Energy Balance Models - viewed from Stochastic Dynamics

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Abstract. In this article the bottom part in the hierarchy of climate models - energy balance models - is revisited by a mathematician working in stochastic dynamics. The review of mostly deterministic 0- to 2- dimensional models focuses on the mathematical problems of equilibria, stability and bifurcations. Stochastic extensions can profit from the availability of well developed mathematical theories. To give an example, we review an approach of stochastic resonance from the theory of large deviations for dynamical systems. Stochastic resonance was born in the area of energy balance models, in an attempt to find a simple explanation of glaciation cycles. It still plays a role, as is shown by very recent applications to the ENSO system in another simple two-dimensional model.

1. Introduction

Energy balance models are at the bottom end of the hierarchy of climate models. Since the climate system is probably the most complex system physical and mathematical theories are tried on, this hierarchy consists of a big number of models of highly different complexity. General circulation models at the top end are based on most of our knowledge about physical and chemical processes in the atmosphere, the oceans, and their interface, we can describe in mathematical equations. Of course, using the increasing power of modern computational facilities, there is no way to do more realistic simulations and climate predictions than by using these models.

However, even the virtual reality thus created is hard to interpret and understand for the human mind. And this is the very reason why there is need for simpler models. We might be able to distill essentials of real or even only virtual phenomena by cutting out details of the model equations, and retaining only simple, but mathematically to a satisfactory degree feasible equations. Especially if one is interested in qualitative features, as for example the presence of non-linear phenomena such as attractors or bifurcations, and not so much in their precise structure, or their exact place of occurence, one may catch these features already in a significant simplification of the model equations. As we shall see, nonlinear phenomena are observed already in the simplest energy balance models.

When introducing stochastic processes or just *noise* in climate models, one follows in principle the same philosophy, as is more precisely discussed elsewhere in this volume. Hasselmann's idea to represent fast fluctuating processes such as wind above the ocean surface as stochastic processes, could be paraphrased very roughly by saying that many complex details on fast scales are concentrated into a few parameters, by applying arguments of mixing, ergodic theory and central limit theorems. This way, complex sets of equations may be replaced by rather simple ones containing stochastic forcing terms.

So the idea of stepping down in the hierarchy of climate models, and of including stochasticity, originates in the same desire of focusing on some qualitative principles. It is therefore not surprising that also in simple energy balance models it makes sense to include stochastic input.

This paper is written not by a specialist in the field of climate dynamics, but by a mathematician trying to read and understand specialists' papers and books. The process of understanding usually goes hand in hand with a transscription of the readings into one's own language. In this sense the part of the paper in which I review energy balance models from the point of view of a mathematician working in the area of stochastic differential equations and random dynamical systems, is written mostly for mathematicians who have, as I still do, the need to getting used to an unfamiliar world. Stochastic analysis and stochastic dynamics is not lacking very good paradigms from many areas - from the semimartingale in the popular area of stochastic finance to the Ising model in statistical physics. One aim of this paper is to look for good paradigms in the area of climate models, preferrably simple mathematically feasible models for which the tools developed in the last two decades in this dynamical area are well suited. In this sense the paper, especially the part on the possible extensions of deterministic results into stochastics, and on the approach of the problem of stochastic resonance by large deviations, is written also for physicists curious to hear about some new mathematical developments which could be interesting for them.

The organization of the paper is as follows.

In section 2, we introduce the simplest energy balance models by revisiting one of the early paradigms of the area: the example of the glaciation cycles which initiated the discovery of stochastic resonance.

In section 3 we review mathematical results about deterministic energy balance models. We start with the 0-dimensional *toy* models, discuss them focussing on questions of stochastic dynamics such as equilibria, stability and bifurcations. We then carry this discussion over by analogy to the more complex 1- and 2-dimensional energy balance models.

Section 4 is devoted to state a few possible extensions of the problems of the preceding section to the setting in which there is additional stochastic forcing in the model equations.

In section 5, we present an approach of stochastic resonance initiated by the Freidlin-Wentzell theory of large deviations for dynamical systems perturbed by

noise, described in diffusion equations. We explain how the optimal tuning effect responsible for stochastic resonance can be understood in mathematical rigor. In the first two subsections, we exploit the asymptotics of exit times due to Freidlin-Wentzell, to get a lower bound for the good tuning parameter. In the third subsection, for an embedded Markov chain also upper bounds for optimal tuning are deduced which are believed to remain correct for the diffusion.

2. The paradigm of stochastic resonance

We begin this review of energy balance models with an example which for some time played the role of an important paradigm. It stimulated research not only in the area of simple climate models, but was at the cradle of a research direction in physics which subsequently took important examples from various domains of biology, chemistry and neurology: it was one of the first examples for which the phenomenon now well known under the name of *stochastic resonance* was used to explain transitions between different stable states of physical systems. For a good overview of this rapidly growing area see Gammaitoni et al. [22] or Jung [40].

In the end of the 70's, Nicolis [53] and Benzi et al. [9] almost simultaneously tried *stochastic resonance* as a rough and qualitative explanation for the glaciation cycles in earth's history. They were looking for a simple mathematical model appropriate to explain experimental findings according to which the earth has seen ten ice ages during the last million years, alternating with warm ages rather regularly in periods of about 10^5 years. Mean temperature shifts between warm and ice age are reported to be of the order of 10 K, and relaxation times, i.e. transition times between two relatively stable mean temperatures as rather short, of the order of only 100 years. Mathematically, their explanation was based on an equation stating the global energy balance in terms of the average temperature $T(t)$, where the global average is taken meridionally (i.e. over all latitudes), zonally (i.e. over all longitudes), and annually around time t. The global energy change at time t is equated to the difference between incoming solar (short wave) radiative energy R_{in} and outgoing (long wave) radiative energy R_{out}.

R_{in} is proportinal to the global average of the solar constant $Q(t)$ at t. To model the periodicity in the glaciation cycles, one assumes that Q undergoes periodic variations due to one of the so-called *Milankovich cycles*, which is based on a periodic eccentricity of the earth's orbit around the sun of a period of, indeed, about 10^5 years, and is caused by gravitational influences of other planets of our solar system. In formulas, Q was assumed to be of the form

$$Q(t) = Q_0 + b \ \sin \omega t,$$

with some constants Q_0, b and a frequency $\omega = 10^{-5}[\frac{1}{y}]$.

The other component determining the absorbed energy R_{in} is a rough and difficult to model averaged *surface albedo* of the earth, i.e. the proportion of the solar energy absorbed. It is supposed to be just (average) temperature dependent.

For temperatures below $\underline{T}$, for which the surface water on earth is supposed to have turned into ice, and the surface is thus constantly bright, the albedo is assumed to be constantly equal to $\underline{a}$, for temperatures above $\overline{T}$, for which all ice has melted, and the surface constantly brown, it is assumed to be given by a constant $\overline{a} < \underline{a}$. For temperatures between $\underline{T}$ and $\overline{T}$, the two constant values $\underline{a}$ and $\overline{a}$ are simply linearly interpolated. The rough albedo function has therefore the *ramp function* shape depicted in Figure 1.

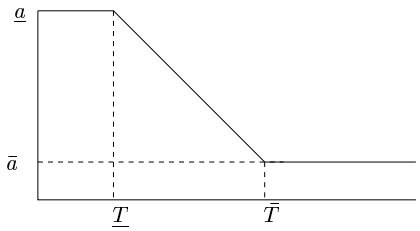

FIGURE 1

For R_{out}, the earth is simply assumed to behave approximately as a *black body radiator*, for which the energy is given by the Stefan-Boltzmann law. According to this basic law, the emitted power is proportional to the energy of the electromagnetic field in equilibrium, which is given by $\gamma T^4(t)$, with a constant γ proportional to the Stefan constant.

