3. Calculus of variations

3.1 Introduction

The calculus of variations deals with functionals mostly being formed as integrals involving an unknown function and its derivatives. For example in many physical application such functionals can be interpreted as the energy of the system. In the equilibrium such energy should be minimal among all other states. This is the reason why we are interested on minimizers of such functionals One of the simplest case is the Dirichlet integral given by

$$F(v) := \frac{1}{2}\int\limits_{\Omega} |\nabla v|^2 + fv dx, \quad v \in W_0^{1,2}(\Omega), \quad f \in L^2(\Omega).$$

As we will see below there exists exactly one $u \in W_0^{1,2}(\Omega)$ such that

$$F(u) = \min F(v).$$

It is easy to see that u satisfies the following Poisson equation

$$\Delta u = f \quad \text{in} \quad \Omega, \quad u|\partial\Omega = 0.$$

The above functional is of the form

$$F(v) = \int\limits_{\Omega} f(x, v, \nabla v) dx,$$

where F is convex and bounded with $F(v) \to +\infty$ as $\|v\|_{W_0^{1,2}(\Omega)} \to +\infty$. As we will see below for such functionals there exists a minimizer.

3.2 Lower semi-continuous functionals

Let X be a normed space.

Definition 3.1 1) $F : M \subseteq X \to \mathbb{R} \cup \{+\infty\}$ is called *sequentially weakly lower semi-continuous* (short: *sequ. w-lsc*) in $x \in X$ if

$$\forall (u_k) \subset X, u_k \rightharpoonup u \quad \Rightarrow \quad F(u) \leq \liminf F(u_k).$$

F is called *sequentially weakly lower semi-continuous* (short: *sequ. w-lsc*) on M if F is sequ. w-lsc in each point $x \in M$.

2) $M \subset X$ is called *weakly sequentially closed* [*weakly sequentially compact*] if

$$\forall (u_k) \subset M, \quad u_k \rightharpoonup u \quad \Rightarrow \quad u \in M$$

$$[\forall (u_k) \subset M \quad \exists (u_{k_l}), u \in M : \quad u_{k_l} \rightharpoonup u.]$$

Remark If M is closed and convex then M is weakly sequentially closed (cf. Th. Banach-Saks for X Hilbert space).

Example 1 For $C \subseteq X$ nonempty we set

$$I_C(u) := \begin{cases} 0 & \text{for} \quad u \in C \\ +\infty & \text{for} \quad u \in X \setminus C. \end{cases}$$

Then $I_C : X \to \mathbb{R} \cup \{+\infty\}$ is sequ. w-lsc iff C is weakly sequentially closed. Indeed, assume I_C is sequ. w-lsc. Let $(u_k) \subset C, u \in X$ such that $u_k \rightharpoonup u$ in X. Then $I_C(u) \leq \liminf u_k = 0$. Thus, $u \in C$ and hence C is weakly sequentially closed.

On the other hand, assume C is weakly sequentially closed. Let $(u_k) \subset X, u \in X$ such that $u_k \rightharpoonup u$ in X. In case $\liminf I_C(u_k) < +\infty$ by the definition of I_C there exists a subsequence $(u_{k_l}) \subset C$. Thus, $u \in C$ and therefore $I_C(u) = \liminf I_C(u_k)$. In case $\liminf I_C(u_k) = +\infty$ then $I_C(u) \leq \liminf I_C(u_k)$ is trivially fulfilled.

In particular, if C is closed and convex then I_C is sequ. w-lsc.

Theorem 3.2 *Let $M \subset X$ be weakly sequentially compact, let $F : M \to \mathbb{R} \cup \{+\infty\}$ be sequ. w-slc on M. Then*

$$\exists u_0 \in M : \quad F(u_0) = \min_{u \in M}.$$

Proof The case $\inf F = +\infty$ is trivial. Assume $\inf F \neq +\infty$. Let $(u_k) \subset M$ minimizing sequence for F, i.e. $\lim F(u_k) = \inf F$. Then there exists $(u_{k_l}) \subset M, u_0 \in M : \quad u_{k_l} \rightharpoonup u_0$. Thus

$$\inf F \leq F(u_0) \leq \liminf F(u_{k_l}) = \lim F(u_k) = \inf F.$$

■

Theorem 3.3 *Let X be a reflexive normed space. Let $F : X \to \mathbb{R} \cup \{+\infty\}$ sequ. w-slc on X. Assume that*

