REGULARIZED DETERMINANTS OF LAPLACE TYPE OPERATORS, ANALYTIC SURGERY AND RELATIVE DETERMINANTS

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Abstract. Let M be a compact Riemannian manifold in which Y is an embedded hypersurface separating M into two parts. Assume that the metric is a product on a tubular neighborhood N of Y. Let Δ be a Laplace type operator on M adapted to the product structure on N. Under certain additional assumptions on Δ, we establish an asymptotic expansion for the logarithm of the regularized determinant $\det \Delta$ of Δ if the tubular neighborhood N is stretched to a cylinder of infinite length. We use the asymptotic expansions to derive adiabatic splitting formulas for regularized determinants.

1. Introduction

In this paper we study the behaviour of the regularized determinant of a Laplace type operator on a closed manifold M with respect to a special type of singular deformations of the underlying Riemannian metric. The singular deformations that we consider are defined as follows. Let $Y \subset M$ be a hypersurface such that $M - Y$ consists of two components M_1 and M_2. Assume that the metric in a collar neighborhood N of Y is a product. Then we consider the family of Riemannian manifolds (M_r, g_r), $r \geq 1$ obtained by the stretching of the collar neighborhood N to a cylinder of infinite length. The singular limit of this family is the disjoint union of two manifolds with cylindrical ends $M_{1,\infty}$ and $M_{2,\infty}$.

A modified version of this stretching process has been called "analytic surgery" by Mazzeo and Melrose [MM] and we will use the same name for the singular deformations described above. Analytic surgery was used, for example, by Douglas and Wojciechowski [DW], Wojciechowski [Wo], Mazzeo and Melrose [MM] and Hassell, Mazzeo and Melrose [HMM] to derive adiabatic limit and surgery formulas for eta-invariants.

For regularized determinants, Burghelea, Friedlander and Kappeler [BFK] were the first to establish a surgery formula. It involves, however, the determinant of some pseudodifferential operator on the dividing hypersurface which makes it more difficult to apply the formula. Analytic surgery leads to adiabatic versions of surgery formulas which can be applied successfully in a number of problems. So, for example, analytic surgery has been

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used by Hassell [Ha] to give another proof of the equality of analytic torsion and Reidemeister torsion of a closed manifold and by Park and Wojciechowski [PW1], [PW2], [PW3] to prove various adiabatic decomposition formulas for regularized determinants.

Our work is related to these results. Let $\Delta\colon C^\infty(M,E)\to C^\infty(M,E)$ be a Laplace type operator on M which is adapted to the product structure on N. Let M_r, $r\ge 1$ be the family of Riemannian manifolds attached to M as described above. Then there is a canonically associated family of Laplace type operators Δ_r on M_r and the main purpose of this paper is to study the behaviour of $\det(\Delta_r)$ as $r\to\infty$. Under some additional assumptions on Δ we will show that $\log\det\Delta_r$ has an asymptotic expansion and the main ingredient of the constant term of this expansion are the relative determinants $\det(\Delta_{i,\infty},\Delta_0)$, $i=1,2$, associated to the Laplacian on the manifolds with cylindrical ends $M_{1,\infty}$ and $M_{2,\infty}$, respectively. Here the relative determinants are defined as in [Mu1]. For surfaces our results are related to the work of Bismut and Bost [BB] who studied the Quillen metric on the determinant line bundle associated to a family of complex curves with singular fibers.

We also consider the analogous problem for a compact manifold with boundary where we stretch a collar neighborhood of the boundary to an infinite half-cylinder. The singular limit of the associated family of Riemannian manifolds with boundary is a manifold with a cylindrical end. The relative determinant of the Laplacian on the manifold with cylindrical end arises in the same manner as above in the asymptotic expansion of the determinant of the Dirichlet Laplacians. This gives a new interpretation of relative determinants.

If we compare the asymptotic expansions of the determinants of the Laplacians on the manifolds M_r, $M_{1,r}$ and $M_{2,r}$, respectively, obtained by stretching the corresponding collar neighborhoods of Y, we recover the adiabatic decomposition formulas of Park and Wojciechowski [PW1], [PW3]. We also establish a gluing formula for relative determinants of Laplace type operators on manifolds with cylindrical ends.

In the present paper we consider Laplace operators of two types. First we assume that the induced Laplace operator on Y is invertible and that the Laplacians $\Delta_{i,\infty}$, $i=1,2$, on $M_{i,\infty}$ have no nonzero L^2-solutions. This simplifies the constructions. The second case that we consider are Bochner-Laplace operators. In a followup paper we will study the case of Dirac-Laplace operators $\Delta=D^2$.

Now we describe the content of the paper in more detail. Let (X,g) be a Riemannian manifold and let $E\to X$ be a Hermitian vector bundle. First recall that a Laplace type operator

$$\Delta\colon C^\infty(X,E)\to C^\infty(X,E)$$

is a second order elliptic differential operator which is symmetric, nonnegative and whose principal symbol is given by

$$\sigma_\Delta(x,\xi)=\parallel\xi\parallel^2\operatorname{Id}_{E_x}.$$

Suppose that X is a compact manifold with boundary ∂X, which may be empty. We impose Dirichlet boundary conditions at ∂X and denote the corresponding selfadjoint extension by

Δ_D. This is a selfadjoint nonnegative operator in $L^2(X,E)$. The regularized determinant $\det \Delta_D$ of Δ_D is defined in the usual way by

$$\det \Delta_D = \exp\left(-\frac{d}{ds}\zeta_{\Delta_D}(s)\Big|_{s=0}\right),$$

where $\zeta_{\Delta_D}(s)$ is the zeta function of Δ_D.

Our first result is a gluing formula for relative determinants of Laplace type operators on a manifold X with a cylindrical end. By definition, X has a decomposition

$$X = M \cup_Y Z, \quad Z = \mathbb{R}^+ \times Y,$$

where M is a compact manifold with boundary Y and the metric g^X of X is a product on $\mathbb{R}^+ \times Y$. Let $E \to X$ be a hermitian vector bundle. We assume that there exist a hermitian vector bundle $E_0 \to Y$ such that $E|Z \cong \mathrm{pr}_Y^* E_0$ and that the fiber metric h^E of E is a product on $\mathbb{R}^+ \times Y$. Let $\Delta\colon C^\infty(X,E) \to C^\infty(X,E)$ be a Laplace type operator on X. We assume that the restriction of Δ to Z satisfies

$$\Delta|_Z = -\frac{\partial^2}{\partial u^2} + \Delta_Y, \tag{1.1}$$

where Δ_Y is a Laplace type operator on Y. This implies that Δ_X is essentially selfadjoint in L^2. We will denote the unique selfadjoint extension of Δ_X by the same letter. Consider the operator

$$-\frac{\partial^2}{\partial u^2} + \Delta_Y\colon C_c^\infty(\mathbb{R}^+ \times Y, E) \to L^2(\mathbb{R}^+ \times Y, E).$$

and impose Dirichlet boundary conditions at $\{0\} \times Y$. Let Δ_0 be the corresponding selfadjoint extension. Then (Δ, Δ_0) is a pair of self-adjoint operators which satisfies conditions 1)–3) in [Mu1, p.312] which are needed to define the relative regularized determinant $\det(\Delta, \Delta_0)$.

Let Δ_M denote the restriction of Δ to M and let $\Delta_{M,D}$ be the selfadjoint extension obtained by imposing Dirichlet boundary conditions at ∂M. We assume that $\Delta_{M,D}$ is invertible. This assumption is satisfied in many cases. Suppose, for example, that $D\colon C^\infty(X,E) \to C^\infty(X,E)$ is a Dirac operator and $\Delta = D^2$. Then it follows from [Ba] that Δ_D is invertible. In particular, if $\Delta_p\colon \Lambda^p(X) \to \Lambda^p(X)$ is the Laplacian on p-forms on a compact manifold with boundary, then $\Delta_{p,D}$ is invertible. Other examples are Bochner-Laplace operators.

If $\Delta_{M,D}$ is invertible, then the Dirichlet-to-Neumann operator R with respect to the hypersurface $Y \cong \{0\} \times Y \subset X$ can be defined in the usual way. This is a pseudo-differential operator of order 1 on Y which is selfadjoint and nonnegative. So R has a well-defined determinant $\det R$.

The last ingredient of the gluing formula is defined in terms of the space $\mathcal{H}$ of extended L^2-solutions of Δ. Recall that a section $\varphi \in C^\infty(X,E)$ is called an extended L^2-solution of Δ, if φ is a bounded solution of $\Delta\varphi = 0$ and its restriction to $\mathbb{R}^+ \times Y$ has the form

$$\varphi(u,y) = \phi(y) + \psi(u,y),$$

where ψ is in L^2 and $\phi \in \ker \Delta_Y$. In this case, ϕ is called the limiting value of φ. Let $V^+ \subset \ker \Delta_Y$ be the space of all limiting values of extended L^2-solutions of Δ. Given $\phi \in V^+$, let $E(\phi, \lambda)$ be the associated generalized eigensection of Δ (cf. [Mu4]). Then $E(\phi, \lambda)$ is holomorphic at $\lambda = 0$ and $E(\phi, 0)$ is an extended L^2 solution of Δ with limiting value 2ϕ. Let $\rho_Y \colon C^\infty(X, E) \to C^\infty(Y, E|Y)$ denote the restriction map and set $\mathcal{H}_Y := \rho_Y(\mathcal{H})$. We show that $\rho_Y : \mathcal{H} \to \mathcal{H}_Y$ is an isomorphism. Let $\varphi_1, ..., \varphi_k$ be an orthonormal basis of $\ker \Delta$ and let $\phi_1, ..., \phi_l$ be an orthonormal basis of V^+. Put $\psi_i = \rho_Y(\varphi_i)$, if $1 \le i \le k$, and $\psi_{k+j} = \frac{1}{2}\rho_Y(E(\phi_j, 0))$, if $1 \le j \le l$. Put $a_{ij} = \langle \psi_i, \psi_j \rangle_Y$, $1 \le i, j \le k + l$ and let A be the $(k + l) \times (k + l)$-matrix with entries a_{ij}. We are now ready to state our first main result which is the following theorem.

Theorem 1.1. *Assume that $\Delta_{M,D}$ is invertible. Let $h_Y = \dim \ker \Delta_Y$ and denote by $\zeta_Y(s)$ is the zeta function of Δ_Y. Then*

$$\frac{\det(\Delta, \Delta_0)}{\det(\Delta_{M,D})} = 2^{-\zeta_Y(0) - h_Y} \frac{\det R}{\det A}.$$

The same result has been proved independently by Loya and Park [LP].

Now assume that (M, g) is an oriented closed connected n-dimensional Riemannian manifold and let Y be a hypersurface of M such that $M - Y$ consists of two components. We denote the closure of the components of $M - Y$ by M_1 and M_2. Thus M_1 and M_2 are compact manifolds with common boundary Y such that

$$M = M_1 \cup_Y M_2, \quad \partial M_1 = \partial M_2 = Y. \tag{1.2}$$

Let $E \to M$ be a Hermitian vector bundle and let

$$\Delta_M \colon C^\infty(M, E) \to C^\infty(M, E)$$

be a Laplace type operator. We assume that there exists a tubular neighborhood N of Y which is diffeomorphic to $[-1, 1] \times Y$ such that all geometric structures are products over N, i.e., $g|_N = du^2 + g^Y$, there exists a Hermitian vector bundle $E_0 \to Y$ such that $E|N = \mathrm{pr}_Y^*(E_0)$ and

$$\Delta_M\big|_N = -\frac{\partial^2}{\partial u^2} + \Delta_Y, \tag{1.3}$$

where $\Delta_Y \colon C^\infty(Y, E_0) \to C^\infty(Y, E_0)$ is a Laplace type operator on Y. Let Δ_{M_i} be the restriction of Δ_M to M_i, $i = 1, 2$. We assume that $\Delta_{M_1,D}$ and $\Delta_{M_2,D}$ are invertible (see the above remark).

We define a family of Riemannian manifolds (M_r, g_r), $r > 0$, as follows. Given $r > 0$, let $N_r = [-r, r] \times Y$ and set

$$M_r = M_1 \cup_Y N_r \cup_Y M_2, \tag{1.4}$$

where ∂M_1 is identified with $\{-r\} \times Y$ and ∂M_2 with $\{r\} \times Y$. Since g is a product in a neighborhood of Y, it has a canonical extension to a metric g_r on M_r such that $g_r|_{N_r} = du^2 + g^Y$. Similarly, $E \to M$ and Δ_M have natural extensions $E_r \to M_r$ and Δ_{M_r}

to M_r. Our main purpose is to study the asymptotic behavior of $\det(\Delta_{M_r})$ as $r \to \infty$. To describe the result we need some more notation. Set

$$M_{i,\infty} = M_i \cup_Y (\mathbb{R}^+ \times Y), \quad i = 1, 2.$$

This is a manifold with a cylindrical end $Z = \mathbb{R}^+ \times Y$. The disjoint union of $M_{1,\infty}$ and $M_{2,\infty}$ may be regarded as the singular limit of M_r as $r \to \infty$. Let $\Delta_{i,\infty}$ be the canonical extension of $\Delta_M\big|_{M_i}$ to $M_{i,\infty}$ which is defined by

$$\Delta_{i,\infty}\big|_{M_i} = \Delta_M\big|_{M_i}, \quad \Delta_{i,\infty}\big|_{\mathbb{R}^+\times Y} = -\frac{\partial^2}{\partial u^2} + \Delta_Y.$$

Then $\Delta_{i,\infty}$ is essentially selfadjoint in L^2. We denote the unique selfadjoint extension of $\Delta_{i,\infty}$ by the same letter. Let Δ_0 be as in Theorem 1.1 and let $\det(\Delta_{i,\infty}, \Delta_0)$ be the relative determinant [Mu1].

Let

$$\xi_Y(s) := \frac{\Gamma(s - 1/2)}{\sqrt{\pi}\Gamma(s)} \zeta_Y(s - 1/2), \tag{1.5}$$

where $\zeta_Y(s)$ is the zeta function of Δ_Y. Our first result concerning the asymptotic behaviour of the determinant of a Laplace type operator is obtained under the assumption that all involved operators are invertible.

Theorem 1.2. *Suppose that* $\ker \Delta_Y = \{0\}$ *and* $\ker \Delta_{i,\infty} = \{0\}$, $i = 1, 2$. *Then* Δ_{M_r} *is invertible for* $r \geq r_0$ *and*

$$\lim_{r\to\infty} e^{r\xi_Y'(0)} \det \Delta_{M_r} = (\det \Delta_Y)^{-1/2} \prod_{i=1}^{2} \det(\Delta_{i,\infty}, \Delta_0). \tag{1.6}$$

In particular, the assumption of Theorem 1.2 are satisfied for the operator $\Delta_M + \lambda$, where $\lambda > 0$. Let $\xi_Y(s, \lambda)$ be defined as in (1.5) with $\zeta_Y(s)$ replaced by the zeta function $\zeta_Y(s, \lambda)$ of $\Delta_Y + \lambda$. Then we get

Corollary 1.3. *Let* $\lambda > 0$. *Then*

$$\lim_{r\to\infty} e^{r\xi_Y'(0,\lambda)} \det(\Delta_{M_r} + \lambda) = \det(\Delta_Y + \lambda)^{-1/2} \prod_{i=1}^{2} \det(\Delta_{i,\infty} + \lambda, \Delta_0 + \lambda).$$

We note that (1.6) also holds if M has a nonempty boundary ∂M. In this case we impose Dirichlet boundary conditions at ∂M.

In particular, we may consider a separating hypersurface which is parallel to the boundary. This is a special case which we consider separately. Let X_0 be a compact manifold with boundary Y and assume that all geometric structures are products in a collar neighborhood of Y. Let $X_r = X_0 \cup_Y ([0, r] \times Y)$ and let $\Delta_{X_r,D}$ be the selfadjoint extension of the corresponding Laplace operator with respect to Dirichlet boundary conditions. Then the analogous statement to Theorem 1.2 is

Proposition 1.4. *Assume that Δ_Y and Δ_∞ are invertible. Then $\Delta_{X_r,D}$ is invertible for $r \geq r_0$ and*

$$\lim_{r\to\infty} e^{r\xi'_Y(0)/2} \det \Delta_{X_r,D} = (\det \Delta_Y)^{-1/2} \det(\Delta_\infty, \Delta_0), \quad r \to \infty. \tag{1.7}$$

Remark. Under the same assumptions as in Theorem 1.2 and Proposition 1.4, respectively, Lee [Le4] has also obtained asymptotic expansions for $\log \det \Delta_{M_r}$ and $\log \det \Delta_{X_r,D}$, which are different from ours. The relation between [Le4] and our results is given by Theorem 1.1.

Especially consider the manifolds with boundary M_1 and M_2 of the decomposition (1.2) of M. Let $M_{i,r} = M_i \cup_Y ([0, r] \times Y)$, $i = 1, 2$. If we apply (1.7) to $\Delta_{M_{i,r},D}$ and compare it to (1.6) , we obtain

Corollary 1.5. *Let M be closed. Assume that Δ_Y and $\Delta_{i,\infty}$, $i = 1, 2$, are invertible. Then Δ_{M_r} and $\Delta_{M_{i,r},D}$, $i = 1, 2$, are invertible for $r \geq r_0$ and*

$$\lim_{r\to\infty} \frac{\det \Delta_{M_r}}{\det \Delta_{M_{1,r},D} \det \Delta_{M_{2,r},D}} = (\det \Delta_Y)^{1/2}. \tag{1.8}$$

This is the "adiabatic decomposition formula" established by Park and Wojciechowski in [PW1].

