Start from columns rather than rows
  - Identify possible parametric reductions
  - Not yet complete but not too difficult!
- Block reductions are implemented for small blocks