Hence the simple energy balance equation with periodic input Q on which the model is built is given by

$$c\,\frac{dT(t)}{dt} = Q(t)\,(1 - a(T(t))) - \gamma\,T(t)^4, \tag{1}$$

where the constant c describes a global thermal inertia. According to (1), (quasi-) stationary states of average temperature should be given by the solutions of $\frac{dT(t)}{dt} = 0$. If the model is good, they should reasonably well interpret ice and warm age temperatures. Graphically, they are given by the intersections of the curves of absorbed and emitted energy.

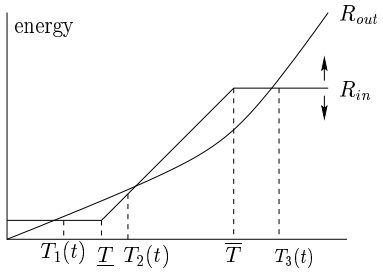

FIGURE 2

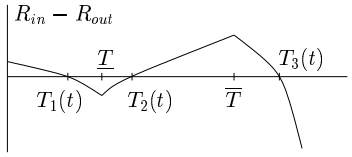

FIGURE 3

As we shall more carefully explain in section 5, the lower ($T_1(t)$) and upper ($T_3(t)$) quasi-equilibria are stable, while the middle one ($T_2(t)$) is unstable.

$T_1(t)$ should represent an ice age temperature, $T_3(t)$ a warm age, while $T_2(t)$ is not observed over noticeably long periods. In their dependence on t they should describe small fluctuations due to the variations in the solar constant. But here one encounters a serious problem with this purely deterministic model.

If the fluctuation amplitude of Q is small, then we will observe the two disjoint branches of stable solutions T_1 and T_3.

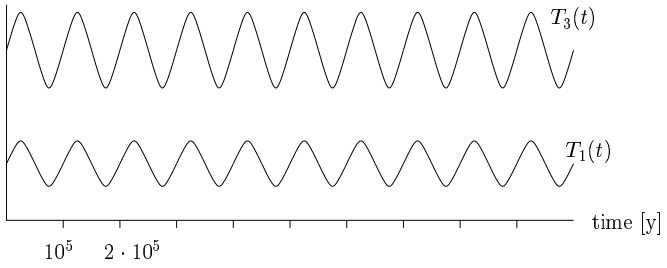

FIGURE 4

But for both branches alone - besides being unrealistically low or high - the difference between minimal and maximal temperature can by no means account for the observed shift of about 10 K, and also the relaxation times are much too long. But the most important shortcoming of the model is the lacking possibility of transitions between the two branches.

If we allow the fluctuation amplitude b to be large, the picture is still very unrealistic:

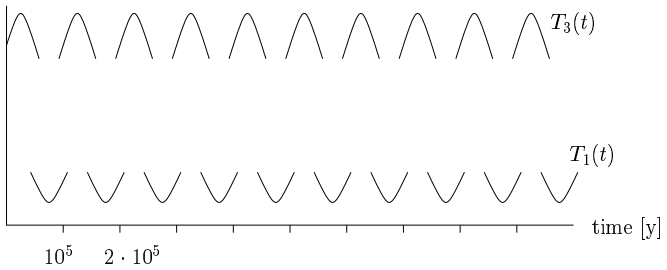

FIGURE 5

There are intervals during which one of the two branches T_1 or T_3 vanishes completely, and transitions are still impossible, unless one is willing to accept discontinuous behaviour.

For this reason, Nicolis [53] and Benzi et al. [9] proposed to add a noise term in (1). Despite the fact that then negative temperatures become possible, they worked with the equation

$$c\,\frac{dT(t)}{dt} = Q(t)\,(1 - a(T(t))) - \gamma\,T(t)^4 + \sqrt{\epsilon}\;\dot{W}_t, \tag{2}$$

$\epsilon > 0$, where $\dot{W}$ is a white noise. Passing to (2) immediately makes transitions between the metastable states $T_1(t)$ and $T_3(t)$ possible, due to the unboundedness

of the Wiener process W. In fact, the *random hopping* between the metastable states immediately exhibits two features which make the model based on (2) much more attractive for a qualitative explanation of glaciation cycles: a) the transitions between T_1 and T_3 allow for far more realistic temperature shifts, b) relaxation times are random, but very short compared to the periods the process solving (2) spends in the stable states themselves.

But now a new problem arises, which actually provided the name *stochastic resonance.*

If, seen on the scale of the period of Q, ϵ is too small, the solution may be trapped in one of the states T_1 or T_3. By the periodic variation of Q, there are well defined periodically returning time intervals during which $T_1(t)$ is the *more probable* state (see section 5 for a more careful explanation in terms of diffusions with potential drift), while $T_3(t)$ takes this role for the rest of the time. So if ϵ is small, the process, initially in T_1, may for example fail to leave this state during a whole period while the other one is more probable. The solution trajectory may then look as in Figure 6.

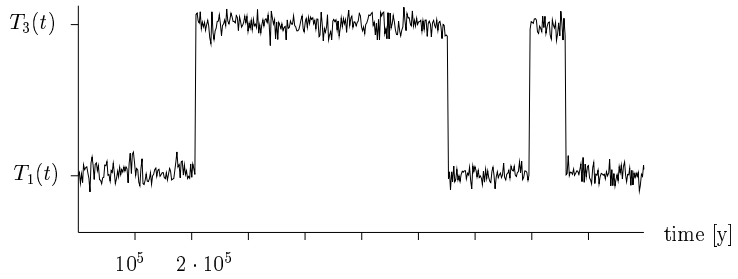

FIGURE 6

If, on the other hand, ϵ is too large, the big random fluctuation may lead to eventual excursions from the actually more probable equilibrium during its domination period to the other one. The trajectory then typically looks like this:

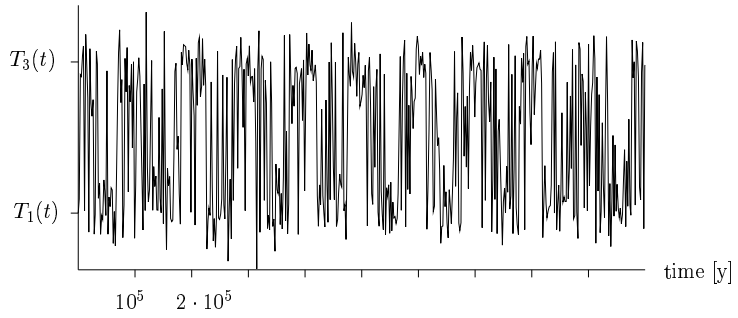

FIGURE 7

In both cases it will be hard to speak of a random periodic curve. Good tuning with the periodic forcing by Q is destroyed by a random mechanism being too slow or too fast to follow. It turned out in numerous simulations in a number

of similar systems that there is, however, an *optimal* parameter value ϵ for which the solution curves are well tuned with the periodic input. A typical well tuned curve is shown in Figure 8.