Next we study the case of a Bochner-Laplace operator. Let ∇ is a metric connection on E which is a product on N. Let $\Delta_M = \nabla^*\nabla$ be the associated Bochner-Laplace operator. Then ∇ has canonical extensions to a connection ∇^r on $E_r \to M_r$ and $\nabla^{i,\infty}$ on $E_{i,\infty} \to M_{i,\infty}$, respectively, and Δ_{M_r} and $\Delta_{i,\infty}$ are the corresponding Bochner-Laplace operators. We need to introduce some further notation. Let

$$S_i(0)\colon \ker \Delta_Y \to \ker \Delta_Y, \quad i = 1, 2,$$

denote the on-shell scattering operator at energy zero associated to $(\Delta_{i,\infty}, \Delta_0)$ (see e.g. [Mu4]). This operator satisfies $S_i(0)^2 = \mathrm{Id}$. Let

$$\ker \Delta_Y = V_i^+ \oplus V_i^-, \quad i = 1, 2,$$

be the decomposition of $\ker \Delta_Y$ into the ± 1-eigenspaces of $S_i(0)$. Let C_{12} denote the restriction of $S_1(0)S_2(0)$ to the orthogonal complement of $(V_1^+ \cap V_2^+) \oplus (V_1^- \cap V_2^-)$ in $\ker \Delta_Y$. Then our next result is

Theorem 1.6. *Let $\Delta_M = \nabla^*\nabla$ be a Bochner-Laplace operator. Let $h_Y = \dim \ker \Delta_Y$ and $h = \dim V_1^+ + \dim V_2^+ - 2 \dim V_1^+ \cap V_2^+$. Then*

$$\begin{aligned} \lim_{r\to\infty} r^{h-h_Y} e^{r\xi'_Y(0)} \det \Delta_{M_r} =& 2^{2h_Y - h} (\det \Delta_Y)^{-1/2} \\ & \cdot \det\left((\mathrm{Id} - C_{12})/2\right) \prod_{i=1}^{2} \det(\Delta_{i,\infty}, \Delta_0). \end{aligned} \tag{1.9}$$

We note that in the case of the Laplacian on p-forms, the term $\det((\mathrm{Id} - C_{12})/2)$ is closely related to the contribution of the reduced normal operator to the surgery formula for analytic torsion in [Ha]. See, for example, Proposition 5 of [Ha].

If we specialize Theorem 1.6 to the case of the Laplacian $\Delta = d^*d$ on functions on a closed surface M, we obtain

$$\det \Delta_r \sim 2 \det(\Delta_{1,\infty}, \Delta_0) \det(\Delta_{2,\infty}, \Delta_0) r e^{-\pi r/3}$$

as $r \to \infty$. This is Theorem 13.7 of [BB] with an explicit constant expressed in terms of relative determinants. See the end of section 7 for details.

As in Proposition 1.4, we may also consider the case of a compact Riemannian manifold X_0 with boundary Y. For a Bochner-Laplace operator on X_0 it follows from [Ba] that $\Delta_{X_r,D}$ is invertible. Let $V^+ \subset \ker \Delta_Y$ be the $+1$-eigenspace of the scattering operator $S(0)$ and let $h^+ = \dim V^+$. The analogous result to (1.7) is

$$\lim_{r\to\infty} r^{h^+ - h_Y} e^{r\xi_Y'(0)/2} \det \Delta_{X_r,D} = 2^{h_Y} (\det \Delta_Y)^{-1/2} \det(\Delta_\infty, \Delta_0). \tag{1.10}$$

Now we apply this again to the manifolds with boundary M_1 and M_2 of the decomposition (1.2) of M and compare it to (1.9). In this way we get

Theorem 1.7. *Let the notation be as in Theorem 1.6 and let* $h_{12} = \dim V_1^+ \cap V_2^+$. *Then*

$$\lim_{r\to\infty} \frac{r^{h_Y - 2h_{12}} \det \Delta_{M_r}}{\det \Delta_{M_1,r,D} \det \Delta_{M_2,r,D}} = 2^{-h} (\det \Delta_Y)^{1/2} \det((\mathrm{Id} - C_{12})/2).$$

Remark. This result was first proved by by Park and Wojciechowski [PW3] under an additional assumption, called Condition A [PW3, p.4], which rules out the existence of exponentially decreasing eigenvalues of Δ_{M_r}. As pointed out by Park and Wojciechowski, their assumption implies that 1 is not an eigenvalue of $S_1(0)S_2(0)$. This has the consequence that $V_1^\pm \cap V_2^\pm = \{0\}$, which in turn implies that $h = h_Y$ and $h_{12} = 0$ and Theorem 1.7 specializes to Theorem 0.1 of [PW3].

Next we explain some of the main ideas of the proofs. The strategy to prove Theorem 1.1 is analogous to the proof of the surgery formula in [HZ]. Let $z \in \mathbb{C} - \mathbb{R}_-$. Then the relative determinant $\det(\Delta + z, \Delta_0 + z)$ and the determinant $\det(\Delta_{M,D} + z)$ are defined. Moreover the Dirichlet-to-Neumann operator $R(z)$ with respect to $\Delta + z$ and the hypersurface $Y \subset X$ exists and the determinant $\det R(z)$ can be defined. Then by Theorem 4.2 of [Ca] there is a polynomial $P(z)$ with real coefficients of degree $\leq (n-1)/2$ such that

$$\frac{\det(\Delta + z, \Delta_0 + z)}{\det(\Delta_{M,D} + z)} = e^{P(z)} \det(R(z)). \tag{1.11}$$

Both sides of this equality have an expansion in z as $z \to 0$. We determine these expansions and compare the constant terms. This proves Theorem 1.1.

To prove Theorems 1.2 and 1.6, we apply the Mayer-Vietoris formula of [BFK] to $\det(\Delta_{M_r} + \lambda)$, $\lambda > 0$, with respect to the decomposition (1.4) and take the limit $\lambda \to 0$. To this end we assume that $\Delta_{M_1,D}$ and $\Delta_{M_2,D}$ are invertible. Under this assumption the Dirichlet-to-Neumann operator R_r with respect to the hypersurface $\Sigma_r := (\{-r\} \times Y) \sqcup (\{r\} \times Y)$ exists and we get a splitting formula for $\det \Delta_{M_r}$. We compare this splitting formula with the splitting formulas for $\det(\Delta_{i,\infty}, \Delta_0)$ given by Theorem 1.1. Then we study the limit of $\det R_r$ as $r \to \infty$ and compare it to $\det R_{1,\infty} \det R_{2,\infty}$. Let $\Delta_{N_r,D}$ be the Laplace operator on N_r with Dirichlet boundary conditions. Under the assumptions of Theorem 1.2 or 1.6 the limit as $r \to \infty$ of $r^h \det \Delta_{M_r} (\det \Delta_{N_r,D})^{-1}$ exists and

$$\lim_{r\to\infty} \frac{r^h \det \Delta_{M_r}}{\det \Delta_{N_r,D}} = 2^{-h} \det\left((\operatorname{Id} - C_{12})/2\right) \prod_{i=1}^{2} \det(\Delta_{i,\infty}, \Delta_0).$$

Finally we determine the asymptotic behaviour of $\det \Delta_{N_r,D}$ as $r \to \infty$. This completes the proof of Theorem 1.2 and Theorem 1.6.

2. Expansion of relative determinants

Let X be a manifold with a cylindrical end and let Δ be a Laplace type operator on X as above. In this section we consider the asymptotic expansion of $\log\det(\Delta + z, \Delta_0 + z)$ as $z \to 0$. We use the framework introduced in [Mu1]. Let H, H_0 be two self-adjoint nonnegative linear operators in a separable Hilbert space $\mathcal{H}$ such that $e^{-tH} - e^{-tH_0}$ is a trace class operator for all $t > 0$. Suppose that the following two conditions are satisfied:

1) As $t \to 0+$, there exists an asymptotic expansion of the form

$$\operatorname{Tr}(e^{-tH} - e^{-tH_0}) \sim \sum_{j=0}^{\infty} a_j t^{\alpha_j},$$

where $-\infty < \alpha_0 < \alpha_1 < \cdots$ and $\alpha_j \to \infty$.

2) There exist $b_0 \in \mathbb{C}$, $\rho > 0$ such that

$$\operatorname{Tr}(e^{-tH} - e^{-tH_0}) \sim b_0 + O(t^{-\rho})$$

as $t \to \infty$.

Set

$$\zeta_1(s, H, H_0) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \operatorname{Tr}(e^{-tH} - e^{-tH_0})\, dt, \quad \operatorname{Re}(s) > -\alpha_0;$$
$$\zeta_2(s, H, H_0) = \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \operatorname{Tr}(e^{-tH} - e^{-tH_0})\, dt, \quad \operatorname{Re}(s) < 0.$$

Then $\zeta_1(s, H, H_0)$ admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$. Similarly $\zeta_2(s, H, H_0)$ has a meromorphic extension to the half-plane $\operatorname{Re}(s) < \rho$ which is also holomorphic at $s = 0$. It is given by

$$\zeta_2(s, H, H_0) = -\frac{b_0}{\Gamma(s+1)} + \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} \left(\operatorname{Tr}(e^{-tH} - e^{-tH_0}) - b_0\right)\, dt. \tag{2.1}$$

The relative zeta function $\zeta(s,H,H_0)$ is then defined by

$$\zeta(s,H,H_0)=\zeta_1(s,H,H_0)+\zeta_2(s,H,H_0),$$

and the relative determinant by

$$\det(H,H_0)=\exp\left(-\frac{d}{ds}\zeta(s,H,H_0)\big|_{s=0}\right).$$

Let $\lambda>0$ and define $\det(H+\lambda,H_0+\lambda)$ similarly.

Remark. Instead of using the relative determinant, one could also use the b-calculus of Melrose [Me] to regularize the determinant of H. One simply has to replace the relative trace $\mathrm{Tr}(e^{-tH}-e^{-tH_0})$ by the b-trace to define a b-determinant in the same way as the relative determinant is defined. This definition was used, for example, by Hassell [Ha]. The two definitions are closely related. There is a simple formula which expresses the relative trace $\mathrm{Tr}(e^{-tH}-e^{-tH_0})$ in terms of the b-trace. See [Mu1, p.337] for details.

Proposition 2.1. *As $\lambda\to 0+$, we have*

$$\log\det(H+\lambda,H_0+\lambda)=b_0\log\lambda+\log\det(H,H_0)+o(1).$$

Proof. From the construction of the analytic continuation of $\zeta_1(s,H+\lambda,H_0+\lambda)$ and $\zeta_1(s,H,H_0)$, respectively, it follows immediately that

$$\lim_{\lambda\to 0}\zeta_1'(0,H+\lambda,H_0+\lambda)=\zeta_1'(0,H,H_0).$$

Let $\mathrm{Re}(s)<\rho$. Using (2.1) we get

$$\begin{aligned}\zeta_2(s,H+\lambda,H_0+\lambda)&=\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}e^{-t\lambda}\,\mathrm{Tr}(e^{-tH}-e^{-tH_0})\,dt\\&=\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}e^{-t\lambda}\left(\mathrm{Tr}(e^{-tH}-e^{-tH_0})-b_0\right)\,dt\\&\quad+\frac{b_0}{\Gamma(s)}\int_1^\infty t^{s-1}e^{-t\lambda}\,dt\\&=\zeta_2(s,H,H_0)+\frac{b_0}{\Gamma(s+1)}+b_0\lambda^{-s}-\frac{b_0}{\Gamma(s)}\int_0^1 t^{s-1}e^{-t\lambda}\,dt+o(1)\end{aligned}$$

as $\lambda\to 0+$. This implies that

$$\zeta_2'(0,H+\lambda,H_0+\lambda)=\zeta_2'(0,H,H_0)-b_0\log\lambda+o(1)$$

as $\lambda\to 0+$. □

In order to apply this result to our case, we need to compute b_0. Let $\xi(\lambda)$ be the spectral shift function of (Δ,Δ_0) [Mu1, pp. 315]. By (2.16) of [Mu1], we have

$$b_0=-\xi(0+). \tag{2.2}$$

So we are reduced to the study of the spectral shift function near zero. Recall that the spectral shift function is a real valued function in $L^2_{\mathrm{loc}}(\mathbb{R})$ which is uniquely determined by the following two properties

(1) $\xi(\lambda) = 0$ for all $\lambda < 0$.
(2) For every $f \in C_c^\infty(\mathbb{R})$, $f(\Delta) - f(\Delta_0)$ is a trace class operator and

$$\operatorname{Tr}(f(\Delta) - f(\Delta_0)) = \int_R f'(\lambda)\xi(\lambda)\, d\lambda.$$

Let Δ_d and Δ_{ac} denote the restriction of Δ to the subspace of $L^2(X, E)$ corresponding to the point spectrum and the absolutely continuous spectrum of Δ, respectively. By [Do], the eigenvalues of Δ have no finite point of accumulation. Hence $f(\Delta_d)$ is a trace class operator for every $f \in C_c^\infty(\mathbb{R})$. This implies that $f(\Delta_{ac}) - f(\Delta_0)$ is also a trace class operator for every $f \in C_c^\infty(\mathbb{R})$. Let $\xi_c(\lambda)$ be the spectral shift function of (Δ_{ac}, Δ_0) and let $N(\lambda)$ denote the counting function of the eigenvalues of Δ. Then it follows from (1) and (2) that

$$\xi(\lambda) = -N(\lambda) + \xi_c(\lambda). \tag{2.3}$$

The spectral shift function $\xi_c(\lambda)$ can be determined in the same way as in Chapter IX of [Mu3]. The manifolds considered in [Mu3] are manifolds with fibered cusps which are different from the manifolds in the present paper. However, the structure of the continuous spectrum is similar and everything said about the continuous spectrum in [Mu3] applies with minor modifications in our case as well. Let $\mu_1 > 0$ be the smallest positive eigenvalue of Δ_Y. Let

$$S(s)\colon \ker \Delta_Y \to \ker \Delta_Y, \quad |s| < \sqrt{\mu_1},$$

be the scattering matrix [Mu4]. It is an analytic function. Then it follows as in the proof of Theorem 9.25 of [Mu3] that

$$\xi_c(\lambda) = -\frac{1}{4}(\operatorname{Tr}(S(0)) + \dim\ker\Delta_Y) + \frac{i}{2\pi}\int_0^\lambda \operatorname{Tr}(S'(s)S(-s))\, d\lambda$$

for $0 \le \lambda < \sqrt{\mu_1}$. Hence we get

$$\xi_c(0+) = -\frac{1}{4}(\operatorname{Tr}(S(0)) + \dim\ker\Delta_Y).$$

Together with (2.3) we obtain

$$\xi(0+) = -\dim\ker\Delta - \frac{1}{4}(\operatorname{Tr}(S(0)) + \dim\ker\Delta_Y)$$

and by (2.2) it follows that

$$b_0 = \dim\ker\Delta + \frac{1}{4}(\operatorname{Tr}(S(0)) + \dim\ker\Delta_Y).$$

Now observe that $S(0)$ satisfies $S(0)^2 = \mathrm{Id}$. Hence

$$\operatorname{Tr}(S(0)) + \dim\ker\Delta_Y = 2\dim\ker(S(0) - \mathrm{Id}).$$

Combined with Proposition 2.1 we obtain the following corollary.

Corollary 2.2. *Let* $k = \dim\ker\Delta$ *and* $l = \dim\ker(S(0) - \mathrm{Id})$. *Then*

$$\log\det(\Delta+\lambda, \Delta_0+\lambda) = (k + l/2)\log\lambda + \log\det(\Delta, \Delta_0) + o(1)$$

as $\lambda \to 0+$.

3. Expansion of the Dirichlet-to-Neumann operator

Let $X = M \cup_Y Z$ be a manifold with a cylindrical end $Z = \mathbb{R}^+ \times Y$ and let $\Delta\colon C^\infty(Z, E) \to C^\infty(Z, E)$ be a Laplace type operator on X with properties as above. For $z \in \mathbb{C} - \mathbb{R}_-$ let $R(z)$ be the Dirichlet-to-Neumann operator with respect to $\Delta + z$ and the hypersurface $Y = \{0\} \times Y \subset X$. In this section we study the expansion of $\det(R(z))$ as $z \to 0$. To begin with we recall the definition of the Dirichlet-to-Neumann operator. Let $z \in \mathbb{C} - R_-$ and $\varphi \in C^\infty(Y, E|Y)$. There exists a unique section $\psi \in C^\infty(X - Y, E) \cap L^2(X, E)$ such that

$$(\Delta + z)\psi = 0 \quad \text{on } \; X - Y;$$
$$\psi = \varphi \quad \text{on } \; Y.$$

The solution ψ is obtained as follows. Let $\widetilde{\varphi} \in C_c^\infty(X, E)$ be any extension of φ. Let Δ_D be the operator Δ with Dirichlet boundary conditions along Y. Then

$$\psi = \widetilde{\varphi} - (\Delta_D + z)^{-1}\big((\Delta + z)(\widetilde{\varphi})\big). \tag{3.1}$$

Furthermore, ψ is continuous on X and smooth on $\overline{M}$ and $\overline{Z}$. Its normal derivative has a jump along Y. Then $R(z)\varphi$ is defined by

$$R(z)\varphi = \frac{\partial}{\partial u}\big(\psi|_M\big)\big|_{\partial M} - \frac{\partial}{\partial u}\big(\psi|_Z\big)\big|_{\partial Z}. \tag{3.2}$$

By Theorem 2.1 of [Ca], $R(z)$ is an invertible pseudo-differential operator of order 1. Its principal symbol is given by

$$\sigma(R(z))(x, \xi) = 2\sqrt{g_x(\xi, \xi)}\,\mathrm{Id}_{E_x}, \quad \xi \in T_x^*Y.$$

Furthermore, $z \in \mathbb{C} - R_- \mapsto R(z)$ is a holomorphic function with values in the space of pseudo-differential operators. Let $G(x, y, z)$ denote the kernel of $(\Delta + z)^{-1}$. Then $G(x, y, z)$ is smooth in the complement of the diagonal and for $x \neq y$, $G(x, y, z) \in \mathrm{Hom}(E_y, E_x)$. As shown in the proof of Theorem 2.1 of [Ca], we have

$$R(z)^{-1}\varphi(x) = \int_Y G(x, y, z)\varphi(y)\, dy, \quad x \in Y,\ \varphi \in C^\infty(Y, E|Y). \tag{3.3}$$

In other words

$$R(z)^{-1} = \rho_Y \circ (\Delta + z)^{-1}(\cdot \otimes \delta_Y),$$

where ρ_Y is the restriction map to Y and δ_Y is the Dirac δ-function along Y. Especially, if $\lambda > 0$ then $R(\lambda)$ is an elliptic pseudodifferential operator of order 1 which is selfadjoint and positive definite. Hence its regularized determinant $\det(R(\lambda))$ is defined.

Under the assumption that $\Delta_{M,D}$ is invertible, we can also define the Dirichlet-to-Neumann operator with respect to Δ and Y. For this purpose we need the following lemma.

Lemma 3.1. *For every $\varphi \in C^\infty(Y, E|Y)$ there exists a unique $\psi \in C^\infty(X - Y, E) \cap C^0(X, E)$, which is bounded and satisfies*

$$\begin{aligned} \Delta\psi &= 0 \quad \text{on } \ X - Y; \\ \psi|_Y &= \varphi. \end{aligned} \tag{3.4}$$

Proof. Since $\Delta_{M,D}$ is invertible, the Dirichlet problem on M has a unique solution, i.e., for every $\varphi \in C^\infty(Y, E|Y)$ there exists a unique $\psi_1 \in C^\infty(M, E) \cap C^0(\overline{M}, E)$ such that

$$\begin{aligned} \Delta_M\psi_1 &= 0 \quad \text{in } \ M; \\ \psi_1|_Y &= \varphi. \end{aligned}$$

Next we show that the Dirichlet problem on Z has also a unique solution. Let $\{\phi_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2(Y, E|Y)$ consisting of eigenfunctions of Δ_Y with eigenvalues $0 \le \lambda_0 \le \lambda_1 \le \cdots$. Let $\varphi \in C^\infty(Y, E|Y)$. Then φ has an expansion of the form

$$\varphi = \sum_{i=1}^\infty a_j\phi_j.$$

Set

$$\psi_2(u, y) = \sum_{j=1}^\infty a_j e^{-u\sqrt{\lambda_j}}\phi_j(y).$$

Then $\psi_2 \in C^\infty(Z, E)$ is bounded and satisfies

$$\Delta\psi_2 = 0 \text{ and } \psi_2(0, y) = \varphi(y), \quad y \in Y. \tag{3.5}$$

This proves existence. Now suppose that $\widetilde{\psi}_2$ is a second bounded solution of (3.4). Set $g = \psi_2 - \widetilde{\psi}_2$. Then $g \in C^\infty(Z, E)$ is bounded and satisfies

$$\begin{aligned} \left(-\frac{\partial^2}{\partial u^2} + \Delta_Y\right) g &= 0; \\ g(u, y) &= 0, \ \ y \in Y. \end{aligned}$$

If we expand g in the orthonormal basis $\{\phi_j\}_{j\in\mathbb{N}}$ it follows that

$$g(u, y) = \sum_{j=1}^m (b_j u + a_j)\phi_j(y) + \sum_{j=m+1}^\infty (b_j e^{\sqrt{\lambda_j}u} + a_j e^{-\sqrt{\lambda_j}u})\phi_j(y),$$

where $m = \dim\ker\Delta_Y$. Since g is bounded, it follows that $b_j = 0$ for all $j \in \mathbb{N}$. Using that $g(0, y) = 0$, we obtain $a_j = 0$ for all $j \in \mathbb{N}$. This proves uniqueness. □

Now we can proceed as above. Given $\varphi \in C^\infty(Y, E|Y)$, let $\psi \in C^\infty(X - Y, E) \cap C^0(X, E)$ be the unique solution of (3.4). Then the Dirichlet-to-Neumann operator is defined by

$$R\varphi = \frac{\partial}{\partial u}(\psi|_M)\big|_{\partial M} - \frac{\partial}{\partial u}(\psi|_Z)\big|_{\partial Z}. \tag{3.6}$$

Next we establish some properties of R.

Lemma 3.2. *There exist a smoothing operator K such that*

$$R = 2\sqrt{\Delta_Y} + K.$$

Proof. Since $X - Y = M \sqcup Z$, R can be written as

$$R = R_{\text{int}} + R_{\text{ext}},$$

where R_{int} is the Neumann jump operator on M. It is defined as follows. Given $\varphi \in C^\infty(Y, E|Y)$, let $\psi_1 \in C^\infty(M, E) \cap C^0(\overline{M}, E)$ be the unique solution of

$$\Delta\psi_1 = 0 \quad \text{on} \quad M, \quad \psi_1|_Y = \varphi.$$

Then R_{int} is defined as

$$R_{\text{int}}\varphi := \frac{\partial\psi_1}{\partial u}\Big|_Y.$$

Similarly let $\psi_2 \in C^\infty(Z, E) \cap C^0(\overline{Z}, E)$ be the unique bounded solution of

$$\Delta\psi_2 = 0 \quad \text{on} \quad Z, \quad \psi_2|_Y = \varphi.$$

Set

$$R_{\text{ext}}(\varphi) := -\frac{\partial\psi_2}{\partial u}\Big|_Y.$$

As explained above, ψ_2 is given by

$$\psi_2(u, y) = \sum_{j=1}^{\infty} \langle\varphi, \phi_j\rangle e^{-\sqrt{\lambda_j}u}\phi_j(y).$$

Hence we get

$$-\frac{\partial\psi_2}{\partial u}(0, y) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\langle\varphi, \phi_j\rangle\phi_j(y) = (\sqrt{\Delta_Y}\varphi)(y).$$

Thus $R_{\text{ext}} = \sqrt{\Delta_Y}$. By Theorem 2.1 of [Le3] it follows that $R_{\text{int}} = \sqrt{\Delta_Y} + K$, where K is a smoothing operator. This proves the lemma. □

In particular, it follows that R is an elliptic pseudodifferential operator of order 1.

Lemma 3.3. *For every $\phi \in C^\infty(Y, E|Y)$, $R(\lambda)\phi$ is a continuous function of $\lambda \in [0, \infty)$ with values in $L^2(Y, E|Y)$ and*

$$\lim_{\lambda \to 0+} R(\lambda)\phi = R\phi.$$

Proof. Let $\lambda \geq 0$. As above, $R(\lambda)$ can be written as

$$R(\lambda) = R_{\text{int}}(\lambda) + R_{\text{ext}}(\lambda).$$

Given $\phi \in C^\infty(Y, E|Y)$, let $\psi_1(\lambda) \in C^\infty(M - Y, E) \cap C^0(M, E)$ be the unique section which satisfies $(\Delta + \lambda)\psi_1(\lambda) = 0$ and $\psi_1(\lambda)|_Y = \phi$. Let $\tilde{\phi} \in C^\infty(M, E)$ be any extension of ϕ which is smooth up to the boundary. Then

$$\psi_1(\lambda) = \tilde{\phi} - (\Delta_{M,D} + \lambda)^{-1}((\Delta_M + \lambda)(\tilde{\phi})).$$

Since $\Delta_{M,D}$ is invertible, this formula also holds for $\lambda = 0$. From this representation of $\psi_1(\lambda)$ it follows immediately that $R_{\text{int}}(\lambda)\phi$ converges to $R_{\text{int}}\phi$ as $\lambda \to 0+$. Next observe that the unique bounded solution $\psi_2(\lambda) \in C^\infty(Z, E) \cap C^0(Z, E)$ of

$$(\Delta + \lambda)\psi_2(\lambda) = 0 \quad \text{on} \quad Z, \quad \psi_2(\lambda)|_Y = \phi$$

is given by

$$\psi_2(\lambda, u, y) = \sum_{j=1}^{\infty} \langle \varphi, \phi_j \rangle e^{-(\lambda_j+\lambda)^{1/2}u} \phi_j(y).$$

Then $R_{\text{ext}}(\lambda)\phi := \partial\psi_2(\lambda, u.y)/\partial u|_{u=0}$ and it follows that $R_{\text{ext}}(\lambda)\phi$ is continuous in $\lambda \in [0, \infty)$ and $R_{\text{ext}}(\lambda)\phi$ converges to $R_{\text{ext}}\phi$ as $\lambda \to 0+$. □

Corollary 3.4. *The operator R is formally selfadjoint and nonnegative.*

Proof. As explained above, for every $\lambda > 0$, the operator $R(\lambda)$ is formally selfadjoint and positive, and therefore the claim follows immediately from Lemma 3.3. □

Together we have proved that R is a first order elliptic pseudo-differential operator which is formally selfadjoint and nonnegative. Hence the regularized determinant $\det R$ is well-defined.

Our next purpose is to study the behaviour of the bounded operator $R(\lambda)^{-1}$ as $\lambda \to 0$. First we recall some facts about the spectral resolution of Δ. For more details we refer to [Mu4]. We have

$$L^2(X, E) = L^2_d(X, E) \oplus L^2_c(X, E),$$

where

$$L^2_d(X, E) = \bigoplus_j \mathcal{E}(\lambda_j)$$

is the discrete sum of the eigenspaces of Δ with eigenvalues $0 \leq \lambda_1 < \lambda_1 < \cdots$. Each eigenspace is finite dimensional. The orthogonal complement $L^2_c(X, E)$ of $L^2_d(X, E)$ is the absolutely continuous subspace for Δ. It can be described in terms of generalized eigensections $E(\phi_j, \lambda)$ attached to the eigensections ϕ_j of Δ_Y. Each $E(\phi_j, \lambda)$ is a smooth section of E and satisfies

$$\Delta E(\phi_j, \lambda) = \lambda E(\phi_j, \lambda).$$

Of particular importance for our purpose are the generalized eigensections $E(\phi, \lambda)$ attached to $\phi \in \ker \Delta_Y$. Let $\mu_1 > 0$ be the smallest positive eigenvalue of Δ_Y. If we put $\lambda = s^2$ and regard $E(\phi, \lambda)$ as a function of s, then $E(\phi, s)$ has an analytic continuation to the disc $|s| < \mu_1$. Let

$$S(s) : \ker \Delta_Y \to \ker \Delta_Y, \quad |s| < \mu_1,$$

be the corresponding scattering matrix. It is also holomorphic for $|s| < \mu_1$ and on $\mathbb{R}^+ \times Y$ we have

$$E(\phi, s, (u, y)) = e^{ius}\phi(y) + e^{-isu}(S(s)\phi)(y) + \psi(s), \tag{3.7}$$

where $\psi(s)$ is in L^2. Let $0 < \mu < \mu_1$ and let P_μ be the spectral projection of Δ onto $[0, \mu]$. By (3.3) we have

$$R(\lambda)^{-1} = \rho_Y \circ P_\mu(\Delta + \lambda)^{-1}(\cdot \otimes \delta_Y) + \rho_Y \circ (\mathrm{Id} - P_\mu)(\Delta + \lambda)^{-1}(\cdot \otimes \delta_Y). \tag{3.8}$$

First we study the second operator on the right. Let

$$i_Y : L^2(Y, E|Y) \to H^{-1}(X, E)$$

be the map which is defined by $i_Y(\varphi) = \varphi \delta_Y$. Then i_Y is continuous. Furthermore the restriction map ρ_Y defines a continuous map

$$\rho_Y : H^1(X, E) \to L^2(Y, E|Y).$$

Since $(\Delta + \lambda)^{-1} : H^{-1}(X, E) \to H^1(X, E)$ is continuous, we get a continuous map

$$\rho_Y \circ (\mathrm{Id} - P_\mu)(\Delta + \lambda)^{-1} \circ i_Y : L^2(Y, E|Y) \to L^2(Y, E|Y).$$

Lemma 3.5. *There exists $C > 0$ such that*

$$\| \rho_Y \circ (\mathrm{Id} - P_\mu)(\Delta + \lambda)^{-1} \circ i_Y \|_{L^2} \leq C$$

for all $\lambda \geq 0$.

Proof. Let $\varphi \in H^{-1}(X, E)$. Then $\| \varphi \|_{H^{-1}} = \| (\Delta + \mathrm{Id})^{-1/2} \varphi \|_{L^2}$. Hence we get

$$\begin{aligned} \| (\mathrm{Id} - P_\mu)(\Delta + \lambda)^{-1} \varphi \|_{H^1} &= \| (\Delta + \mathrm{Id})(\mathrm{Id} - P_\mu)(\Delta + \lambda)^{-1} (\Delta + \mathrm{Id})^{-1/2} \varphi \|_{L^2} \\ &\leq \| (\Delta + \mathrm{Id})(\mathrm{Id} - P_\mu)(\Delta + \lambda)^{-1} \|_{L^2} \cdot \| \varphi \|_{H^{-1}} . \end{aligned}$$

Using the spectral theorem we get

$$\| (\Delta + \mathrm{Id})(\mathrm{Id} - P_\mu)(\Delta + \lambda)^{-1} \|_{L^2} \leq 1 + 1/\mu$$

for $\lambda \geq 0$. This implies

$$\| (\mathrm{Id} - P_\mu)(\Delta + \lambda)^{-1} \|_{L(H^{-1}, H^1)} \leq 1 + 1/\mu$$

for $\lambda \geq 0$. Since i_Y and ρ_Y are continuous, the lemma follows. □

It remains to consider the first operator on the right hand side of (3.8). This is a smoothing operator whose kernel $R(y_1, y_2, \lambda)$ can be described as follows. Let $\{\varphi_j\}$ be an orthonormal basis of eigensections of Δ with eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ and let $\phi_1, \ldots, \phi_m$ be an orthonormal basis of $\ker \Delta_Y$. Then it follows from the explicit description of the spectral resolution of Δ (see [Gu], [Mu4]) that

$$\begin{aligned} R(y_1, y_2, \lambda) &= \sum_{\lambda_j \leq \mu} (\lambda_j + \lambda)^{-1} \varphi_j(y_1) \otimes \varphi_j(y_2) \\ &\quad + \frac{1}{2\pi} \sum_{j=1}^{m} \int_0^\mu (s^2 + \lambda)^{-1} E(\phi_j, s, y_1) \otimes E(\phi_j, -s, y_2) ds. \end{aligned} \tag{3.9}$$

We shall now determine the behaviour of this kernel as $\lambda \to 0$. The behaviour of the first sum is obvious and we only need to investigate the second sum.

Lemma 3.6. *Let* $\phi_1, ..., \phi_m$ *be an orthonormal basis of* $\ker \Delta_Y$. *Then*

$$\sum_{j=1}^{m} E(\phi_j, s, y_1) \otimes E(\phi_j, -s, y_2)$$

is an even function of s, $|s| < \mu_1$.