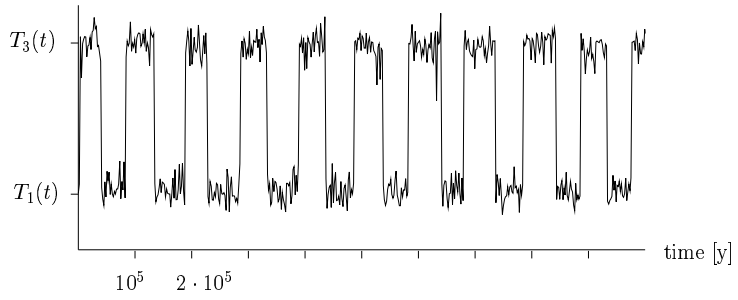

FIGURE 8

The optimally tuned system is then said to be in *stochastic resonance.* From the point of view of mathematical rigor, this notion is still poorly understood. In section 5, we shall give an overview of the main ideas of a mathematically sound treatment based on the methods of *large deviations* for random dynamical systems in the framework of the Freidlin-Wentzell theory.

Summarizing, Nicolis [53] and Benzi et al. [9], by tuning the noise parameter ϵ to appropriate values, were able to give qualitative explanations for glaciation cycles based on the phenomenon of stochastic resonance.

3. Deterministic Energy Balance Models

By far the biggest part of the mathematical work on EBM done so far is using a deterministic framework. In this section we shall briefly review this work, while focusing on particular mathematical questions belonging to the core of the areas of (random) dynamical systems and their asymptotics: invariant measures, stability, and bifurcations (see Arnold [1], Freidlin and Wentzell [20] for an overview). We shall come back to stochastic models in the following section, to present possible extensions of the problems of the deterministic setting. To be able to reason by easy analogy arguments in the multidimensional setting, we first present the ideas in the very simple framework of *0-dimensional (toy)* models.

3.1. 0-dimensional deterministic models

The paradigm presented in the previous section belongs to this class of models, if noise is turned off. In fact, these simple models differ from the deterministic (1) just by some refinements of the appearing functions in the balanced energies R_{in} and R_{out}. They were introduced by Budyko [11] and Sellers [66] around '70, and then intensively studied towards the end of the 70's by many authors: Bar-Eli, Field [6], Bhattacharya et al. [10], Fraedrich [16], [17], Ghil [24], [25], Ghil and Childress [23], Frankignoul, Hasselmann [18] Hasselmann [27], Held, Suarez [28], Hetzer [32], Hetzer et al. [33], Nicolis [50], [51], [52], [53], Nicolis, Nicolis [54], [55], North [57],

[58], North et al. [59], Olbers [60]. For a still more detailed list of references we refer to the book of Ghil and Childress [23], the presentation of which we shall partially follow here. In many of the papers, the crucial dependence of R_{in} on the solar constant, appearing as a periodic function $Q(t)$ above, is parametrized in the following way:

$$c\,\frac{dT(t)}{dt} = \mu\, Q_0\ (1 - a(T(t))) - g(T(t))\, T(t)^4. \tag{3}$$

The explicit time dependence of $Q(t)$ disappears as it is replaced by $\mu\, Q_0$.The main reason for this is of mathematical nature and will emerge in the following. For different values of μ the system will have different sets of equilibria (*climates*) with different stability properties. As μ varies, the *structural stability* of the system changes. Essential changes in the stability picture are considered as bifurcations of the climate system, originally formulated in the context of early catastrophe theory (see Fraedrich [16], [17]).

Q_0 is a time independent basic solar constant. The new function g (considered as constant in the preceding section) is supposed to model a temperature dependent *greyness* factor in the black body radiation law, which takes into account phenomena such as the *greenhouse effect*, i.e. reduced emission of radiative energy due, for example, to higher CO_2 concentration in the atmosphere going along with higher temperatures.

We shall next briefly summarize the main variants for the different ingredient functions in (3) as used in the literature.

The most frequently used variants for the albedo function date back to the early models by Budyko [11] and Sellers [66]. They employ either the *ramp function* albedo discussed before, or a still simpler one in which the linear interpolation of the ramp function is replaced by a piecewise constant function taking the values $\underline{a}$ and $\overline{a}$ each for the left half and the right half of the interval between $\underline{T}$ and $\overline{T}$ ($T_0 = \frac{1}{2}\,(\underline{T} + \overline{T})$):

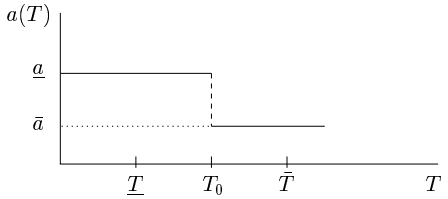

FIGURE 9

In other variants (see Fraedrich [17]) for more local questions in restricted ranges of temperatures simple linear or quadratic feedback functions are used, formally given by $a(T) = a_1 - a_2\, T$ or $a(T) = b_1 - b_2\, T^2$ with positive constants $a_1, ..., b_2$.

It is easy to imagine that the complex surface structure of earth makes it very hard to design a realistic albedo function. The up to date still unsatisfactory knowledge of the relationship between albedo and cloud formation adds another

difficulty. One effect of cloud formation is included in an interesting variant of the albedo function presented in Bhattacharya et al. [10]. It is experimentally observed that near the *ice margin* there is an increasing cloudiness due to the formation of mid-latitude storms. This effect is taken into account by adding a piecewise linear kink into the albedo function near $\overline{T}$. The resulting albedo function is shown in Figure 10.

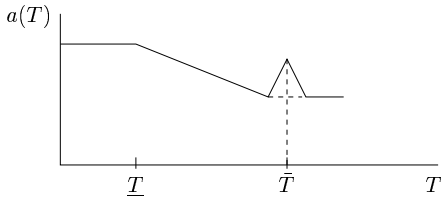

FIGURE 10

Variants of the greyness function g modelling the greenhouse effect considered in the literature range from a constant in the simplest models, via local definitions with quadratic temperature dependence $g(T) = c_1 - c_2\,T^2$ (see Fraedrich [17]) to the most frequently accepted function $g(T) = 1 - m\,\tanh(\frac{T^6}{T_0^6})$ with constants m and T_0. Since we are mainly interested in mathematical qualitative results, we shall not specify the values of various physical constants, and argue with dimension free variables.

So the mathematical essence of 0-dimensional deterministic energy balance models is given by a *parametrized ordinary differential equation* of the form

$$\frac{dT(t)}{dt} = f(\mu, T(t)), \tag{4}$$

with a parameter μ ranging in some real interval, and a continuous function f for example of the form appearing in (3). The main mathematical questions which one can expect to yield some interesting qualitative features of real behaviour of the climate system are the following:

(Q1) What are the possible equilibria (*climates*) $T_i(\mu)$ of (4)?
(Q2) Are the $T_i(\mu)$ *stable/unstable*?
(Q3) What can be said about the structural stability of the system described by (4) as μ varies? At which values of μ does the stability structure change, in other words does the system exhibit *bifurcations*?

In the framework of 0-dimensional models these questions are easy to treat. Our presentation of the key points in their treatment just aims at setting the stage for a simple presentation of the multi-dimensional case in the following subsection. We focus on the ramp function albedo, and suppose that the greenhouse effect is described by the most popular (tanh −)variant.

Concerning (Q1), we can refer to the remarks made in section 2. In the usual range of μ, we obtain three equilibrium temperatures: $T_1(\mu)$, the *deep freeze*

temperature, $T_3(\mu)$, the *warm age* temperature, and the *intermediate* temperature $T_2(\mu)$.