Proof. We recall that the generalized eigensections and the scattering matrix satisfy the following functional equations. Let $\phi \in \ker \Delta_Y$. Then

$$\begin{aligned} E(\phi, -s) &= E(S(-s)\phi, s), \\ S(s)S(-s) = \mathrm{Id}, \quad S(s)^t &= S(s), \quad |s| < \mu_1. \end{aligned} \tag{3.10}$$

Let $\phi_1, ..., \phi_m$ be an orthonormal basis of $\ker \Delta_Y$. Then there exist analytic functions $a_{ij}(s)$, $i, j = 1, ..., m$, defined in $|s| < \mu_1$, such that

$$S(s)\phi_i = \sum_{j=1}^{m} a_{ij}(s)\phi_j, \quad i = 1, ..., m. \tag{3.11}$$

Using (3.10) and (3.11) we get

$$\begin{aligned} \sum_{j=1}^{m} E(\phi_j, -s, y_1) \otimes E(\phi_j, s, y_2) &= \sum_{j=1}^{m} E(S(-s)\phi_j, s, y_1) \otimes E(S(s)\phi_j, -s, y_2) \\ &= \sum_{j=1}^{m} \sum_{k,l=1}^{m} a_{jk}(-s)a_{jl}(s)E(\phi_k, s, y_1) \otimes E(\phi_l, -s, y_2). \end{aligned}$$

By (3.10) the matrix $A(s) = (a_{ij}(s))_{i,j}$ is symmetric and satisfies $A(-s)A(s) = \mathrm{Id}$. This implies

$$\sum_{j=1}^{m} E(\phi_j, -s, y_1) \otimes E(\phi_j, s, y_2) = \sum_{j=1}^{m} E(\phi_j, s, y_1) \otimes E(\phi_j, -s, y_2)$$

as claimed. □

By Lemma 3.6 there exists a smooth section $\widetilde{E}(s)$ of $E \boxtimes E$ over $X \times X$ which is holomorphic for $|s| < \mu$ such that

$$\begin{aligned} \sum_{j=1}^{m} E(\phi_j, s, y_1) \otimes E(\phi_j, -s, y_2) &= \sum_{j=1}^{m} E(\phi_j, 0, y_1) \otimes E(\phi_j, 0, y_2) \\ &\quad + s^2 \widetilde{E}(s, (y_1, y_2)), \quad |s| < \mu. \end{aligned} \tag{3.12}$$

Note that

$$\int_0^{\mu} \frac{ds}{s^2 + \lambda} = \frac{\pi}{2\sqrt{\lambda}} - \frac{1}{\mu} + O(\lambda)$$

as $\lambda \to 0$. Together with (3.12) we get

$$\frac{1}{2\pi}\sum_{j=1}^{m}\int_0^{\mu}(s^2+\lambda)^{-1}E(\phi_j,s,y_1)\otimes E(\phi_j,-s,y_2)ds$$
$$=\frac{1}{4\sqrt{\lambda}}\sum_{j=1}^{m}E(\phi_j,0,y_1)\otimes E(\phi_j,0,y_2)+O(1)$$

as $\lambda \to 0$. To continue we consider the the scattering matrix $S(0)$ at zero energy. It satisfies

$$S(0)^2 = \mathrm{Id}\,.$$

Let $\phi \in \ker \Delta_Y$. If $S(0)\phi = \phi$ then it follows from (3.7) that on $\mathbb{R}^+ \times Y$ we have

$$E(\phi, 0) = 2\phi + \psi,$$

where $\psi \in L^2(\mathbb{R}^+ \times Y, E)$. If $S(0)\phi = -\phi$, then $E(\phi, 0) = 0$ [Mu2, p. 209]. Let

$$\ker \Delta_Y = V^+ \oplus V^-$$

be the decomposition of $\ker \Delta_Y$ in the ± 1- eigenspaces of $S(0)$. Then V^+ equals the space of limiting values of extended solutions of Δ [Mu2]. Let $\phi_1, \dots, \phi_l$ be an orthonormal basis of V^+ and let $\varphi_1, ..., \varphi_m$ be an orthonormal basis of $\ker \Delta$. Define the kernel R_1 by

$$R_1(y_1,y_2,\lambda)=\frac{1}{\lambda}\sum_{j=1}^{m}\varphi_j(y_1)\otimes\varphi_j(y_2)+\frac{1}{4\sqrt{\lambda}}\sum_{j=1}^{l}E(\phi_j,0,y_1)\otimes E(\phi_j,0,y_2). \tag{3.13}$$

Let $R_1(\lambda)\colon L^2(Y, E|Y) \to L^2(Y, E|Y)$ be the operator defined by this kernel. Together with Lemma 3.4 we obtain

Proposition 3.7. *There exists a bounded operator $R_2(\lambda) : L^2(Y, E|Y) \to L^2(Y, E|Y)$ such that*

$$R(\lambda)^{-1} = R_1(\lambda) + R_2(\lambda), \quad \lambda > 0,$$

and $\| R_2(\lambda) \|$ is uniformly bounded as $\lambda \to 0$.

Let $\mathcal{H} \subset C^\infty(X, E)$ be the subspace spanned by $\ker \Delta \cap L^2$, and $E(\phi_1, 0), \dots, E(\phi_l, 0)$. Then $\mathcal{H}$ is the subspace of all bounded sections $\phi \in C^\infty(X, E)$ such that $\Delta\phi = 0$. Set

$$\mathcal{H}_Y = \rho_Y(\mathcal{H}).$$

Lemma 3.8. *The restriction map $\rho_Y : \mathcal{H} \to \mathcal{H}_Y$ is an isomorphism.*

Proof. Let $\phi \in \mathcal{H}$. Then $\Delta\phi = 0$ and ϕ is bounded. Suppose that $\rho_Y(\phi) = 0$. This means that $\phi|_Y = 0$. By the uniqueness of the Dirichlet problem, it follows that $\phi = 0$. Thus ρ_Y is injective and hence an isomorphism. □

Lemma 3.9. $\ker R = \mathcal{H}_Y$.

Proof. Let $\varphi \in \mathcal{H}_Y$. Then there exists $\psi \in \mathcal{H}$ with $\psi|_Y = \varphi$. Moreover ψ is bounded and $\Delta\psi = 0$. Thus ψ is a solution of the Dirichlet problem (3.4). Since ψ is smooth on X, it follows that $R\varphi = 0$. Now suppose that $\varphi \in \ker R$. Then there exists a bounded solution ψ of (3.4) such that

$$\frac{\partial}{\partial u}(\psi|_M)\big|_{\partial M} = \frac{\partial}{\partial u}(\psi|_Z)\big|_{\partial Z}.$$

This implies that $\Delta\psi = 0$ in the sense of distributions. By elliptic regularity it follows that $\psi \in C^\infty(X, E)$ and $\Delta\psi = 0$. If we expand $\psi|_Z$ in the orthonormal basis $\{\phi_j\}_{j\in\mathbb{N}}$ we get

$$\psi(u, y) = \sum_{j=1}^{m} a_j\phi_j(y) + \sum_{j=m+1}^{\infty} a_j e^{-u\sqrt{\lambda_j}}\phi_j(y),$$

where $m = \dim\ker\Delta_Y$. Let $\phi = \sum_{j=1}^m a_j\phi_j$. Then we get

$$\psi|_Z = \phi + \psi_1,$$

where $\psi_1 \in L^2$. Put $\tilde{\psi} = \psi - \frac{1}{2}E(\phi, 0)$. Then it follows that $\tilde{\psi} \in \ker\Delta$. This implies that $\psi \in \mathcal{H}$. □

Let $\langle\cdot,\cdot\rangle_Y$ be the inner product in $\mathcal{H}_Y$ induced by the inner product in $L^2(Y, E|Y)$. Let $\varphi_1, \ldots, \varphi_k$ be an orthonormal basis of $\ker\Delta$. Set $\psi_i = \rho_Y(\varphi_i)$, if $1 \le i \le k$, and $\psi_{k+j} = \frac{1}{2}\rho_Y(E(\phi_j), 0))$, if $1 \le j \le l$. Set $a_{ij} = \langle\psi_i, \psi_j\rangle_Y$, $1 \le i, j \le k + l$ and let A be the $(k + l) \times (k + l)$-matrix with entries a_{ij}, $i, j = 1, ..., k + l$. Then the main result of this section is the following theorem.

Theorem 3.10. *Let $k = \dim\ker\Delta$ and $l = \dim V^+$. Then*

$$\log\det R(\lambda) = (k + l/2)\log\lambda - \log\det A + \log\det R + O(\lambda)$$

as $\lambda \to 0+$.

Proof. The proof is analogous to the proof of Theorem B of [Le1]. Let

$$0 \le \mu_1(\lambda) \le \cdots \le \mu_{k+l}(\lambda) < \mu_{k+l+1}(\lambda) \le \cdots$$

be the eigenvalues of $R(\lambda)$. By Lemma 3.9 it follows that

$$\lim_{\lambda\to 0}\mu_j(\lambda) = 0 \quad \text{for } 1 \le j \le k + l,$$

and $\mu_j(\lambda) \ge c > 0$ for $j > k + l$. Then

$$\log\det R(\lambda) = \log(\mu_1(\lambda)\cdots\mu_{k+l}(\lambda)) + \log\det R + O(\lambda) \tag{3.14}$$

as $\lambda \to 0$. So it remains to determine the behaviour of $\log(\mu_1(\lambda)\cdots\mu_{k+l}(\lambda))$ as $\lambda \to 0$. Let $\eta_1(\lambda), ..., \eta_{k+l}(\lambda)$ be an orthonormal set of eigensections of $R(\lambda)$ corresponding to the eigenvalues $\mu_1(\lambda), ..., \mu_{k+l}(\lambda)$. Let $1 \le j \le k + l$. By Proposition 3.7 we get

$$\mu_i(\lambda)^{-1}\delta_{ij} = \langle R(\lambda)^{-1}\eta_i(\lambda), \eta_j(\lambda)\rangle = \langle R_1(\lambda)\eta_i(\lambda), \eta_j(\lambda)\rangle + \langle R_2(\lambda)\eta_i(\lambda), \eta_j(\lambda)\rangle,$$

and the second term on the right remains bounded as $\lambda \to 0+$. By (3.13) the first term equals

$$\begin{aligned}\langle R_1(\lambda)\eta_i(\lambda),\eta_j(\lambda)\rangle &= \frac{1}{\lambda}\sum_{p=1}^{k}\langle \varphi_p,\eta_i(\lambda)\rangle_Y\langle\varphi_p,\eta_j(\lambda)\rangle_Y \\ &+ \frac{1}{4\sqrt{\lambda}}\sum_{q=1}^{l}\langle E(\phi_q,0),\eta_i(\lambda)\rangle_Y\langle E(\phi_q,0),\eta_j(\lambda)\rangle_Y.\end{aligned} \tag{3.15}$$

Set

$$\widetilde{\psi}_i(\lambda) = \begin{cases}\rho_Y(\varphi_i), & \text{if } 1\le i\le k,\\ \frac{\lambda^{1/4}}{2}\rho_Y(E(\phi_{i-k},0)), & \text{if } k+1\le i\le k+l.\end{cases}$$

Let $\widetilde{a}_{ij}(\lambda) = \langle\widetilde{\psi}_i(\lambda),\eta_j(\lambda)\rangle$ and let $\widetilde{A}(\lambda)$ be the matrix with entries $\widetilde{a}_{ij}(\lambda)$, $1\le i,j\le k+l$. Then (3.15) can be written as

$$\langle R_1(\lambda)\eta_i(\lambda),\eta_j(\lambda)\rangle = \frac{1}{\lambda}(\widetilde{A}(\lambda)^t\widetilde{A}(\lambda))_{ij}$$

and we get

$$\mu_i(\lambda)^{-1}\delta_{ij} = \frac{1}{\lambda}(\widetilde{A}(\lambda)^t\widetilde{A}(\lambda))_{ij} + O(1)$$

as $\lambda\to 0+$. Hence for $i\neq j$ we get $(\widetilde{A}(\lambda)^t\widetilde{A}(\lambda))_{ij} = O(\lambda)$ as $\lambda\to 0+$. This implies

$$(\mu_1(\lambda)\cdots\mu_{k+l}(\lambda))^{-1} = \lambda^{-(k+l)}\det(\widetilde{A}(\lambda)^t\widetilde{A}(\lambda))(1+O(\lambda)) \tag{3.16}$$

as $\lambda\to 0+$. Now observe that $\widetilde{A}(\lambda)\widetilde{A}(\lambda)^t$ is equal to the matrix with entries $\langle\widetilde{\psi}_i(\lambda),\widetilde{\psi}_j(\lambda)\rangle$, $1\le i,j\le k+l$. Let

$$C(\lambda) = \begin{pmatrix}\mathrm{Id}_k & 0\\ 0 & \lambda^{1/4}\,\mathrm{Id}_l\end{pmatrix}.$$

Then it follows from the definition of A that

$$\widetilde{A}(\lambda)\widetilde{A}(\lambda)^t = C(\lambda)\cdot A\cdot C(\lambda).$$

Together with (3.16) we obtain

$$(\mu_1(\lambda)\cdots\mu_{k+l}(\lambda))^{-1} = \lambda^{-(k+l/2)}\det(A)(1+O(\lambda)).$$

Taking the logarithm and inserting the result in (3.14), the theorem follows.

□

4. Proof of Theorem 1.1

Let $\lambda > 0$. By Theorem 4.2 of [Ca] there is a polynomial $P(\lambda)$ with real coefficients of degree $\leq (n-1)/2$ such that

$$\frac{\det(\Delta+\lambda, \Delta_0+\lambda)}{\det(\Delta_{M,D}+\lambda)} = e^{P(\lambda)} \det(R(\lambda)). \tag{4.1}$$

All terms have asymptotic expansions as $\lambda \to 0$. Since $\Delta_{M,D}$ is invertible, $\det(\Delta_{M,D}+\lambda)$ is continuous at $\lambda = 0$ and $\lim_{\lambda\to 0}\det(\Delta_{M,D}+\lambda) = \det(\Delta_{M,D})$. Next consider the polynomial $P(\lambda)$. In the proof of Proposition 4.7 of [Ca], Carron has shown that the polynomial $P(\lambda)$ can be computed in terms of the coefficients of the asymptotic expansion of $\operatorname{Tr}(e^{-t\Delta_Y})$ as $t \to 0$. Let

$$\operatorname{Tr}(e^{-t\Delta_Y}) \sim \sum_{j=0}^{\infty} a_j t^{-(n-1)/2+j}, \quad t \to 0+,$$

be the heat expansion. If n is even, we have $P = 0$, and if $n = 2p+1$ then

$$P(\lambda) = -\log(2) \sum_{j=0}^{p} \frac{(-1)^{p-j}}{(p-j)!} a_j \lambda^{p-j}.$$

In particular, it follows that

$$P(0) = -\log(2)(h_Y + \zeta_Y(0)),$$

where $h_Y = \dim\ker \Delta_Y$ and $\zeta_Y(s)$ is the zeta function of Δ_Y. Together with Corollary 2.2 and Theorem 3.10, Theorem 1.1 follows.

5. Regularized determinants on a finite cylinder

In this section we study the regularized determinant of a Laplace type operator on a finite cylinder over a closed Riemannian manifold Y. Let $\Delta_Y : C^\infty(Y, E_0) \to C^\infty(Y, E_0)$ be a Laplace type operator on Y. For $r > 0$ set

$$Z_r = [0, r] \times Y.$$

Let $E \to Z_r$ be the pull back bundle of E_0, i.e., $E = [0, r] \times E_0$. Let

$$\Delta = \Delta_{Z_r} := -\frac{\partial^2}{\partial u^2} + \Delta_Y : C^\infty(Z_r, E) \to C^\infty(Z_r, E).$$

Then Δ is a Laplace type operator on Z_r. Impose Dirichlet boundary conditions at ∂Z_r and let Δ_D be the corresponding self-adjoint extension. Then Δ_D is positive definite. Let

$$0 \leq \mu_1 \leq \mu_2 \leq \cdots \to +\infty$$

be the eigenvalues of Δ_Y, counted with multiplicity. Let $\zeta_Y(s)$ be the zeta function of Δ_Y and set

$$\xi_Y(s) = \frac{\Gamma(s-1/2)}{\sqrt{\pi}\,\Gamma(s)} \zeta_Y(s-1/2). \tag{5.1}$$

Sine $\zeta_Y(s)$ has at most a simple pole at $s = -1/2$, $\xi_Y(s)$ is holomorphic at $s = 0$. The main result of this section is the following proposition.

Proposition 5.1. *Let* $h_Y = \dim\ker\Delta_Y$. *Then*

$$\det(\Delta_D) = (2r)^{h_Y} e^{-r\xi_Y'(0)/2}(\det\Delta_Y)^{-1/2}\cdot\prod_{\mu_j>0}(1-e^{-2r\sqrt{\mu_j}}). \tag{5.2}$$

Proof. The eigenvalues of Δ_D are given by

$$\lambda_{k,l} = \mu_l + \left(\frac{\pi}{r}\right)^2 k^2, \quad k,l\in\mathbb{N}.$$

Hence the zeta function of Δ_D equals

$$\zeta_{\Delta_D}(s) = \sum_{k,l\in\mathbb{N}}\left(\mu_l + \left(\frac{\pi}{r}\right)^2 k^2\right)^{-s}, \quad \operatorname{Re}(s) > \frac{d+1}{2},$$

where $d = \dim Y$. Let $\zeta(s)$ denote the Riemann zeta function. Then

$$\zeta_{\Delta_D}(s) = h_Y\left(\frac{\pi}{r}\right)^{-2s}\zeta(2s) + \sum_{k\in\mathbb{N}}\sum_{\mu_l>0}\left(\mu_l + \left(\frac{\pi}{r}\right)^2 k^2\right)^{-s}, \quad \operatorname{Re}(s) > \frac{d+1}{2}. \tag{5.3}$$

Recall that $\zeta(0) = -1/2$ and $\zeta'(0) = -1/2\log(2\pi)$. Hence we get

$$\frac{d}{ds}\left\{\left(\frac{\pi}{r}\right)^{-2s}\zeta(2s)\right\}\bigg|_{s=0} = -\log 2 - \log r. \tag{5.4}$$

Set

$$\zeta_1(s) := \sum_{k\in\mathbb{N}}\sum_{\mu_l>0}\left(\mu_l + \left(\frac{\pi}{r}\right)^2 k^2\right)^{-s}, \quad \operatorname{Re}(s) > \frac{d+1}{2}.$$

By the Poisson summation formula we get

$$\begin{aligned}
\Gamma(s)\zeta_1(s) &= \sum_{\mu_l>0}\int_0^\infty e^{-\mu_l t}\sum_{k\in\mathbb{N}} e^{-(\pi/r)^2k^2t}t^{s-1}dt\\
&= \sum_{\mu_l>0}\int_0^\infty e^{-\mu_l t}\left(\frac{r}{\sqrt{\pi t}}\sum_{k\in\mathbb{N}}e^{-r^2k^2/t} + \frac{1}{2}\left(\frac{r}{\sqrt{\pi t}} - 1\right)\right)t^{s-1}dt\\
&= \frac{r}{\sqrt{\pi}}\sum_{k\in\mathbb{N}}\sum_{\mu_l>0}\int_0^\infty e^{-(\mu_l t + r^2k^2/t)}t^{s-3/2}dt\\
&\qquad + \frac{1}{2}\frac{r}{\sqrt{\pi}}\Gamma(s-1/2)\zeta_Y(s-1/2) - \frac{1}{2}\Gamma(s)\zeta_Y(s).
\end{aligned} \tag{5.5}$$

Denote by $T(s)$ the integral-series on the right hand side. For $a,b,c\neq 0$ and $s\in\mathbb{C}$ set $K_s(a,b) = \int_0^\infty e^{-(a^2t+b^2/t)}t^{s-1}dt$ and $K_s(c) = \int_0^\infty e^{-c(t+1/t)}t^{s-1}dt$.

It is proved in [La, p.270f] that the following relations hold

$$K_s(c) = K_{-s}(c),\ K_s(a,b) = \left(\frac{b}{a}\right)^s K_s(ab),\ K_{1/2}(c) = \sqrt{\frac{\pi}{c}}e^{-2c}. \tag{5.6}$$

Furthermore, for every $x_0 > 0$ and $\sigma_0 < \sigma_1$ there exists $C = C(x_0, \sigma_0, \sigma_1)$ such that

$$|K_s(x)| \le Ce^{-2x} \tag{5.7}$$

for all $x \ge x_0$ and $\mathrm{Re}(s) \in [\sigma_0, \sigma_1]$ [La]. With this notation we have

$$T(s) = \frac{r}{\sqrt{\pi}} \sum_{k\in\mathbb{N}} \sum_{\mu_l>0} K_{s-1/2}(\sqrt{\mu_l}, rk).$$

Using (5.6) and (5.7) it follows that $T(s)$ is an entire function of s. Especially it is holomorphic at $s = 0$. Since by (5.6) we have $K_{-1/2}(a,b) = \frac{\sqrt{\pi}}{b}e^{-2ab}$, we get

$$T(0) = \frac{r}{\sqrt{\pi}} \sum_{k\in\mathbb{N}} \sum_{\mu_l>0} \frac{\sqrt{\pi}}{rk} e^{-2r\sqrt{\mu_l}k} = -\sum_{\mu_l>0} \log(1 - e^{-2r\sqrt{\mu_l}}). \tag{5.8}$$

Thus by (5.5) we have

$$\zeta_1(s) = \frac{1}{\Gamma(s)}T(s) + \frac{1}{2}\frac{r}{\sqrt{\pi}}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta_Y(s-1/2) - \frac{1}{2}\zeta_Y(s).$$

Using that $\xi_Y(s)$ is holomorphic at $s = 0$, we obtain

$$\zeta_1'(0) = T(0) + r\xi_Y'(0) - \frac{1}{2}\zeta_Y'(0).$$

Together with (5.3), (5.4) and (5.8) we get

$$\zeta_{\Delta_D}'(0) = -\sum_{\mu_k>0} \log\left(1 - e^{-2r\sqrt{\mu_k}}\right) + r\xi_Y'(0) - \frac{1}{2}\zeta_Y'(0) - h_Y(\log 2 + \log r).$$

This implies the claimed equality. □

6. The decomposition formula

Let (M, g) be a closed connected n-dimensional Riemannian manifold and let $Y \subset M$ be a separating hypersurface as in the introduction such that

$$M = M_1 \cup_Y M_2, \quad M_1 \cap M_2 = Y.$$

We assume that the metric g is a product on a tubular neighborhood N of Y. For $r \ge 0$ let

$$M_{1,r} = M_1 \cup ([-r, 0] \times Y), \quad M_{2,r} = M_2 \cup ([0, r] \times Y),$$

where we identify Y with $\{-r\} \times Y$ in the first case and with $\{r\} \times Y$ in the second case. Set

$$M_r = M_{1,r} \cup_{\{0\}\times Y} M_{2,r}, \quad N_r = [-r, r] \times Y.$$

Then

$$M_r = M_1 \cup_Y N_r \cup_Y M_2, \tag{6.1}$$

where ∂M_1 is identified with $\{-r\} \times Y$ and ∂M_2 with $\{r\} \times Y$. The metric g on M has an obvious extension to a metric on M_r. Furthermore, let

$$M_{i,\infty} = M_i \cup_Y (\mathbb{R}^+ \times Y), \quad i = 1, 2.$$

Let $\Delta_M : C^\infty(M, E) \to C^\infty(M, E)$ be a Laplace type operator as in the introduction. and let Δ_{M_r} be its canonical extension to a Laplace type operator on M_r, i.e. Δ_{M_r} is uniquely defined by

$$\Delta_{M_r}\big|_{M_i} = \Delta|_{M_i}, \quad \Delta_{M_r}\big|_{N_r} = -\frac{\partial^2}{\partial u^2} + \Delta_Y.$$

Let $\Delta_{M_i} = \Delta|_{M_i}$ and let $\Delta_{M_i,D}$ be the selfadjoint extension of $\Delta_{M_i} : C_c^\infty(M_i, E) \to L^2(M_i, E)$ with respect to Dirichlet boundary conditions. We assume that $\Delta_{M_1,D}$ and $\Delta_{M_2,D}$ are invertible. Let $\Delta_{N_r,D}$ denote the selfadjoint extension of

$$-\frac{\partial^2}{\partial u^2} + \Delta_Y : C_c^\infty(N_r, E) \to L^2(N_r, E)$$

with respect to Dirichlet boundary conditions. Let $Y_{\pm r} := \{\pm r\} \times Y$ and denote by $\Sigma_r \subset M_r$ the hypersurface

$$\Sigma_r := Y_{-r} \sqcup Y_r.$$

Given $z \in \mathbb{C} - \mathbb{R}^-$, let $R_r(z)$ be the Dirichlet-to-Neumann operator associated to $(\Delta_{M_r} + z)$ and the hypersurface Σ_r. We recall the definition of $R_r(z)$. Let $\phi \in C^\infty(\Sigma_r, E_r|\Sigma_r)$. There exists a unique section $\varphi \in C^\infty(M_r - \Sigma_r, E_r) \cap C^0(M_r, E_r)$ such that

$$\begin{aligned} (\Delta_{M_r} + z)\varphi &= 0 \quad \text{on } M_r - \Sigma_r; \\ \varphi &= \phi \quad \text{on } \Sigma_r. \end{aligned} \tag{6.2}$$

Then $R_r(z)(\phi)$ is given by

$$\begin{aligned} R_r(z)(\phi)\big|_{Y_{-r}} &= \frac{\partial}{\partial u}\big(\varphi|_{M_1}\big)\big|_{\partial M_1} - \frac{\partial}{\partial u}\big(\varphi|_{N_r}\big)\big|_{Y_{-r}}, \\ R_r(z)(\phi)\big|_{Y_r} &= \frac{\partial}{\partial u}\big(\varphi|_{N_r}\big)\big|_{Y_r} - \frac{\partial}{\partial u}\big(\varphi|_{M_2}\big)\big|_{\partial M_2}. \end{aligned} \tag{6.3}$$

Now we apply the Mayer-Vietoris formula of [BFK], specialized to our case. We note that Theorem 1.4 of [Ca] also holds in our case, where $M_r - \Sigma_r$ consists of three components. Thus there exists a polynomial $P(z)$ with real coefficients of degree $< (n-1)/2$ such that for every $z \in \mathbb{C} - \mathbb{R}^-$:

$$\frac{\det(\Delta_{M_r} + z)}{\det(\Delta_{N_r,D} + z)\det(\Delta_{M_1,D} + z)\det(\Delta_{M_2,D} + z)} = e^{P(z)} \det R_r(z).$$

Since we assume that the metric of M_r is a product on a tubular neighborhood of Σ_r, the polynomial depends only on Y and can be computed as follows. Let $\zeta_Y(s, z)$ be the zeta function of $\Delta_Y + z$. Then it follows from [PW1, Theorem 6.3] and also from the proof of Proposition 4.7 of [Ca] that

$$P(z) = -2\zeta_Y(0, z).$$

Thus

$$\frac{\det(\Delta_{M_r} + z)}{\det(\Delta_{N_r,D} + z)\det(\Delta_{M_1,D} + z)\det(\Delta_{M_2,D} + z)} = 2^{-2\zeta_Y(0,-z)} \det R_r(z). \tag{6.4}$$

Now take $z = \lambda > 0$ and consider the limit as $\lambda \to 0$ of the left and right hand side of (6.4). Since $\Delta_{M_i,D}$, $i = 1, 2$, and $\Delta_{N_r,D}$ are invertible, it follows that

$$\lim_{z\to 0} \det(\Delta_{M_i,D} + \lambda) = \det \Delta_{M_i,D}, \quad \lim_{z\to 0} \det(\Delta_{N_r,D} + \lambda) = \det \Delta_{N_r,D}. \tag{6.5}$$

Let $h_r = \dim \ker \Delta_{M_r}$. Then

$$\det(\Delta_{M_r} + \lambda) = \lambda^{h_r} \det(\Delta_{M_r}|_{(\ker \Delta_{M_r})^\perp} + \lambda)$$

and therefore we get

$$\lim_{\lambda\to 0} \det(\Delta_{M_r} + \lambda)\lambda^{-h_r} = \det \Delta_{M_r}. \tag{6.6}$$

Also note that

$$\lim_{\lambda\to 0} \zeta_Y(0, \lambda) = \zeta_Y(0) + h_Y, \tag{6.7}$$

where $h_Y = \dim \ker \Delta_Y$. It remains to consider the limit of $\det R_r(\lambda)$ as $\lambda \to 0$. Let

$$\rho_r \colon C^\infty(M_r, E_r) \to C^\infty(\Sigma_r, E|\Sigma_r)$$

denote the restriction operator. Let $\mathcal{H}_r := \rho_r(\ker \Delta_{M_r})$.

Lemma 6.1.

$$\rho_r : \ker \Delta_{M_r} \to \mathcal{H}_r$$

is an isomorphism.

Proof. Let $\phi \in \ker \Delta_{M_r}$ and suppose that $\phi|_{\Sigma_r} = 0$. Let $\psi = \phi|_{N_r}$. Then $\Delta_{N_r}\psi = 0$ and $\psi|_{\partial N_r} = 0$. Since $\Delta_{N_r,D}$ is invertible, it follows that $\psi = 0$. In the same way we get $\phi|_{M_i} = 0$, $i = 1, 2$, and hence $\phi = 0$. Thus ρ_r is injective and therefore an isomorphism. □

Let $\Delta_{M_r,D}$ be the selfadjoint extension of

$$\Delta_{M_r} : C_c^\infty(M_r - \Sigma_r, E_r) \to L^2(M_r, E_r)$$

with respect to Dirichlet boundary conditions. By our assumption, $\Delta_{M_r,D}$ is invertible and hence, the Dirichlet-to-Neumann operator R_r associated to Δ_{M_r} with respect to $\Sigma_r \subset M_r$ can be defined in the same way as $R_r(z)$.

Lemma 6.2. *We have*

$$\ker R_r = \rho_r(\ker \Delta_{M_r}).$$

Proof. Let $\varphi \in \ker \Delta_{M_r}$ and let $\phi = \rho_r(\varphi)$. Then φ is a solution of the Dirichlet problem with boundary value ϕ. Since φ is smooth on M_r, it follows that $R_r(\phi) = 0$. Now suppose that $\phi \in \ker R_r$. Then there exists $\varphi \in C^\infty(M_r - \Sigma_r) \cap C^0(M_r)$ such that $\Delta_{M_r}\varphi = 0$ on $M_r - \Sigma_r$, $\varphi|_{\Sigma_r} = \phi$ and

$$\frac{\partial}{\partial u}(\varphi|_{M_1})\big|_{\partial M_1} = \frac{\partial}{\partial u}(\varphi|_{N_r})\big|_{Y_{-r}}, \quad \frac{\partial}{\partial u}(\varphi|_{N_r})\big|_{Y_r} = \frac{\partial}{\partial u}(\varphi|_{M_2})\big|_{\partial M_2}.$$

This implies that $\Delta_{M_r}\varphi = 0$ in the distributional sense. By elliptic regularity we conclude that $\varphi \in \ker \Delta_{M_r}$ and $\rho_r(\varphi) = \phi$. □

Let $\varphi_1, ..., \varphi_p$ be an orthonormal basis of $\ker \Delta_{M_r}$. Set

$$b_{ij} = \langle \rho_r(\varphi_i),\ \rho_r(\varphi_j)\rangle_{\Sigma_r}, \quad i, j = 1, ..., p$$

and let

$$B_r = (b_{ij})_{i,j=1}^p.$$

Then B_r is a symmetric invertible matrix.

Proposition 6.3. *Let $h_r = \dim \ker \Delta_{M_r}$. Then*

$$\log \det R_r(\lambda) = h_r \log \lambda - \log \det B_r + \log \det R_r + O(\lambda)$$

as $\lambda \to \infty$.

Proof. We use Lemma 6.2 and proceed in the same way as in the proof of Theorem (3.10). □

Combining (6.4)-(6.7) and Proposition (6.3) we obtain

$$\begin{aligned} \log \det(\Delta_{M_r}) =& \log \det \Delta_{N_r,D} + \log \det \Delta_{M_1,D} + \log \det \Delta_{M_2,D} \\ & - \log \det B_r + \log \det R_r - 2(\zeta_Y(0) + h_Y) \log 2. \end{aligned} \tag{6.8}$$

Let $Z = \mathbb{R}^+ \times Y$ and let Δ_0 be the selfadjoint extension of the symmetric operator

$$-\frac{\partial^2}{\partial u^2} + \Delta_Y : C_c^\infty(Z, E) \to L^2(Z, E)$$

with respect to Dirichlet boundary conditions at $\partial Z = \{0\} \times Y$. Let $R_{i,\infty}$ be the Dirichlet-to-Neumann operator for $\Delta_{i,\infty}$ with respect to the hypersurface $Y = \{0\} \times Y \subset M_{i,\infty}$. Let A_i be the Grahm matrix defined by the restrictions of the extended L^2-solutions of $\Delta_{i,\infty}$ to Y as in Theorem 3.10. By Theorem (1.1) we have

$$\begin{aligned} \log \det(\Delta_{i,\infty}, \Delta_0) = & \log \det R_{i,\infty} + \log \det \Delta_{M_i,D} - \log \det A_i \\ & - (\zeta_Y(0) + h_Y) \log 2. \end{aligned}$$

Together with (6.8) we get

Proposition 6.4. *Let the notation be as above. Then*

$$\begin{aligned}\log\det\Delta_{M_r} =& \log\det\Delta_{N_r,D} - \log\det B_r + \log\det R_r \\ &+ \sum_{i=1}^{2}(\log\det(\Delta_{i,\infty},\Delta_0) - \log\det R_{i,\infty} + \log\det A_i).\end{aligned} \tag{6.9}$$

Our next purpose is to study the behaviour of the various terms in this equality as $r \to \infty$. This, of course, will require additional assumptions. We begin with the consideration of $\det R_r$.

To this end we need to describe the operator R_r more explicitly. Let Q_i denote the Neumann jump operator on M_i. In the proof of Lemma (3.2) we established the following equality

$$R_{i,\infty} = Q_i + \sqrt{\Delta_Y}, \quad i = 1, 2. \tag{6.10}$$

Let $P_0 : L^2(Y, E|Y)$ denote the orthogonal projection onto $\ker\Delta_Y$. Let

$$h_r(x) = \frac{\sqrt{x}}{\sinh(2\sqrt{x}r)}.$$

Define

$$K_r : L^2(Y, E|Y) \oplus L^2(Y, E|Y) \to L^2(Y, E|Y) \oplus L^2(Y, E|Y)$$

by

$$K_r := \left(\frac{1}{2r}P_0 + h_r(\Delta_Y)P_0^{\perp}\right)\begin{pmatrix} e^{-2\sqrt{\Delta_Y}r} & -\mathrm{Id} \\ -\mathrm{Id} & e^{-2\sqrt{\Delta_Y}r}\end{pmatrix}. \tag{6.11}$$

Set

$$R_\infty = \begin{pmatrix} R_{1,\infty} & 0 \\ 0 & R_{2,\infty}\end{pmatrix}. \tag{6.12}$$

Recall that $\Sigma_r \cong Y \sqcup Y$. Using (5.8) and the formula at the bottom of p. 4104 of [Le3], it follows that

$$R_r : C^\infty(Y, E|Y) \oplus C^\infty(Y, E|Y) \to C^\infty(Y, E|Y) \oplus C^\infty(Y, E|Y)$$

is given by

$$R_r = R_\infty + K_r. \tag{6.13}$$

Next observe that K_r is a trace class operator and its trace norm $\| K_r \|_1$ satisfies

$$\| K_r \|_1 \underset{r\to\infty}{\longrightarrow} 0. \tag{6.14}$$

By Corollary 3.4, $R_{i,\infty}$, $i = 1, 2$, are selfadjoint nonnegative operators in $L^2(Y, E|Y)$.