To formulate the problem of determining stability of equilibria mathematically, we consider the linearization of the parametrized ODE (4) near the equilibria $T_i(\mu)$ for some small variation $\theta = T - T_i(\mu)$ of temperatures:

$$\frac{d\theta(t)}{dt} = f'(\mu, T_i(\mu))\,\theta(t). \tag{5}$$

It is immediate from (5) that stability/instability of the equilibria are completely determined by the sign of $f'(\mu, T_i(\mu))$. For the purposes of a smoother presentation later, we reformulate this simple fact as a trivial eigenvalue problem

$$f'(\mu, T_i(\mu)\,\psi = \lambda\,\psi, \tag{6}$$

for some nontrivial 'eigenvector' ψ. In these terms the solutions of (5) are given by

$$\theta(t) = \theta_0\ \exp(\lambda\, t).$$

Hence the sign of $\lambda_i = f'(\mu, T_i(\mu))$ determines the stability of the equilibria: $\lambda_i < 0$ means that $T_i(\mu)$ is *stable*, $\lambda_i(\mu) > 0$ means that it is *unstable*. Looking at Figure 2, we see that the slope of $f(\mu, .)$ at $T_i(\mu)$ is negative for $i = 1, 3$ and positive for $i = 2$. This way we found the precise justification of the above mentioned fact that equilibria $T_i(\mu)$ is stable for $i = 1, 3$ and unstable for $i = 2$.

To approach (Q3), imagine μ to vary in (4). For the curve of the absorbed energy this just means to lower and lift the plateaus given by the constance intervals of the albedo function. Remember that the equilibria are just the intersections of the curves. So, as μ increases, $T_3(\mu)$ moves to the right, while $T_1(\mu)$ and $T_2(\mu)$ move in opposite directions and approach each other:

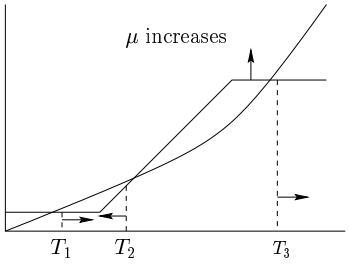

FIGURE 11

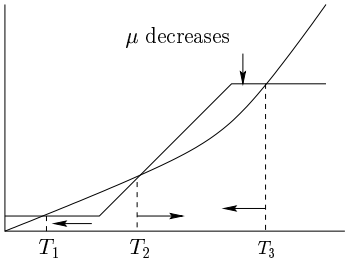

FIGURE 12

As μ decreases, $T_1(\mu)$ moves to the left, whereas $T_2(\mu)$ and $T_3(\mu)$ move in opposite directions and approach each other.

So if we plot the equilibria against μ, we obtain the following branches of graphs:

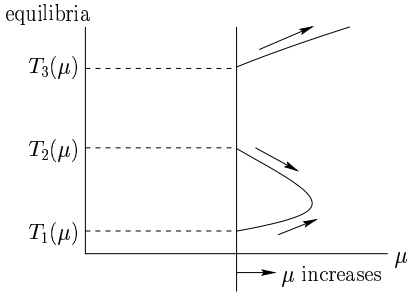

FIGURE 13

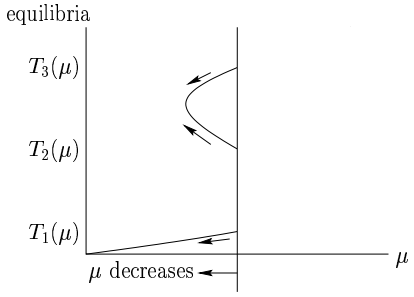

FIGURE 14

Putting them together, we obtain the following *S-shaped* curve

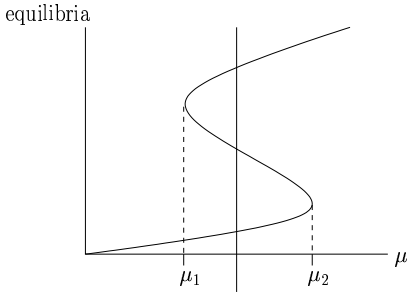

FIGURE 15

in which the two *turning points* μ_1, μ_2 mark critical parameter values for which branches of equilibria meet and vanish. To the left of μ_1, only the branch $T_1(\mu)$, the *deep freeze* temperature, persists. Therefore, μ_1 deserves the name of *deep freeze bifurcation.* To the right of μ_2, only the branch $T_3(\mu)$ survives. This is why μ_2 is also called *desert heat bifurcation* point.

So the bifurcation scenario of the ramp function albedo has two main bifurcation points of three different climates. For the other possible albedo functions, more climates and correspondingly more bifurcations are possible. For example, for the function of Figure 6 with the increasing cloudiness effect near the ice margin, two more bifurcation points appear (see Ghil and Childress [23], or Fraedrich [16]).

3.2. Multi-dimensional deterministic models

In the toy models of the preceding subsection, the surface temperature of the earth was globally averaged. Still keeping several considerable oversimplifications, we shall now let this and other quantities of our models depend on a position parameter on the earth's surface. Continuing to work with non-dimensional variables, we shall idealize the surface by the unit sphere S^2, and introduce for $x \in S^2$ the zonal component θ and the meridional component ϕ.

We first have to take into account that measurements for the zonal averages of the energy components in the energy budget appearing in our EBM indicate

that a third term has to be included into our equation: a heat diffusion term which is due to net energy transport from the equator to the poles, caused by an asymmetry in the balance of absorbed and emitted radiative energy. Due to the decreasing angle of incidence the absorption of radiation from the sun decreases rapidly when passing from the equator to the poles. The meridional dependence of long wave emitted radiation is far less pronounced. The schematic picture is Figure 16.

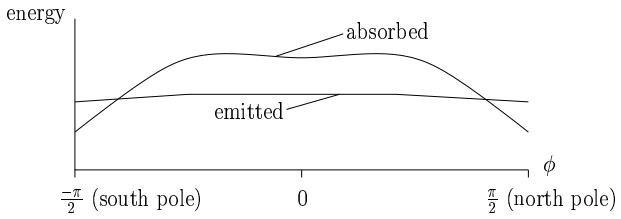

FIGURE 16

The net heat flux from the equator to the poles is reflected in form of a diffusion term in our EBM. In the great variety of 1- and 2-dimensional models studied since the early 80's, another important effect not present in the 0-dimensional models is taken into account. In particular in the albedo function, but also in the thermal inertia coefficient time delay influences have to be considered. The *delay factor* is mostly modeled by means of a *memory function* $\beta(s), -t_0 \leq s \leq 0$, where $\beta(s)$ describes the weight given to time point s in the past from some oldest memory time $-t_0$ up to the present. If $T(t,x)$ denotes the (still eventually temporally averaged) temperature at $x \in S^2, t \geq 0$, the delay factor is then given by

$$D_\beta(T(t,.)) = \int_{-t_0}^{0} \beta(s)\, T(t+s,.)ds.$$

Then multi-dimensional deterministic EBM are described by the equation

$$\begin{aligned} c(x, D_\beta(T(t,x)))\ \frac{\partial}{\partial t}T(t,x) \qquad\qquad (7)\\ = \operatorname{div}(k(.)\ \operatorname{grad}T(t,.))(x)\\ +\mu\, Q_0(x)\,[1 - a(x, T(t,x), D_\beta(T(t,x))] - g(T(t,x))\ T(t,x)^4, \end{aligned}$$

$t \geq 0, x \in S^2$. Some more comments on quantities appearing in (7) are in order. k is a space dependent diffusion coefficient. The albedo function a this time may depend on space, temperature and delay, the thermal inertia coefficient c on space and delay, and the solar constant Q_0 on space. The dependence on the delay factor in many papers is supposed to model *delayed responses* of the functions due to the big *heat storage* capacities of continental and polar ice sheets, sea ice etc. In the big number of mathematical papers written on models described by (7) the structure of the model functions T, a, c, and k propagates along the lines of increasing complexity. For example, T is first supposed to depend besides time t just on the meridional variable ϕ, while it is zonally averaged. This leads to 1-dimensional

deterministic models. As for a for example, first the dependence on x and D_β may be suppressed, then first x (or just ϕ) is allowed, finally delay factors with memory functions of different complexity are admitted. Besides part of the papers already mentioned in the preceding subsection we refer to the following main contributions: Ghil [26], [24], Hetzer [29], [30], [31], [32], Hetzer et al. [33], Hetzer, Schmidt [34], [35], [36], Lions et al. [44], [45], [46], B. Schmidt [71], Wang [76], [77].