Lemma 6.5. *Suppose that $R_{i,\infty} > 0$, $i = 1, 2$. Then*

$$\lim_{r\to\infty} \det R_r = \det R_{1,\infty} \cdot \det R_{2,\infty}.$$

Proof. This is proved in [Le3, Lemma 4.1]. For the convenience of the reader we recall the proof. It follows from (6.12) and the assumptions that $R_\infty > 0$. By (6.14) it follows that there exists $r_0 > 0$ such that the operator $R_\infty + tK_r$ is invertible for $0 \le t \le 1$ and $r \ge r_0$. Thus

$$\begin{aligned}\log\det(R_\infty + K_r) - \log\det R_\infty &= \int_0^1 \frac{d}{dt}\log\det(R_\infty + tK_r)dt\\ &= \int_0^1 \operatorname{Tr}((R_\infty + tK_r)^{-1}K_r)dt \le \frac{1}{2\lambda_0}\parallel K_r \parallel_1,\end{aligned}$$

where $\lambda_0 > 0$ is the smallest eigenvalue of R_∞. The lemma follows from (6.14). □

Let $\mathcal{H}_i$, $i = 1, 2$, be the space of extended L^2-solutions of $\Delta_{i,\infty}$. By Lemma (3.8) and Lemma (3.9) it follows that $R_{i,\infty}$ is invertible if and only if $\mathcal{H}_i = \{0\}$, and the latter condition is a consequence of $\ker\Delta_Y = \{0\}$ and $\ker\Delta_{i,\infty} = \{0\}$. Furthermore, if R_∞ is invertible, it follows from (6.13) and (6.14) that R_r is invertible for $r \ge r_0$. By Lemma 6.1 and Lemma 6.2, R_r is invertible if and only if $\ker\Delta_{M_r} = \{0\}$.

Using these observation together with Proposition (6.4) and Lemma (6.5), we obtain

Corollary 6.6. *Suppose that* $\ker\Delta_Y = \{0\}$ *and* $\ker\Delta_{i,\infty} = 0$, $i = 1, 2$. *Then* Δ_{M_r} *is invertible for* $r \ge r_0$ *and*

$$\lim_{r\to\infty}\frac{\det\Delta_{M_r}}{\det\Delta_{N_r,D}} = \det(\Delta_{1,\infty}, \Delta_0)\cdot\det(\Delta_{2,\infty}, \Delta_0).$$

The asymptotic behaviour of $\det\Delta_{N_r,D}$ as $r \to \infty$ is described by Proposition 5.1. Using this result, Theorem 1.2 follows.

Next we consider a compact Riemannian manifold (X_0, g) with a nonempty boundary Y. We assume that the metric is a product on a collar neighborhood $N = (-\epsilon, 0] \times Y$ of Y in X_0. Let

$$\Delta_{X_0}\colon C^\infty(X_0, E) \to C^\infty(X_0, E)$$

be Laplace type operator as above such that on N it equals $-\partial^2/\partial u^2 + \Delta_Y$. For $r > 0$ set

$$Z_r = [0, r] \times Y, \quad \text{and} \quad X_r = X_0 \cup_Y Z_r,$$

where $\{0\} \times Y \subset Z_r$ is identified with $\partial X_0 = Y$. Let $X_\infty = X_0 \cup_Y (\mathbb{R}^+ \times Y)$ be the corresponding manifold with a cylindrical end. We extend Δ_{X_0} in the obvious to Laplace type operators Δ_{X_r} and Δ_∞ on X_r and X_∞, respectively. Let $\Delta_{X_r,D}$ and $\Delta_{Z_r,D}$ denote the Dirichlet Laplacians associated to Δ_{X_r} and Δ_{Z_r}, respectively. Furthermore, let R_r be the Dirichlet-to-Neumann operator associated to the decomposition $X_r = X_0 \cup Z_r$. Let $\lambda > 0$. Then by Theorem 4.2 of [Ca] we have

$$\frac{\det(\Delta_{X_r,D} + \lambda)}{\det(\Delta_{X_0,D} + \lambda)\det(\Delta_{Z_r,D} + \lambda)} = 2^{-\zeta_Y(0,\lambda)}\det R_r(\lambda). \tag{6.15}$$

As above, we let $\lambda \to 0$. Note that $\Delta_{Z_r,D}$ is invertible. Assume that $\Delta_{X_r,D}$ is invertible. Then as $\lambda \to 0$, the determinants converge to the determinants of Δ_{X_r} and Δ_{Z_r}, respectively. Furthermore as in Lemma 6.2 it follows that

$$\ker R_r = \rho_Y(\ker \Delta_{M_r,D}). \tag{6.16}$$

Hence R_r is also invertible and

$$\log\det R_r(\lambda) = \log\det R_r + O(\lambda)$$

as $\lambda \to 0$. Thus taking the limit $\lambda \to 0$ of both sides of (6.15), we get

$$\frac{\det \Delta_{X_r,D}}{\det \Delta_{X_0,D} \det \Delta_{Z_r,D}} = 2^{-\zeta_Y(0)} \det R_r. \tag{6.17}$$

Now recall that by Theorem 1.1 we have

$$\log\det(\Delta_\infty, \Delta_0) = \log\det R_\infty + \log\det \Delta_{X_r,D} - \log\det A - \log(2)\ (\zeta_Y(0) + h_Y).$$

Combining this equality with (6.17) we obtain

$$\begin{aligned} \log\det \Delta_{X_r,D} =& \log\det \Delta_{N_r,D} + \log\det R_r - \log\det R_\infty \\ &+ \det(\Delta_\infty, \Delta_0) + \log\det A. \end{aligned} \tag{6.18}$$

To study the behaviour of $\det R_r$ as $r \to \infty$, we proceed as above. Let

$$f_r(x) = \frac{\sqrt{x}}{\sinh(r\sqrt{x})}, \quad x \in \mathbb{R}.$$

Define the operator

$$L_r\colon L^2(Y, E|Y) \to L^2(Y, E|Y)$$

by

$$L_r := \left(\frac{1}{r} P_0 + f_r(\Delta_Y) P_0^\perp\right) e^{-r\sqrt{\Delta_Y}}, \tag{6.19}$$

where P_0 denotes the orthogonal projection onto $\ker \Delta_Y$. Then

$$R_r = R_\infty + L_r.$$

Suppose that $\ker \Delta_Y = \{0\}$ and $\ker \Delta_\infty = \{0\}$. Then it follows from Lemma 3.9 that $\ker R_\infty = \{0\}$ and by Lemma 4.1 of [Le3] we get

$$\lim_{r\to\infty} \det R_r = \det R_\infty.$$

Since $\| L_r \| \to 0$ as $r \to \infty$, it follows that R_r is invertible for $r \geq r_0$. By (6.16) this implies that $\Delta_{M_r,D}$ is invertible for $r \geq r_0$. Under the same assumptions we have $\det A = 1$. Together with (6.18) we get

Proposition 6.7. *Suppose that* $\ker \Delta_Y = \{0\}$ *and* $\ker \Delta_\infty = \{0\}$. *Then*

$$\lim_{r\to\infty} \frac{\det \Delta_{X_r,D}}{\det \Delta_{Z_r,D}} = \det(\Delta_\infty, \Delta_0).$$

Using Proposition 5.1, it follows that as $r \to \infty$,

$$\det \Delta_{X_r,D} \sim e^{-r\xi'_Y(0)/2} \left(\det \Delta_Y\right)^{-1/2} \det(\Delta_\infty, \Delta_0). \tag{6.20}$$

We apply (6.20) to $\det \Delta_{M_{i,r},D}$, $i = 1, 2$, and compare the asymptotic behaviour with (1.11). In this way we get

$$\lim_{r\to\infty} \frac{\det \Delta_{M_r}}{\det \Delta_{M_{1,r},D} \det \Delta_{M_{2,r},D}} = \left(\det \Delta_Y\right)^{1/2}$$

which is the statement of Corollary 1.5.

7. Bochner-Laplace operators

In this section we study the case where Δ is a connection Laplacian. To begin with we consider a manifold with a cylindrical end $X = M \cup_Y Z$, $Z = \mathbb{R}^+ \times Y$. Let $F \to X$ be a Hermitian vector bundle over X such that $F|_{\mathbb{R}^+\times Y} = \mathrm{pr}_Y^*(F_0)$ for some Hermitian vector bundle F_0 over Y. Let ∇ be a metric connection in F such that on $\mathbb{R}^+ \times Y$ it has the form

$$\nabla = d_u \otimes \mathrm{Id} + d \otimes \nabla^Y, \tag{7.1}$$

where ∇^Y is a metric connection on F_0. Let

$$\Delta = \nabla^*\nabla, \quad \Delta_Y = (\nabla^Y)^*\nabla^Y$$

be the associated Bochner-Laplace operators. Then

$$\Delta = -\frac{\partial^2}{\partial u^2} + \Delta_Y \quad \text{on } \mathbb{R}^+ \times Y. \tag{7.2}$$

Let $\overline{\Delta}$ be the unique selfadjoint extension of Δ_X in L^2.

Let $\{\phi_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2(Y, F|Y)$ consisting of eigensections of Δ_Y with eigenvalues

$$0 \le \mu_1 \le \mu_2 \le \cdots \to +\infty.$$

Lemma 7.1. *We have*

$$\ker \overline{\Delta} = \{0\}.$$

Proof. Let $\varphi \in C^\infty(X, F)$ be a square integrable solution of $\Delta\varphi = 0$. Then

$$0 = \langle \nabla^*\nabla\varphi, \varphi\rangle = \parallel \nabla\varphi \parallel^2 .$$

Thus $\nabla\varphi = 0$. Since φ is square integrable, it has the following expansion on $\mathbb{R}^+ \times Y$ in terms of the orthonormal basis $\{\varphi_i\}_{i\in\mathbb{N}}$:

$$\varphi(u, y) = \sum_{\mu_i>0} c_i e^{-\sqrt{\mu_i}u} \phi_i(y), \quad u \in \mathbb{R}^+, y \in Y. \tag{7.3}$$

Furthermore,

$$\frac{\partial\varphi}{\partial u}(u, y) = \nabla_{\frac{\partial}{\partial u}}\varphi = 0.$$

Using (7.3), it follows that the restriction of φ to $\mathbb{R}^+ \times Y$ vanishes. Since $\nabla\varphi = 0$ and ∇ is a metric connection, it follows that $d \parallel \varphi \parallel^2 = 0$ and hence $\varphi = 0$. ∎

Let

$$\ker \Delta_Y = V^+ \oplus V^-$$

be the decomposition into the ± 1-eigenspaces of the scattering matrix $S(0)$ (cf. §2) and let

$$R : C^\infty(Y, F|Y) \to C^\infty(Y, F|Y)$$

be the Dirichlet-to-Neumann operator with respect to the hypersurface

$$Y = \{0\} \times Y \subset X.$$

Lemma 7.2. *We have*

$$\ker R = V^+.$$

Proof. Let $\mathcal{H} \subset C^\infty(X, E)$ be the space of bounded solutions of $\Delta\varphi = 0$. By Lemma (3.9) we have $\ker R = \rho_Y(\mathcal{H})$. So it suffices to prove that

$$\rho_Y(\mathcal{H}) = V^+.$$

Let $\varphi \in \mathcal{H}$. For $r > 0$ let $X_r = M \cup_Y ([0, r] \times Y)$. Using integration by parts, we get

$$\begin{aligned} 0 &= \int_{X_r} \langle \nabla^* \nabla \varphi(x), \varphi(x) \rangle dx \\ &= \int_{X_r} |\nabla \varphi(x)|^2 dx - \int_Y \left\langle \frac{\partial}{\partial u} \varphi(u, y), \varphi(u, y) \right\rangle \Big|_{u=r} dy. \end{aligned} \tag{7.4}$$

Since φ is bounded and satisfies $\Delta\varphi = 0$, it has the following expansion on $\mathbb{R}^+ \times Y$:

$$\varphi(u, y) = \sum_{i=1}^{h_Y} a_i \phi_i(y) + \sum_{i=h_Y+1}^{\infty} b_i e^{-\sqrt{\mu_i} u} \phi_i(y), \tag{7.5}$$

where $h_Y = \dim \ker \Delta_Y$. This implies that the second integral on the right of (7.4) is exponentially decreasing as $r \to \infty$. Hence $\nabla\varphi = 0$. In particular, it follows that

$$\frac{\partial}{\partial u} \varphi(u, y) = 0, \quad u \in \mathbb{R}^+, \ y \in Y.$$

Together with (7.5) we get

$$\varphi|_Z = \phi \in \ker \Delta_Y. \tag{7.6}$$

Thus $\rho_Y(\mathcal{H}) \subset \ker \Delta_Y$. Now recall that $\varphi \in \mathcal{H}$ if and only if there exist $\phi \in V^+$ and $\psi \in L^2(Z, F)$ such that $\varphi|_Z = \phi + \psi$. By (7.6) it follows that $\psi = 0$. This proves that $\rho_Y(\mathcal{H}) = V^+$. □

Let A be the matrix that occurs in Theorem 1.1.

Corollary 7.3. *We have*

$$\det A = 1.$$

Proof. Recall the definition of A. Given $\phi \in V^+$, let $\frac{1}{2}E(\phi,0)$ be the extended solution of Δ_X with limiting value ϕ. Let $\phi_1,\dots,\phi_p$ be an orthonormal basis of V^+. Let $\psi_j = \frac{1}{2}\rho_Y(E(\phi,0))$. Since by Lemma (7.1) $\ker \overline{\Delta} = \{0\}$, it follows that the entries of A are $a_{ij} = \langle \psi_i, \psi_j\rangle_Y$. By Lemma (7.2), we have $\frac{1}{2}\rho_Y(E(\phi,0)) = \phi$ for $\phi \in V^+$. Hence $a_{ij} = \delta_{ij}$. □

Now consider a compact Riemannian manifold (M,g) and a Hermitian vector bundle $E \to M$ as in the previous section. Let ∇ be a metric connection on E such that on the tubular neighborhood $N = [-1,1]\times Y$ of Y in M

$$\nabla = d_u \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^Y, \tag{7.7}$$

where ∇^Y is a metric connection on $E_0 = E|Y$. Let

$$\Delta_M = \nabla^*\nabla.$$

Let $E_r \to M_r$ and $E_{i,\infty} \to M_{i,\infty}$ be the canonical extensions of the vector bundle $E \to M$ to vector bundles over M_r and $M_{i,\infty}$, respectively. By (7.7), ∇ has a canonical extension to a connection ∇^r on E_r and $\nabla^{i,\infty}$ on $E_{i,\infty}$, $i = 1,2$, respectively. Then

$$\Delta_{M_r} = (\nabla^r)^*\nabla^r, \quad \Delta_{i,\infty} = (\nabla^{i,\infty})^*\nabla^{i,\infty}, \quad i = 1,2.$$

Recall that the Dirichlet-to-Neumann operator R_r is a selfadjoint operator in $C^\infty(Y,E|Y)\oplus C^\infty(Y,E|Y)$.

Next we determine $\ker R_r$. Let $V_i^+ \subset \ker \Delta_Y$ be the subspace of limiting values of extended L^2-sections of $\Delta_{i,\infty}$, $i = 1,2$.

Lemma 7.4. *We have*

$$\ker R_r = \{(\phi,\phi) \mid \phi \in V_1^+ \cap V_2^+\}.$$

Proof. By Lemma 6.2 we have

$$\ker R_r = \rho_r(\ker \Delta_{M_r}).$$

Let $\varphi \in \ker \Delta_{M_r}$. Then

$$0 = \langle \nabla^*\nabla\varphi, \varphi\rangle = \parallel \nabla\varphi \parallel^2 .$$

Thus

$$\nabla\varphi = 0 \quad \text{for all } \varphi \in \ker \Delta_{M_r}. \tag{7.8}$$

Next observe that the restriction of φ to N_r satisfies

$$\left(-\frac{\partial^2}{\partial u^2} + \Delta_Y\right)\varphi(u,y) = 0, \quad u \in [-r,r],\ y \in Y,$$

and hence the expansion of $\varphi|_{N_r}$ in the orthonormal basis $\{\phi_i\}_{i\in\mathbb{N}}$ is of the form

$$\varphi(u,y) = \sum_{i=1}^{h_Y}(a_i + b_i u)\phi_i(y) + \sum_{i=h_Y+1}^{\infty}(c_i e^{-\sqrt{\mu_i}u} + d_i e^{\sqrt{\mu_i}u})\phi_i(y).$$

By (7.8) it follows that $\frac{\partial}{\partial u}\varphi(u,y)=0$, $u\in[-r,r]$. Hence we get

$$\varphi(u,y)=\sum_{i=1}^{h_Y} a_i\phi_i(y),\quad (u,y)\in N_r. \tag{7.9}$$

Actually, by our assumptions this holds on a slightly larger collar neighborhood of Y. Denote the right hand side by ϕ. Then $\phi\in\ker\Delta_Y$ and it follows that

$$\rho_r(\varphi)=(\varphi(-r,\cdot),\ \varphi(r,\cdot))=(\phi,\phi).$$

Furthermore, let $\varphi_i=\varphi|M_i$, $i=1,2$. Then φ_i satisfies

$$\Delta_{M_i}\varphi_i=0\quad\text{and}\quad \varphi_i|_{(-\varepsilon,0]\times Y}=\phi.$$

Thus φ_i, $i=1,2$, has a unique extension $\hat{\varphi}_i$ to an extended L^2-solution of $\Delta_{i,\infty}$ with limiting value ϕ. This implies $\phi\in V_1^+\cap V_2^+$. On the other hand, suppose that $\phi\in V_1^+\cap V_2^+$. Let $\hat{\varphi}_i\in C^\infty(M_{i,\infty},E_{i,\infty})$, $i=1,2$, be an extended L^2-solution with limiting value ϕ. By (7.6) we have

$$\hat{\varphi}_i|_{(-\varepsilon,0]\times Y}=\phi.$$

Thus we can patch together $\hat{\varphi}_1|M_1$ and $\hat{\varphi}_2|M_2$ to a section $\varphi\in\ker\Delta_{M_r}$ with $\rho_r(\varphi)=(\phi,\phi)$. □

Lemma 7.5. *For all $r>0$ there exists an isomorphism*

$$j_r:\ker\Delta_{M_r}\to\ker\Delta_M.$$

Proof. Let $\varphi\in\ker\Delta_{M_r}$. By (7.9), there exists $\phi\in\ker\Delta_Y$ such that

$$\varphi(u,y)=\phi(y),\quad (u,y)\in N_r. \tag{7.10}$$

Note that by our assumption, ∇ is the product connection on a slightly larger tubular neighborhood $N_{r+\epsilon}$ of Y and (7.10) continues to hold on $N_{r+\epsilon}$. Set

$$\psi_i=\varphi|_{M_i},\quad i=1,2.$$

By (7.10), it follows that

$$\psi_1\big|_{\partial M_1}=\psi_2\big|_{\partial M_2}. \tag{7.11}$$

Define $\psi\in C^\infty(M-Y,E)\cap C^0(M,E)$ by

$$\psi(x)=\begin{cases}\psi_1(x), & x\in M_1,\\ \psi_2(x), & x\in M_2.\end{cases}$$

By the above observation, there exists a tubular neighborhood $N_\epsilon=(-\epsilon,\epsilon)\times Y$ of Y such that $\psi|_{N_\epsilon}=\phi$. Hence $\psi\in C^\infty(M,E)$ and $\Delta_M\psi=0$. By construction, the map

$$j_r:\varphi\in\ker\Delta_{M_r}\longmapsto\psi\in\ker\Delta_M$$

is injective and the inverse map can be defined in the same way. This proves that j_r is surjective. □

Corollary 7.6. *The dimension of $\ker_{M_r}$ is independent of r.*

Put

$$q := \dim\ker\Delta_{M_r}.$$

Let the matrix B_r be defined as in the previous section. Our next purpose is to study the behaviour of $\det B_r$ as $r\to\infty$. To this end we need some auxiliary result. Let

$$P_i : C^\infty(Y,E|Y)\to C^\infty(M_i,E),\quad i=1,2$$

be the Poisson operator. Recall that for $\phi\in C^\infty(Y,E|Y)$, $P_i(\phi)$ is the unique solution of the Dirichlet problem

$$\Delta_{M_i}\psi=0,\quad \psi|_{\partial M_i}=\phi.$$

Lemma 7.7. *There exists $C>0$ such that*

$$\parallel P_i(\phi)\parallel\le C\parallel\phi\parallel,\quad \phi\in\ker\Delta_Y,\quad i=1,2.$$

Proof. There exists a collar neighborhood $(-\epsilon,0]\times Y$ of Y in M_i such that

$$\Delta_{M_i}=-\frac{\partial^2}{\partial u^2}+\Delta_Y\quad\text{on } (-\epsilon,0]\times Y. \tag{7.12}$$

Let $f\in C^\infty(\mathbb{R})$ be such that $f(u)=1$ for $u\ge-\epsilon/4$ and $f(u)=0$ for $u\le-\epsilon/2$. Given $\phi\in C^\infty(Y,E|Y)$, set

$$\widetilde{\phi}(u,y)=f(u)\phi(y),\quad u\in(-\epsilon,0],\ y\in Y,$$

and extend $\widetilde{\phi}$ by zero to a smooth section of $E\to M_i$. Then

$$P_i(\phi)=\widetilde{\phi}-(\Delta_{M_i,D})^{-1}(\Delta_{M_i}\widetilde{\phi}). \tag{7.13}$$

Let $\phi\in\ker\Delta_Y$. By (7.12) we get

$$\Delta_{M_i}\widetilde{\phi}=-f''\phi.$$

Let $\lambda_1>0$ be the smallest eigenvalue of $\Delta_{M_i,D}$. Then by (7.13) we get

$$\parallel P_i(\phi)\parallel\le C_1\parallel\phi\parallel+\frac{1}{\lambda_1}C_2\parallel\phi\parallel\le C\parallel\phi\parallel,$$

where $C>0$ is independent of $\phi\in\ker\Delta_Y$. □

Lemma 7.8. *Let $q=\dim\ker R_r$. Then*

$$r^q\det B_r=1+O(r^{-1})$$

as $r\to\infty$.

Proof. Let $\psi_{r,1},\dots,\psi_{r,q}\in\ker\Delta_{M_r}$ be an orthonormal basis of $\ker\Delta_{M_r}$. Then B_r is defined as

$$B_r=\left(\langle\rho_r(\psi_{r,i}),\rho_r(\psi_{r,k})\rangle\right)_{i,k=1}^q.$$

By (7.9), for each $r>0$ and $k, k=1,\dots,q$, there exists $\phi_{r,k}\in\ker\Delta_Y$ such that

$$\psi_{r,k}(u,y)=\phi_{r,k}(y),\quad u\in[-r,r],\ y\in Y. \tag{7.14}$$

Let $M_0 = M_1 \sqcup M_2$. Then

$$\delta_{ik} = \langle \psi_{r,i}, \psi_{r,k} \rangle_{M_r} = \left\langle \psi_{r,i}\big|_{M_0}, \psi_{r,k}\big|_{M_0} \right\rangle_{M_0} + 2r\langle \phi_{r,i}, \phi_{r,k} \rangle. \tag{7.15}$$

By (7.14) we have

$$\rho_r(\psi_{r,k}) = (\phi_{k,r}, \phi_{k,r}).$$

Hence by (7.15) we get

$$\langle \rho_r(\psi_{r,i}), \rho_r(\psi_{r,k}) \rangle = 2\langle \phi_{r,i}, \phi_{r,k} \rangle = \frac{1}{r}\left(1 - \left\langle \psi_{r,i}\big|_{M_0}, \psi_{r,k}\big|_{M_0} \right\rangle\right). \tag{7.16}$$

Furthermore, by (7.15)

$$\parallel \phi_{r,k} \parallel^2 \leq \frac{1}{2r}. \tag{7.17}$$

Now observe that by (7.14) we have

$$\psi_{k,r}\big|_{\partial M_i} = \phi_{k,r}, \quad k = 1, \dots, q.$$

Moreover $\Delta_{M_i}\psi_{k,r} = 0$. Thus

$$\psi_{r,k}\big|_{M_0} = \psi_{r,k}\big|_{M_1} + \psi_{r,k}\big|_{M_2} = P_1(\phi_{k,r}) + P_2(\phi_{k,r}).$$

Together with Lemma 7.7 and (7.17) it follows that there exists $C > 0$ such that

$$\parallel \psi_{k,r}\big|_{M_0} \parallel \leq C \parallel \phi_{k,r} \parallel \leq \frac{c}{\sqrt{r}}$$

for all $r > 0$ and $k = 1, \dots, q$. Hence by (7.15) we get

$$\langle \rho_r(\psi_{r,i}), \rho_r(\psi_{r,k}) \rangle = \frac{1}{r}\left(\delta_{ik} + O\left(\frac{1}{r}\right)\right)$$

as $r \to \infty$. This implies $r^q \det B_r = 1 + O(r^{-1})$. □

Our next purpose is to study the behaviour of $\det R_r$ as $r \to \infty$. Recall that by Lemma 7.2

$$\ker R_\infty = V_1^+ \oplus V_2^+. \tag{7.18}$$

Furthermore, by Lemma 7.4 we have

$$\ker R_r = \{(\phi, \phi) \mid \phi \in V_1^+ \cap V_2^+\}. \tag{7.19}$$

To study R_r on the orthogonal complement of $\ker R_r$ we need to introduce some auxiliary subspaces of $L^2(Y, E|Y) \oplus L^2(Y, E|Y)$. First put

$$L = (V_1^+ \cap V_2^+) \oplus (V_1^+ \cap V_2^+). \tag{7.20}$$

By (7.18) we have $L \subset \ker R_\infty$. Furthermore, it follows from (6.11) that on $\ker \Delta_Y \oplus \ker \Delta_Y$ the operator K_r is given by

$$K_r = \frac{1}{2r}\begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix}. \tag{7.21}$$

This implies that L is invariant under K_r. Therefore, L is an invariant subspace for $R_r = R_\infty + K_r$. Let

$$W = \{(\phi, -\phi) \mid \phi \in V_1^+ \cap V_2^+\}.$$

Then by (7.19) we get an orthogonal decomposition

$$L = \ker R_r \oplus W$$

and it follows from (7.20) that W is an invariant subspace of K_r and hence of R_r. Moreover

$$R_r|_W = \frac{1}{r} \operatorname{Id}.$$

Set

$$h_{12} := \dim(V_1^+ \cap V_2^+).$$

Note that $h_{12} = q = \dim \ker R_r$. Let $L^\perp$ be the orthogonal complement of L in $L^2(Y, E|Y) \oplus L^2(Y, E|Y)$. Then it follows that

$$\det R_r = r^{-h_{12}} \det \left(R_r|L^\perp\right). \tag{7.22}$$

So we can continue with the investigation of $R_r|L^\perp$. Let $L_1 \subset V_1^+ \oplus V_2^+$ be the orthogonal complement of L in $V_1^+ \oplus V_2^+$ and $(\ker R_\infty)^\perp$ the orthogonal complement of $\ker R_\infty$ in $L^2(Y, E|Y) \oplus L^2(Y, E|Y)$. Then

$$L^\perp = L_1 \oplus (\ker R_\infty)^\perp \tag{7.23}$$

with $L_1 \subset \ker R_\infty$. This decomposition is invariant under R_∞, however, it is not invariant under K_r and hence, it is not invariant under R_r. In fact, with respect to (7.23) we may write

$$R_r|_{L^\perp} = \begin{pmatrix} A(r) & B(r) \\ C(r) & D(r) \end{pmatrix},$$

where the operators $A(r), ..., D(r)$ are defined as follows. Let Π_1 denote the orthogonal projection of $L^\perp$ onto L_1. Then

$$A(r) = \Pi_1 K_r \Pi_1, \quad B(r) = \Pi_1 K_r (\operatorname{Id} - \Pi_1), \quad C(r) = (\operatorname{Id} - \Pi_1) K_r \Pi_1,$$

and

$$D(r) = R_\infty|_{(\ker R_\infty)^\perp} + (\operatorname{Id} - \Pi_1) K_r (\operatorname{Id} - \Pi_1). \tag{7.24}$$

Recall that K_r is a trace class operator whose trace norm $\| K_r \|_1$ satisfies

$$\| K_r \|_1 = O(r^{-1})$$

as $r \to \infty$. Thus

$$K_{r,1} := (\operatorname{Id} - \Pi_1) K_r (\operatorname{Id} - \Pi_1) \tag{7.25}$$

is also a trace class operator with trace norm satisfying

$$\| K_{r,1} \|_1 = O(r^{-1}), \quad r \to \infty. \tag{7.26}$$

Furthermore, $B(r)$ and $C(r)$ are finite rank operators with

$$\| B(r) \|_1, \ \| C(r) \|_1 = O(r^{-1}). \tag{7.27}$$

Finally, $A(r)$ is a linear operator in the finite dimensional vector space L_1 whose norm is also $O(r^{-1})$. This operator can be described more explicitly as follows. First note that $L_1 \subset \ker \Delta_Y \oplus \ker \Delta_Y$ and hence we can replace Π_1 by the orthogonal projection Π_2 of $\ker \Delta_Y \oplus \ker \Delta_Y$ onto L_1. Let $(V_1^+ \cap V_2^+)_i^\perp \subset V_i^+$ denote the orthogonal complement of $V_1^+ \cap V_2^+$ in V_i^+, $i = 1, 2$, and let

$$P_i : \ker \Delta_Y \to (V_1^+ \cap V_2^+)_i^\perp$$

be the orthogonal projection of $\ker \Delta_Y$ onto $(V_1^+ \cap V_2^+)_i^\perp$. Then $\Pi_2 = (P_1, P_2)$ and by (7.21) it follows that

$$A(r) = \frac{1}{2r} \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \circ \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix} \circ \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} = \frac{1}{2r} \begin{pmatrix} P_1 & -P_1 P_2 \\ -P_2 P_1 & P_2 \end{pmatrix}.$$

Regarded as operator in $(V_1^+ \cap V_2^+)_1^\perp \oplus (V_1^+ \cap V_2^+)_2^\perp$, we get

$$A(r) = \frac{1}{2r} \begin{pmatrix} \mathrm{Id} & -P_1 \\ -P_2 & \mathrm{Id} \end{pmatrix}. \tag{7.28}$$

Suppose that $(\phi, \psi) \in L_1$ is in the kernel of $A(r)$. Then it follows that

$$\phi = P_1 \psi, \quad \psi = P_2 \phi.$$

Since $\phi \in V_1^+$ and $\psi \in V_2^+$, it follows that $\phi, \psi \in V_1^+ \cap V_2^+$ and therefore $\phi = \psi = 0$. Thus $A(r)$ is invertible and its norm satisfies

$$\| A(r) \| = c r^{-1}, \quad r > 0. \tag{7.29}$$

for some constant $c > 0$. Let

$$S \colon (V_1^+ \cap V_2^+)_1^\perp \oplus (V_1^+ \cap V_2^+)_2^\perp \to (V_1^+ \cap V_2^+)_1^\perp \oplus (V_1^+ \cap V_2^+)_2^\perp$$

denote the restriction of the operator

$$\begin{pmatrix} \mathrm{Id} & -P_1 \\ -P_2 & \mathrm{Id} \end{pmatrix} \colon \ker \Delta_Y \oplus \ker \Delta_Y \to \ker \Delta_Y \oplus \ker \Delta_Y$$

to the subspace $(V_1^+ \cap V_2^+)_1^\perp \oplus (V_1^+ \cap V_2^+)_2^\perp$. Set

$$h := \dim V_1^+ + \dim V_2^+ - 2 \dim V_1^+ \cap V_2^+ \quad \text{and} \quad h_{12} := \dim V_1^+ \cap V_2^+. \tag{7.30}$$

Lemma 7.9. *We have*

$$\lim_{r \to \infty} r^{h + h_{12}} \det R_r = 2^{-h} \det(S) \det R_{1,\infty} \det R_{2,\infty}.$$

Proof. Let

$$T_0(r) = \begin{pmatrix} A(r) & 0 \\ 0 & D(r) \end{pmatrix}.$$

Since $A(r)$ is an invertible operator in a finite-dimensional vector space and $D(r)$ is invertible for $r \geq r_0$, it follows that $T_0(r)$ is invertible for $r \geq r_0$ and

$$\det T_0(r) = \det A(r) \det D(r).$$

Let

$$T_1(r) = \begin{pmatrix} 0 & B(r) \\ C(r) & 0 \end{pmatrix}.$$

Then $T_1(r)$ is a trace class operator with $\| T_1(r) \|_1 = O(r^{-1})$ as $r \to \infty$, and

$$R_r = T_0(r) + T_1(r)$$

Moreover $T_0(r) + tT_1(r)$ is invertible for $0 \le t \le 1$ and $r \ge r_0$. Put

$$T_2(r) = T_1(r) T_0(r)^{-1}.$$

Then for $r \ge r_0$ we get

$$\begin{aligned} \log\det R_r - \log\det T_0(r) &= \int_0^1 \frac{d}{dt} \log\det(T_0(r) + tT_1(r))\, dt \\ &= \int_0^1 \operatorname{Tr}\left(T_1(r)(T_0(r) + tT_1(r))^{-1}\right)\, dt \\ &= \int_0^1 \operatorname{Tr}\left(T_2(r)(\operatorname{Id} + tT_2(r))\right)^{-1}\, dt. \end{aligned} \tag{7.31}$$

Set

$$\widetilde{B}(r) = B(r) D(r)^{-1}, \quad \widetilde{C}(r) = C(r) A(r)^{-1}.$$

Using the definition of $T_2(r)$, we get

$$T_2(r) = \begin{pmatrix} 0 & \widetilde{B}(r) \\ \widetilde{C}(r) & 0 \end{pmatrix}$$