The mathematical essence of (7) is a possibly *delayed, parametrized quasi-linear partial differential equation* on a *compact manifold*, also known under the notion *reaction-diffusion equation*. Its general form is given more concisely by

$$\frac{\partial}{\partial t}T(t,x) = L\,T(t,x) + f(\mu, T(t,x)), \tag{8}$$

$t \geq 0, x \in S^2$. L is a second order linear diffusion operator, f a non-linear continuous function of space, temperature and possibly delay describing the difference of absorbed and emitted radiative energy.

As in the 0-dimensional case, we shall now sketch the mathematical approaches presented in the literature of questions (Q1)-(Q3) formulated above. Not to overload this survey, we shall cut out the delay effects, and focus on the better known methods of classical reaction-diffusion equations, following the exposition mainly of the nice survey paper by Hetzer [30] (see also Ghil [26], and Ghil and Childress [23]). We shall illustrate the mathematical results obtained by one particular main theorem, which appears in numerous variations in numerous papers.

First of all, for (Q1)-(Q3) to have a precise meaning, one needs a good existence/uniqueness theory for (8). Denote by $M = S^2$ the unit sphere, and by $C(M)$ the continuous functions on M, by $C_+(M)$ the nonnegative ones among them. Let $(S_t)_{t\geq 0}$ be the semigroup of bounded linear operators on $C(M)$ associated with L as an infinitesimal generator ($S_t = \exp(-t\,L)$). Then the so-called *mild solutions* of (8) are given under simple assumptions on f by the generalized variation of constants formula

$$T(t,.) = S_t\,T_0 + \int_0^t S_{t-s}\,f(\mu, T(s,.))\,ds, \tag{9}$$

where T_0 is an initial temperature distribution. Under further simple assumptions on f and L one has uniqueness and boundedness results for the solutions (9).

Here is the approach of (Q1)-(Q3). We consider *climates*, i.e. equilibria of (8) as pairs $(\mu, T) \in \mathbf{R}_+ \times C_+(M)$. They are defined by satisfying the equilibrium equation

$$L\,T + f(\mu, T) = 0, \tag{10}$$

i.e. they describe the set

$$\mathbf{C} = \{(\mu, T) \in \mathbf{R}_+ \times C_+(M) : (\mu, T) \quad \text{solves} \quad (10)\}. \tag{11}$$

It may well happen that $\mathbf{C}$ has more than one connected component. We are interested in the unbounded one among them, which we call $\mathbf{P}$. In the spirit of our treatment of the 0-dimensional case, the essential answers to (Q1)-(Q3) will

be contained in statements about the *S-shapedness* of $\mathbf{P}$, or rather a reasonable translation of this notion to the framework given here.

Let now an equilibrium $(\mu, T) \in \mathbf{P}$ be given. Information on the stability of $T = T(\mu)$ will be given by studying again the linearization of the PDE near $T(\mu)$. This leads us to the equation

$$\frac{\partial}{\partial t}\psi(t) = L\,\psi(t) + \frac{\partial}{\partial x} f(\mu, T)\,\psi(t). \tag{12}$$

Still as before, discussing the stability of solutions ψ to (12) leads us to consider the eigenvalues of the elliptic operator

$$L + \frac{\partial}{\partial x} f(\mu, T), \tag{13}$$

given as solutions of the elliptic eigenvalue problem

$$L\,\psi + \frac{\partial}{\partial x} f(\mu, T)\,\psi = \lambda\,\psi, \tag{14}$$

for nontrivial eigenfunctions $\psi \in C_+(M)$. The infinite dimensional elliptic operator (13) is compact and therefore has a - say monotonically decreasing - sequence of eigenvalues $(\lambda_i(\mu, T) : i \geq 0)$ with limit $-\infty$. Only finitely many of them will therefore be positive. It is clear that the condition $\lambda_i(\mu, T) < 0$ for all $i \geq 0$ characterizes stability of (μ, T). It is therefore plausible that in order to obtain reasonable answers to (Q2), (Q3), which will be expressed in terms of the S-shapedness of $\mathbf{P}$, one has to work under the following mathematical structural hypotheses

(H1) $\lambda_0(\mu, T) = 0$ for only finitely many $(\mu, T) \in \mathbf{P}$,
(H2) $\lambda_1(\mu, T) < 0$ for all $(\mu, T) \in \mathbf{P}$.

Under these hypotheses, the essential result on *S-shapedness* takes the following typical form. For a function γ with values in a product space with two components we denote by γ_1 the projection onto the first coordinate etc.

Theorem 3.1. *Under the structural hypotheses (H1), (H2), and some technical assumptions concerning the functions $c, a, ..., g$ we have:*

a) $\mathbf{P}$ *is the trace of a Jordan curve in* $\mathbf{R}_+ \times C_+(M)$, *i.e. there exists a* C^1*-homeomorphism* $\gamma : \mathbf{R}_+ \to \mathbf{P}$ *such that* $\gamma'(\rho) \neq 0$ *for* $\rho > 0$,
b) $\mathbf{P}$ *is* S- shaped, *i.e.* $\gamma_1(\rho) \to \infty$ *for* $\rho \to \infty$, γ_1 *has an even number of local extrema,*
c) *for* $(\mu, T) \in \mathbf{P}$, T *is asymptotically stable (unstable) iff*

$$\gamma_1'(\gamma^{-1}(\mu, T)) > 0 (< 0).$$

In particular, the bifurcation points of the equation are given by those $\mu = \gamma_1(\rho)$ for which $\gamma_1'(\gamma^{-1}(\mu, T)) = 0$. The following sketch illustrates the bifurcation scenario

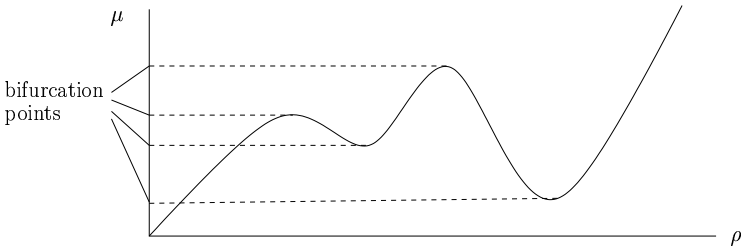

FIGURE 17

4. Stochastic extensions of EBM

Apart from the stochastic resonance example in section 1, we have so far been discussing purely analytical models. According to Hasselmann's approach (see Arnold [2] in this volume), fast fluctuating variables in processes in atmosphere and ocean (or even in the solar atmosphere) may be modeled as stochastic forcing. Taking effects like this into account would make the model equations studied in section 3 stochastic equations. We write *would* because though the mathematical tools are available, the stochastic input into the theory of EBM is rather restricted so far (see, however, Olbers [60], Wolf-Gladrow [79]). Hence this is a section essentially on (mathematically) open problems. We outline the equations to be investigated, the methods involved, starting again with the *toy* 0-dimensional model.