By (7.27) and (7.29) it follows that

$$\| \widetilde{B}(r) \| = O(r^{-1}), \quad \| \widetilde{C}(r) \| = O(1) \tag{7.32}$$

as $r \to \infty$. Thus

$$T_2(r)^2 = \begin{pmatrix} \widetilde{B}(r)\widetilde{C}(r) & 0 \\ 0 & \widetilde{C}(r)\widetilde{B}(r) \end{pmatrix}$$

and by (7.32) we have

$$\| T_2(r)^2 \| = O(r^{-1}), \quad r \to \infty.$$

Let $r_1 > 0$ be such that

$$\| T_2(r)^2 \| \le \frac{1}{2}$$

for $r \ge r_1$. Then

$$\sum_{k=0}^{\infty} T_2(r)^k = (\operatorname{Id} + T_2(r)) \sum_{k=0}^{\infty} T_2(r)^{2k}$$

is absolutely convergent and hence, $\operatorname{Id} + tT_2(r)$ is invertible for $0 \le t \le 1$ and $r \ge r_1$ with

$$(\operatorname{Id} + tT_2(r))^{-1} = \sum_{k=0}^{\infty} t^k T_2(r)^k.$$

Moreover it follows that

$$(\operatorname{Id}+tT_2(r))^{-1} = \operatorname{Id}+tT_2(r) + O(r^{-1}), \quad r \to \infty.$$

Thus

$$T_2(r)(\operatorname{Id}+tT_2(r))^{-1} = T_2(r) + tT_2(r)^2 + O(r^{-1}) = T_2(r) + O(r^{-1}).$$

Since $\operatorname{Tr}(T_2(r)) = 0$, we get

$$\operatorname{Tr}\big(T_2(r)(\operatorname{Id}+tT_2(r))^{-1}\big) = O(r^{-1}).$$

Together with (7.31) this implies

$$\big|\log\det R_r - \log\det T_0(r)\big| \leq Cr^{-1}.$$

Hence we get

$$\frac{\det(R_r|L^\perp)}{\det T_0(r)} = 1 + O(r^{-1}), \quad r \to \infty.$$

As observed above, $\det T_0(r) = \det A(r)\det D(r)$. Using the definition of $D(r)$ by (7.24) and that $R_\infty|(\ker R_\infty)^\perp$ is invertible, it follows as in Lemma 6.5 that

$$\lim_{r\to\infty}\det D(r) = \det R_{1,\infty}\det R_{2,\infty}.$$

Let h and h_{12} be defined by (7.30). Note that

$$h = \dim(V_1^+ \cap V_2^+)_1^\perp + \dim(V_1^+ \cap V_2^+)_2^\perp.$$

Then by definition of $A(r)$

$$\det A(r) = (2r)^{-h}\det S.$$

So combined with (7.22) we get

$$\lim_{r\to\infty} r^{h+h_{12}}\det R_r = 2^{-h}\det(S)\det R_{1,\infty}\det R_{2,\infty}.$$

□

Next we express $\det(S)$ in terms of the scattering matrices $S_1(0)$ and $S_2(0)$. Let $V = \ker\Delta_Y$ and set

$$V_2 = V \ominus ((V_1^+ \cap V_2^+) \oplus (V_1^- \cap V_2^-)).$$

Lemma 7.10. *Let $C_{12} = S_1(0)S_2(0)|V_2$. We have*

$$\det(S) = \det\left((\operatorname{Id}-C_{12})/2\right).$$

Proof. First we consider the following special case: Assume that

(1) $V_1^+ \cap V_2^+ = \{0\}, \quad V_1^- \cap V_2^- = \{0\}$,
(2) $\dim V = 2p$ and $\dim V_i^+ = \dim V_i^- = p,\ i = 1, 2$,
(3) $P_1^+ \colon V_2^+ \to V_1^+$ is an isomorphism.

Let $e_1, ..., e_{2p}$ be an orthonormal basis of $\ker \Delta_Y$ such that $e_1, ..., e_p$ is an orthonormal basis of V_1^+ and $e_{p+1}, ..., e_{2p}$ is an orthonormal basis of V_1^-. Let $f_1, ..., f_p \in V_2^+$ be such that

$$P_1^+(f_i) = e_i, \quad i = 1, ..., p.$$

Then there exists a symmetric matrix $A = (a_{ij}) \in \mathrm{GL}(p, \mathbb{R})$ such that

$$f_i = e_i + \sum_{j=1}^{p} a_{ij} e_{p+j}, \quad i = 1, ..., p.$$

Let $A^{-1} = (b_{kl})$ and put

$$f_{p+k} = e_k + \sum_{l=1}^{p} b_{kl} e_{p+k}, \quad k = 1, ..., p.$$

Then $\langle f_i, f_{p+j} \rangle = 0$, $i, j = 1, ..., p$. Thus $f_{p+j} \in V_2^-$, $j = 1, ..., p$. Furthermore $P_1^+(f_{p+j}) = e_j$. Thus $f_{p+1}, ..., f_{2p}$ is a basis of V_2^-. By definition the matrix T which transforms the basis $(e_1, ..., e_{2p})$ into $(f_1, ..., f_{2p})$ is given by

$$T = \begin{pmatrix} \mathrm{Id} & A \\ \mathrm{Id} & -A^{-1} \end{pmatrix}.$$

Since A is symmetric, it follows that $(A^2 + \mathrm{Id})$ is invertible and one immediately verifies that the inverse of T is given by

$$T^{-1} = \begin{pmatrix} (A^2 + \mathrm{Id})^{-1} & A^2(A^2 + \mathrm{Id})^{-1} \\ \\ A(A^2 + \mathrm{Id})^{-1} & -A(A^2 + \mathrm{Id})^{-1} \end{pmatrix}.$$

Now note that the matrix of $S_1(0)S_2(0)$ with respect to the basis $(e_1, ..., e_{2p})$ is given by

$$\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix} \circ T^{-1} \circ \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix} \circ T \tag{7.33}$$

Hence the matrix of $\mathrm{Id} - S_1(0)S_2(0)$ in the basis $(e_1, ..., e_{2p})$ is equal to

$$\begin{pmatrix} 2A^2(A^2 + \mathrm{Id})^{-1} & -2A(A^2 + \mathrm{Id})^{-1} \\ \\ 2A(A^2 + \mathrm{Id})^{-1} & 2A^2(A^2 + \mathrm{Id})^{-1} \end{pmatrix}.$$

This implies

$$\det(\mathrm{Id} - S_1(0)S_2(0)) = 2^{h_Y} \det(A^2) \det(A^2 + \mathrm{Id})^{-1}.$$

On the other hand $P_2^+ = 1/2(\mathrm{Id} + S_2(0))$. So it follows from (7.33) that in the basis $(e_1, ..., e_{2p})$, P_2^+ is represented by the matrix

$$\begin{pmatrix} (A^2 + \mathrm{Id})^{-1} & A(A^2 + \mathrm{Id})^{-1} \\ \\ A(A^2 + \mathrm{Id})^{-1} & A^2(A^2 + \mathrm{Id})^{-1} \end{pmatrix}.$$

Thus, with respect to the bases $(e_1,...,e_p)$ and $(f_1,...,f_p)$, the operator $P_2^+ : V_1^+ \to V_2^+$ is represented by the matrix $(A^2+\operatorname{Id})^{-1}$. Hence the matrix of S with respect to the basis $(e_1,...,e_p,f_1,...,f_p)$ is given by

$$\begin{pmatrix} \operatorname{Id} & -\operatorname{Id} \\ -(A^2+\operatorname{Id})^{-1} & \operatorname{Id} \end{pmatrix}.$$

Thus

$$\det(S) = \det\begin{pmatrix} \operatorname{Id}-(A^2+\operatorname{Id})^{-1} & 0 \\ -(A^2+\operatorname{Id})^{-1} & \operatorname{Id} \end{pmatrix} = \det(A^2)\det(A^2+\operatorname{Id})^{-1}.$$

Next we reduce the general case to this special one. If we restrict $S_1(0)$ and $S_2(0)$ to V_2, it follows immediately that we can assume condition 1). Now suppose that

$$\dim V_2^+ \leq \dim V_1^+ \quad \text{and} \quad P_1^+ : V_2^+ \to V_1^+$$

is injective. Let $W_1 := P_1^+(V_1^+)$ and let $W_2 \subset V_1^+$ denote the orthogonal complement of W_1 in V_1^+. We claim that $W_2 \subset V_2^-$. To prove this claim let $w \in W_2$ and $v \in V_2^+$ be given. Write $v = v_1 + v_2, \quad v_1 \in V_1^+,\ v_2 \in V_1^-$. By definition we have $\langle w, v_1\rangle = 0$. Since $w \in W_2 \subset V_1^+$, we have $\langle w, v_2\rangle = 0$. Thus $\langle w, v\rangle = 0$, which shows that W_2 is orthogonal to V_2^+, and hence $W_2 \subset V_2^-$. Now

$$S_1(0)|W_2 = \operatorname{Id}, \quad S_2(0)|W_2 = -\operatorname{Id}. \tag{7.34}$$

Thus $S_1(0)S_2(0)|W_2 = -\operatorname{Id}$. Let

$$\tilde{V} = V \ominus W_2.$$

Then by (7.34), $\tilde{V}$ is an invariant subspace for $S_1(0)$ and $S_2(0)$. Let $\tilde{S}_i = S_i(0)|\tilde{V}$, $i=1,2$. Then

$$\operatorname{Id}-S_1(0)S_2(0) = \begin{pmatrix} 2\operatorname{Id} & 0 \\ 0 & \operatorname{Id}-\tilde{S}_1\tilde{S}_2 \end{pmatrix}.$$

Hence we get

$$\det(\operatorname{Id}-S_1(0)S_2(0)) = 2^{\dim W_2}\det(\operatorname{Id}-\tilde{S}_1\tilde{S}_2). \tag{7.35}$$

Let $\tilde{V}_i^\pm \subset \tilde{V}$ be the ± 1-eigenspaces of $\tilde{S}_i$, $i=1,2$. Then it follows that

$$\tilde{V}_1^+ = P_1^+(V_2^+) = W_1, \quad \tilde{V}_2^+ = V_2^+.$$

In particular, $\tilde{P}_1^+ : \tilde{V}_2^+ \to \tilde{V}_1^+$ is an isomorphism. Thus $\dim \tilde{V}_1^\pm = \tilde{V}_2^\pm$. Since $\tilde{V}_1^\pm \cap \tilde{V}_2^\pm = \{0\}$, it follows that $\dim \tilde{V}_i^+ = 1/2\dim\tilde{V}$. Thus conditions 2) and 3) are also satisfied and hence, by the first part of the proof we get

$$\det(\operatorname{Id}-\tilde{S}_1\tilde{S}_2) = 2^{\dim\tilde{V}}\det\begin{pmatrix} \operatorname{Id} & -\tilde{P}_1^+ \\ -\tilde{P}_2^+ & \operatorname{Id} \end{pmatrix}. \tag{7.36}$$

Finally note that with respect to the decomposition $V_1^+ = W_1 \oplus W_2$,

$$P_1^+ : V_2^+ \to V_1^+ \quad \text{and} \quad P_2^+ : V_1^+ \to V_2^+$$

are of the form

$$P_1^+ = (\tilde{P}_1^+, 0), \quad P_2^+ = \tilde{P}_2^+ \oplus 0.$$

Hence

$$\begin{pmatrix} \mathrm{Id} & -P_1^+ \\ -P_2^+ & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 & -\tilde{P}_1^+ \\ 0 & \mathrm{Id} & 0 \\ -\tilde{P}_2^+ & 0 & \mathrm{Id} \end{pmatrix},$$

which shows that

$$\det \begin{pmatrix} \mathrm{Id}_{V_1^+} & -P_1^+ \\ -P_2^+ & \mathrm{Id}_{V_2^+} \end{pmatrix} = \det \begin{pmatrix} \mathrm{Id}_{\tilde{V}_1^+} & -\tilde{P}_1^+ \\ -\tilde{P}_2^+ & \mathrm{Id}_{\tilde{V}_2^+} \end{pmatrix}.$$

Together with (7.35) and (7.36), the lemma follows. □

Combining Proposition 6.4 with Corollary 7.3 and Lemmas 7.8, 7.9 and 7.10, we obtain

$$\begin{aligned} \lim_{r\to\infty} r^h \frac{\det \Delta_{M_r}}{\det \Delta_{N_r,D}} &= \prod_{i=1}^{2} \frac{\det(\Delta_{i,\infty}, \Delta_0)}{\det R_{i,\infty}} \lim_{r\to\infty} \frac{r^{h+h_{12}} \det R_r}{r^{h_{12}} \det B_r} \\ &= 2^{-h} \det\left((\mathrm{Id} - C_{12})/2\right) \prod_{i=1}^{2} \det(\Delta_{i,\infty}, \Delta_0). \end{aligned} \tag{7.37}$$

Using Proposition (5.1), we get

$$\begin{aligned} \det \Delta_{M_r} \sim\ & r^{h_Y - h} e^{-r\xi_Y'(0)} 2^{2h_Y - h} (\det \Delta_Y)^{-1/2} \\ & \cdot \det\left((\mathrm{Id} - C_{12})/2\right) \prod_{i=1}^{2} \det(\Delta_{i,\infty}, \Delta_0). \end{aligned} \tag{7.38}$$

As an example, we consider the case of a closed surface M. Let

$$M_L = M_1 \cup_Y ([0, L] \times Y) \cup_Y M_2, \quad Y = \mathbb{R}/\mathbb{Z},\ L > 0.$$

Then

$$\zeta_Y(s) = (2\pi)^{-2s} 2\zeta(2s),$$

where $\zeta(s)$ denotes the Riemann zeta function. Recall that

$$\zeta(-1) = -\frac{1}{12}, \quad \zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2}\log 2\pi.$$

Since $\Gamma(s - 1/2)$ and $\zeta(2s)$ are analytic at $s = 0$, we get

$$\xi_Y'(0) = \frac{1}{\sqrt{\pi}} \frac{d}{ds} \left(\frac{\Gamma(s-1/2)}{\Gamma(s)} \zeta_Y(s - 1/2) \right) \Big|_{s=0} = -\frac{\sqrt{\pi}}{3} \frac{d}{ds} \left(\frac{\Gamma(s-1/2)}{\Gamma(s)} \right) \Big|_{s=0} = \frac{2}{3}\pi.$$

Similarly

$$\zeta_Y'(0) = -4\log(2\pi)\zeta(0) + 4\zeta'(0) = 0.$$

Thus

$$\det \Delta_Y = e^{-\zeta_Y'(0)} = 1.$$

Furthermore note that $h_Y = h_{12} = 1$, $h = 0$, and $\det\left((\mathrm{Id} - C_{12})/2\right) = 1/2$. Inserting this into (7.38), we get

$$\det \Delta_{M_L} \sim 2L e^{-\pi L/3} \det(\Delta_{1,\infty}, \Delta_0) \cdot \det(\Delta_{2,\infty}, \Delta_0) \tag{7.39}$$

as $L \to \infty$. Bismut and Bost proved in [BB] that $\det \Delta_{M_L} \sim cLe^{-\pi L/3}$, $L \to \infty$, with some constant c. Our result expresses the constant c explicitly as $c = 2\det(\Delta_{1,\infty}, \Delta_0) \cdot \det(\Delta_{2,\infty}, \Delta_0)$.

Next consider a compact Riemannian manifold (X_0, g) with boundary Y as at the end of the previous section. We assume that the connection ∇^E is a product on the collar neighborhood $N = (-\epsilon, 0] \times Y$ of Y in X_0. By (6.18) and Corollary 7.3 we have

$$\log\det \Delta_{X_r,D} = \log\det \Delta_{N_r,D} + \log\det R_r - \log\det R_\infty + \det(\Delta_\infty, \Delta_0). \tag{7.40}$$

Furthermore by Lemma 7.2 we have $\ker R_\infty = V^+$. By (6.19) it follows that $\ker R_\infty$ is invariant under L_r and hence under R_r, and

$$R_r|_{\ker R_\infty} = \frac{1}{r}\,\mathrm{Id}\,.$$

Let $h^+ = \dim V^+$. Then

$$\det R_r = r^{-h^+} \det\left(R_r|(\ker R_\infty)^\perp\right)$$

and by Lemma 4.1 of [Le3] it follows that

$$\lim_{r\to\infty} r^{h^+} \det R_r = \det R_\infty.$$

Using (7.40) we obtain

$$\lim_{r\to\infty} \frac{r^{h^+}\det\Delta_{X_r,D}}{\det\Delta_{N_r,D}} = \det(\Delta_\infty, \Delta_0).$$

Together with Proposition 5.1 we get

$$\det\Delta_{X_r,D} \sim r^{-h^+ + h_Y} e^{-r\xi_Y'(0)/2} 2^{h_Y} \left(\det\Delta_Y\right)^{-1/2} \det(\Delta_\infty, \Delta_0)$$

as $r \to \infty$. If we apply this to $\det\Delta_{M_{i,r},D}$ and use (1.9), we get

$$\lim_{r\to\infty} \frac{r^{h_Y - 2h_{12}}\det\Delta_{M_r}}{\det\Delta_{M_{1,r},D}\det\Delta_{M_{2,r},D}} = 2^{-h}\left(\det\Delta_Y\right)^{1/2}\det\left((\mathrm{Id} - C_{12})/2\right),$$

which proves Theorem 1.7.

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