Instead of (4), we now write an equation which includes periodic effects, and allows a parameter:

$$\frac{dT(t)}{dt} = f(\mu, t, T(t)) + \sigma(T(t))\ \dot{X}_t, \tag{15}$$

with, e.g.

$$f(\mu, t, T) = \frac{1}{c}[\mu\, Q(t)\ (1 - a(T)) - g(T)\, T^4],$$

a, g as above, and

$$Q(t) = Q_0 + b\, \sin \omega\, t,$$

$t \geq 0, Q_0, b$ constants. In the terminology of stochastic analysis, (15) is a *parametrized one-dimensional diffusion equation* or *stochastic differential equation* with periodic coefficient. It is not so easy to make realistic assumptions about the right source of noise, and its coupling function σ. In the spirit of the central limit type arguments leading to the replacement of fast fluctuating terms by random noise one might be tempted to choose a Gaussian noise $\dot{X}$. In papers on stochastic resonance (see Jung [40], Freund et al. [21] in this volume), the noise source chosen is often parametrized white noise $\dot{X} = \sigma\, \dot{W}$. We shall come back to this problem from the point of view of large deviations theory in the following section.

In the multi-dimensional case, instead of (8) we should consider the *stochastic partial differential equation*

$$\frac{\partial}{\partial t}T(t,.) = [L\,T(t,.) + f(\mu,t,T(t,.))]\,dt + \sigma(T(t,.))\;\dot{X}(t,.), \tag{16}$$

a type of *stochastic reaction-diffusion equation*, possibly a functional equation, if delay factors are allowed. Here $\dot{X}$ is a space-time noise field, for example a temporally $\delta-$correlated Gaussian field with a spatially smoother correlation function.

Mathematical tools for dealing with equations of this type are readily available, due for example to the rapid development of the area of *stochastic partial differential equations*. See for example the survey of Zabczyk [81] in this volume, or da Prato, Zabczyk [63], [64], Walsh [73], Rozovskii [65], or Holden et al. [37]. Quasilinear stochastic partial differential equations like the above stochastic reaction-diffusion equations have recently been investigated by means of the theory of backwards stochastic differential equations (see Pardoux, Peng [61]).

To deal with stochastic versions of the basic mathematical questions (Q1)-(Q3) of section 3, mathematical tools are provided or are being developed in several areas of *stochastic analysis* and *random dynamical systems*.

Instead of deterministic *equilibria* we will have to look for *invariant measures*, both in the sense of invariant measures of Markovian semigroups associated with the evolution equations, or in the pathwise sense of random invariant measures of associated stochastic cocycles (see Arnold [1], where also the distinction between invariant measures of semigroups and random invariant measures of cocycles is thoroughly investigated).

In the stochastic setting, *Lyapunov exponents* take the role of *eigenvalues* of deterministic matrices (operators), as is made precise in the multiplicative ergodic theory due to Oseledets (see Arnold [1]).

Structural stability and *bifurcations* of deterministic systems have their stochastic counterparts in the up to date partly developed theory of *stochastic bifurcations* (see Arnold [1] for a survey on the state of the art). The concept of stochastic bifurcations promises to be essentially more complex than its deterministic counterpart. For example, one has to distinguish two different types of bifurcations: *P-bifurcations*, i.e. critical changes of the geometry of the invariant measure of the Markovian semigroup, and *D-bifurcations*, i.e. essential changes of the set of random invariant measures of the random cocycle. P-bifurcations can be seen as critical changes on the level of the laws of the solutions, whereas D-bifurcations describe critical changes in the behaviour of the solution trajectories of our equations.

Questions of the asymptotics of random dynamical systems relevant for the concepts just discussed are still under intensive investigation. We quote some of a big number of papers: Arnold et al. [3], Baxendale [5], [7], Baxendale, Stroock [8], Crauel [12], Crauel et al. [13], [14], Ebeling [15], Horsthemke, Lefever [38], Imkeller, Schmalfuss [39], Keller, Ochs [41], Khasminskii [42], Mohammed [48], Mohammed, Scheutzow [49], Namachchivaya [56], Schenk-Hoppé [67], [68], [69],

Schmalfuss [70], Wihstutz [78]. For a more complete list of references see Arnold [1].

5. Stochastic resonance: Freidlin's approach

In this final section we shall sketch the main ideas of a rigorous mathematical approach of the phenomenon of stochastic resonance which is heavily based on the Freidlin-Wentzell theory of large deviations. Freidlin [20] is able to formulate Kramers' [43] very old seminal approach mathematically rigorously in a very general setting, and this way provides a lower estimate for the good tuning. See also the numerical results by Tretyakov [72]. To obtain an upper estimate, we finally argue by embedding time discrete Markov chains into the diffusion processes, which promises to yield optimal tuning results also in the time continuous case.

To describe the idea of the approach, let us briefly return to the situation of section 2. Recall that the function $f(t,T)$ described a multiple of $R_{in} - R_{out}$, and its periodicity in t was created by the assumption on the solar constant $Q(t) = Q_0 + b \sin\omega t$. Let us compare this quantity, sketched in Figure 3 schematically for two times, say t_1, t_2 such that Q takes its minimum at t_1 and its maximum at t_2. Then the graph of f moves periodically between the two extreme positions given by the following sketches.

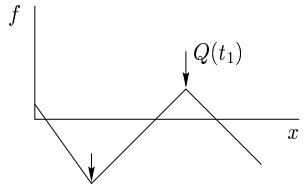

FIGURE 18

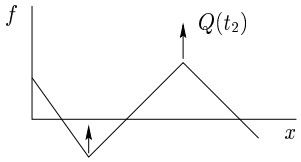

FIGURE 19

We now turn to a more general context, still focusing on a situation as simple as possible. We consider a *potential function* U such that

$$f(t,.) = -\frac{\partial}{\partial x} U(t,.), \quad t \geq 0,$$

then U will oscillate between the two extreme positions depicted schematically in the following sketches.

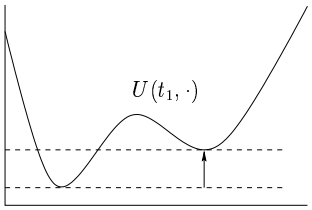

FIGURE 20

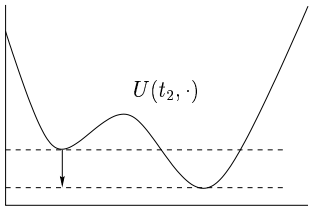

FIGURE 21

In Figure 20, the potential well on left hand side is higher than on the right hand side, in Figure 21 the role of the deeper well has changed. As t varies, we will observe a smoothly time dependent potential with two wells of periodically and smoothly fluctuating relative depth. Just the function describing the position of the deepest well will in general be discontinuous. It will play a crucial role in the analysis now sketched.

We assume in the sequel that $U(t,x), t \geq 0, x \in \mathbf{R}$, is a smooth function such that for all $t \geq 0$ $U(t,.)$ has exactly two minima, one at $x_0 < 0$, the other at $y_0 > 0$, and that the two wells at x_0 and y_0 are separated by the saddle 0, where $U(t,.)$ is assumed to take the value 0. Two moment pictures of the potential may look like this:

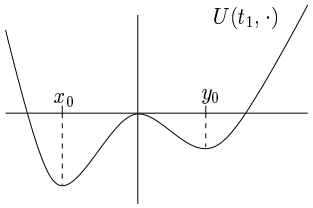

FIGURE 22

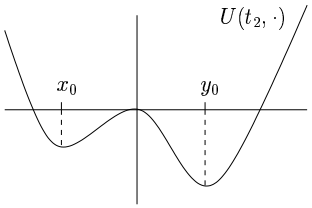

FIGURE 23

We further assume that

$$\begin{aligned} U(t,.) &= U(t+1,.), \\ f(t,.) &= -\tfrac{\partial}{\partial x} U(t,.). \end{aligned}$$

The period of the periodic input will be denoted by some positive number T. We therefore consider the stochastic differential equation

$$\frac{d}{dt} X_t^\epsilon = f(\frac{t}{T}, X_t^\epsilon) + \sqrt{\epsilon}\, \dot{W}_t, \tag{17}$$

with a one-dimensional Wiener process W (white noise $\dot{W}$). In section 2, we described the problem of stochastic resonance like this: given T ($\omega = \frac{1}{T}$), find the parameter $\epsilon = \epsilon(T)$ such that X^ϵ is *optimally tuned* with the periodic input $f(\frac{t}{T},.)$. We now pose the problem in the following (almost equivalent) way: given $\epsilon > 0$, find the good scale $T = T(\epsilon)$ such that optimal tuning of X^ϵ with the periodic input is given, at least in the limit $\epsilon \to 0$.

5.1. Time independent potential

We first study the case, in which $U(t,.)$ is given by some time independent potential function U for all t. Following Freidlin and Wentzell [20], the description of the asymptotics contained in the *large deviations principle* requires the crucial notion of *action functional*. It is defined for $T > 0$ and absolutely continuous functions $\phi : [0,T] \to \mathbf{R}$ with derivative $\dot{\phi}$ by

$$S_{0T}(\phi) = \frac{1}{2} \int_0^T [\dot{\phi}_s - (-\frac{\partial}{\partial x} U)(\phi_s)]^2 \, ds.$$

By means of the action functional we can define the *pseudopotential function*

$$V(x,y) = \inf\{S_{0T}(\phi) : \phi_0 = x, \phi_T = y, T > 0\},$$

for $x, y \in \mathbf{R}$. Intuitively, $V(x,y)$ describes the minimal work to be done in the potential landscape given by U to pass from x to y. Keeping this in mind, the relationship between U and V is easy to understand. If x and y are in the same potential well, we have

$$V(x,y) = 2(U(y) - U(x))^+, \tag{18}$$

where $b^+ = b \vee 0$ denotes the positive part of a real number b. In particular, if $U(y) < U(x)$, then $V(x,y) = 0$, i.e. going downhill in the landscape does not require work. If, however, x and y are in different potential wells, we have (recall $U(0) = 0$)

$$V(x,y) = -2U(x). \tag{19}$$

This equation reflects the fact that the minimal work to do to pass to y consists in reaching the saddle 0, since then one just can go downhill.

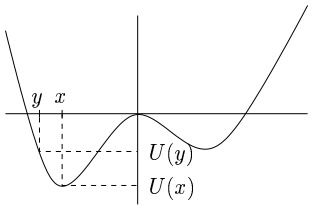

FIGURE 24

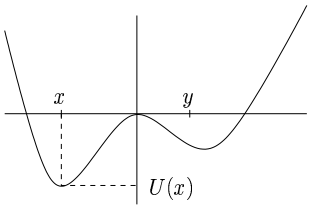

FIGURE 25

Rudiments of the following arguments can also be found in the explanation of stochastic resonance by Mc Namara, Wiesenfeld [47]. The main ingredient is the *exit time law* by Freidlin and Wentzell [20]. For $y \in \mathbf{R}, \epsilon > 0$ the first time y is visited is defined by

$$\tau_y^\epsilon = \inf\{t \geq 0 : X_t^\epsilon = y\}.$$

If P_x denotes the law of the diffusion $(X_t^\epsilon)_{t \geq 0}$ started at x, the exit time law states

Theorem 5.1. *For any $\delta > 0$ we have*

$$P_x(\exp(\frac{1}{\epsilon}[V(x,y) - \delta]) \leq \tau_y^\epsilon \leq \exp(\frac{1}{\epsilon}[V(x,y) + \delta])) \to 1$$

as $\epsilon \to 0$.

In other words, in the limit $\epsilon \to 0$, the process started at x takes approximately time $\exp(\frac{V(x,y)}{\epsilon})$ to reach y, or more roughly

$$\epsilon \ln \tau_y^\epsilon \cong V(x,y)$$

as $\epsilon \to 0$. As a consequence, one finds that as $\epsilon \to 0$, on time scales $T(\epsilon)$ at least as long as $\exp(\frac{V(x,y)}{\epsilon})$ or such that

$$\epsilon \ln T(\epsilon) > V(x,y),$$

we may expect with P_x-probability close to 1 that the process $X^\epsilon_{T(\epsilon)\,t}$ has reached y by time 1. Remembering (18) and (19) one obtains the following theorem stated much more generally by Freidlin.

Theorem 5.2. *Suppose*

$$\lim_{\epsilon\to 0} \epsilon \ln T(\epsilon) > 2 \max\{-U(x_0), -U(y_0)\}, \tag{20}$$

and $U(x_0) < U(y_0)$. *Then the Lebesgue measure of the set*

$$\{t \in [0,1] : |X^\epsilon_{t\,T(\epsilon)} - x_0| > \delta\}$$

tends to 0 as $\epsilon \to 0$ *in* P_x*-probability, for any* $\delta > 0$.

In other words, the process X^ϵ, run in a time scale $T(\epsilon)$ large enough, will spend most of the time in the deeper potential well. Excursions to the other well are *exponentially negligible* on this scale, as $\epsilon \to 0$. The picture is roughly this:

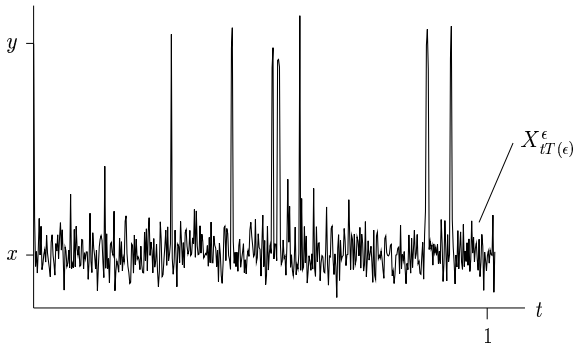

FIGURE 26

5.2. Periodic step potentials

As a rough approximation of temporally continuously varying potential functions we consider periodic *step function* potentials such as

$$U(t,.) = \begin{cases} U_1, & t \in [k, k+\frac{1}{2}[, \\ U_2, & t \in [k+\frac{1}{2}, k+1[, \end{cases} \quad k \in \mathbf{Z}_+. \tag{21}$$

We assume that both U_1 and U_2 are of the type described above, and that U_1 has a deeper well at x_0, U_2 at y_0. Then it is plausible that Theorem 5.2 generalizes to the following theorem of Freidlin

Theorem 5.3. *Suppose*

$$\lim_{\epsilon\to 0} \epsilon \ln T(\epsilon) > 2 \max\{-U_1(x_0), -U_1(y_0), -U_2(x_0), -U_2(y_0)\}, \tag{22}$$

and $U_1(x_0) < U_1(y_0), U_2(x_0) > U_2(y_0)$. *Define*

$$\phi(t) = \begin{cases} x_0, & t \in [k, k+\frac{1}{2}, \\ y_0, & t \in [k+\frac{1}{2}, k+1[, \end{cases} \quad k \in \mathbf{Z}_+.$$

Then the Lebesgue measure of the set

$$\{t \in [0,1] : |X^\epsilon_{t\,T(\epsilon)} - \phi(t)| > \delta\}$$

tends to 0 as $\epsilon \to 0$ *in* P_x*-probability, for any* $\delta > 0$.

Again, this just means that the process X^ϵ, run in a time scale $T(\epsilon)$ large enough, will spend most of the time in the deepest potential well which this time is a function ϕ of time. Excursions to the other well are *exponentially negligible* on this scale, as $\epsilon \to 0$. The picture is typically this:

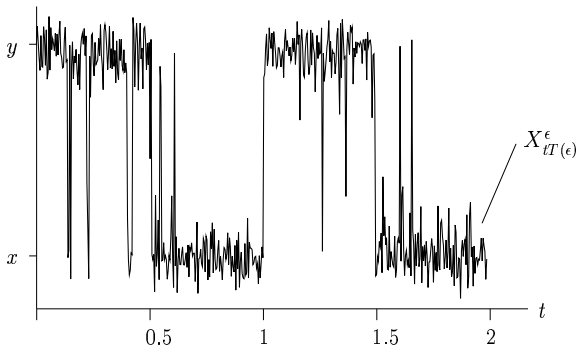

FIGURE 27

Since the function ϕ appearing in the theorem is already discontinuous, it is plausible that the step function potential is in fact a reasonable approximation of the general case of continuously and periodically changing potential functions. It is clear how the result has to be generalized to this situation. One has to define the periodic function ϕ denoting the deepest well position in dependence on t. Then, in a scale large enough, given in terms of the maximal potential depth, the process X^ϵ will spend most of the time near ϕ for small ϵ.

Do Theorems 5.2 and 5.3 explain *stochastic resonance*? The problem is obvious. They just give lower bounds for the scale $T(\epsilon)$ for which noise strength ϵ leads to random switches between the *most probable* potential wells near the (periodic) deterministic times when the role of the deepest well switches. But if $T(\epsilon)$ is too big, occasional excursions into the higher well will destroy a truely periodic tuning with the potential (see Figure 27). Just the duration of the excursions, being exponentially smaller than the periods of dwelling in the deeper well, will not be noticed by the criteria of the Theorems. We therefore also need an upper bound for possible scales. In order to find this optimal tuning scale, we first have to measure *goodness* of periodic tuning of the trajectories of the solution. The physical literature knows several notions of goodness, for example the *signal-to-noise ratio*, or the *amplitude of the first harmonic* in the Fourier decomposition of the solution (see Jung [40], or Gammaitoni et al. [22]). We shall work with the also well known and frequently studied notion of *spectral power amplification.*

5.3. Embedded Markov chain

We shall restrict to the crucial case of periodic step potentials, and approximate our diffusion processes by embedded Markov chains which are able to commute

between the two positions of potential wells. For simplicity of exposition, we shall assume

$$U_1(x_0) = -\frac{\alpha}{2}, U_1(y_0) = -\frac{\beta}{2}, U_2(x_0) = -\frac{\beta}{2}, U_2(y_0) = -\frac{\alpha}{2},$$

with $0 < \beta < \alpha$. With the prefactors $p, q, 0 \leq p, q \leq 1$ we define two possible transition matrices

$$P = \begin{bmatrix} 1 - p\, e^{-\frac{\alpha}{\epsilon}} & p\, e^{-\frac{\alpha}{\epsilon}} \\ q\, e^{-\frac{\beta}{\epsilon}} & 1 - q\, e^{-\frac{\beta}{\epsilon}} \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 - q\, e^{-\frac{\beta}{\epsilon}} & q\, e^{-\frac{\beta}{\epsilon}} \\ p\, e^{-\frac{\alpha}{\epsilon}} & 1 - p\, e^{-\frac{\alpha}{\epsilon}} \end{bmatrix}.$$

For the dynamics of the embedded Markov chain, a transition governed by P happens in a period in which U_1 is switched on, while a transition governed by Q happens when U_2 is on. To make the transition mechanism periodic, we have to set

$$P(k) = \begin{cases} P, & 0 \leq k \leq n-1, \\ Q, & n \leq k \leq 2n-1, \end{cases}$$

periodically continued with period $2n$. Period length $2n$ corresponds to the scale T in the continuous time model (see (17)).

Let $(X_k)_{k \geq 0}$ be the corresponding Markov chain. We next have to define the goodness measure for periodic tuning with the input frequency $\frac{1}{2n}$ for the Markov chain. For $n \in \mathbf{N}, \epsilon > 0$ let

$$\eta(n, \epsilon) = |\frac{1}{2n} \sum_{k=0}^{2n-1} e^{2\pi i \frac{k}{2n}} E_\rho(X_k)|^2,$$

where ρ is the invariant measure of the chain, considered as a time homogeneous Markov chain on the enlarged state space $\{x_0, y_0\} \times \{0, ..., 2n-1\}$. $\eta(n, \epsilon)$ is the *expected spectral power* corresponding to the frequency $\frac{1}{2n}$. Now we can formulate our task precisely: find the scale $n = n(\epsilon)$ such that $\eta(n, \epsilon)$ is maximal. Then the following basic result on optimal tuning holds (joint work with Ilya Pavlyukevitch)

Theorem 5.4. *As $\epsilon \to 0$ $\eta(n, \epsilon)$ has a unique maximum at*

$$n(\epsilon) \cong \frac{1}{\pi} \sqrt{2pq} \sqrt{\frac{\alpha - \beta}{\beta}} \exp(\frac{\alpha + \beta}{2\epsilon}),$$

and we have

$$\eta(n(\epsilon), \epsilon) \to \frac{4}{\pi^2}.$$

Optimal tuning curves can be exhibited and are just as expected from physical papers (see Jung [40]).

The optimal tuning parameter also seems to be characterized by the minimum of the entropies of the invariant measures $\rho = \rho(n)$ of the Markov chain. This very interesting observation seems to indicate that nature itself looks for good tuning.

Many interesting questions, however, still remain open:

1) does the result of Theorem 5.4 persist when passing from the embedded Markov chain to the diffusion process? How do we have to choose the prefactors p, q for this passage?
2) how does the result have to be modified as one passes from a potential step function to continuously varying periodic potentials?
3) what can be said about different goodness measures such as the signal-to-noise ratio? Can we work with pathwise spectral power functions instead of the expected one studied above?
4) can we generalize results to dimension ≥ 2?

Stochastic resonance recently proved to be relevant in other elementary climate models than the primitive one of section 2. In Penland et al. [62], Wang et al. [74], [75], a two-dimensional stochastic model for a qualitative explanation of the ENSO (El Nino Southern Oscillation) phenomenon leads to stochastic resonance effects: for certain parameter ranges the model exhibits random tuned transitions between two stable sea surface temperatures.

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