$$F(u) \to +\infty \quad \text{as} \quad \|u\| \to +\infty. \tag{3.1}$$

Then

$$\exists u_0 \in X : \quad F(u_0) = \min_{u \in X}.$$

Proof The case $\inf F = +\infty$ is trivial. Assume $\inf F \neq +\infty$. Let $(u_k) \subset X$ be a minimizing sequence for F, i.e. $\lim F(u_k) = \inf F$. By (3.1) the sequence (u_k) is bounded. Since X is reflexive the exists $(u_{k_l}) \subset X, u_0 \in X$ such that

$$u_{k_l} \rightharpoonup u_0 \quad \text{in} \quad X \quad \text{as} \quad l \to +\infty.$$

As in the proof of Th. 3.2 we obtain $F(u_0) = \inf F$.

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Example 2 (**Weierstraß** (1870)) $X = C^1([-1,1]), M := \{u \in X | u(-1) = -1, u(1) = 1\}$. Consider

$$F(u) := \int_{-1}^{1} (tu'(t))^2 dt, \quad u \in X.$$

- M closed, convex, unbounded;
- F continuous;
- $F(u) \geq 0 \quad \forall u \in X$;
- $\inf\limits_{u \in M} F(u) = 0$, but there is no $u \in M$ with $F(u) = 0$.

  In order to verify $\inf\limits_{u \in M} F(u) = 0$ we define

$$u_\varepsilon(t) := \frac{\arctan \frac{t}{\varepsilon}}{\arctan \frac{1}{\varepsilon}}, \quad \varepsilon > 0, \quad t \in [-1, 1].$$

  Clearly, $u_\varepsilon \in M$. We calculate

$$u'_\varepsilon = \frac{\varepsilon}{\varepsilon^2 + t^2} \frac{1}{\arctan \frac{1}{\varepsilon}}$$

  and

$$F(u_\varepsilon) = \frac{1}{\arctan \frac{1}{\varepsilon}} \int_{-1}^{1} \frac{\varepsilon^2 t^2}{(\varepsilon^2 + t^2)^2} dt \leq \frac{1}{\arctan \frac{1}{\varepsilon}} \int_{-1}^{1} \frac{\varepsilon^2}{\varepsilon^2 + t^2} dt = 2\varepsilon.$$

  Thus, $\lim\limits_{\varepsilon \to 0} F(u_\varepsilon) = 0$;

  Assume $F(u) = 0$ for $u \in X$. Then $u = u_0 = \text{const}$ and therefore $u \neq M$.

Example 3 $X = C([0,1]), \|u\|_X := \max\limits_{t \in [0,1]} |u(t)|$. $M := \{u \in X \mid \|u\| \leq 1\}$. Consider

$$F(u) := \int_0^1 u(t)dt - u(1), \quad u \in X.$$

- M closed, convex, bounded;
- F linear; continuous;
- $F(u) \geq (-1) \int_0^1 dt - 1 = -2 \quad \forall u \in M$. Thus, $\inf\limits_{u \in M} F(u) \geq -2$. In fact, we have $\inf\limits_{u \in M} F(u) = -2$. For, define

$$u_k(t) := \begin{cases} -1 & \text{for} \quad t \in [0, 1 - 1/k] \\ 2kt - 2k + 1 & \text{for} \quad t \in (1 - 1/k, 1] \end{cases}$$

  Clearly, $u_k \in M$ and $\lim\limits_{k \to \infty} F(u_k) = -2$.

On the other hand, assume $u \in X$ with $F(u) = -2$. Then

$$\int_0^1 u(t)dt = -2 + u(1) \le -1 = \int_0^1 (-1)dt.$$

It follows $\int_0^1 (u(t)+1)dt \le 0$. Taking into account $u(t)+1$ is continuous and nonnegative implies $u \equiv -1$. But $F(-1) = 0$ which is a contradiction.

3.3 The epigraph and its properties

Definition 3.4 Let $F : M \subseteq X \to \mathbb{R} \cup \{+\infty\}$. Then the *epigraph* of F is defined by

$$\operatorname{epi}(F) := \{(u,t) \in X \times \mathbb{R} \,|\, t \ge F(u)\}.$$

Remark For a given functional F on X we set

$$\operatorname{dom}(F) := \{u \in X \,|\, F(u) \in \mathbb{R}\}.$$

From the definition of the epigraph it follows that $\operatorname{epi}(F) \subset \operatorname{dom}(F) \times \mathbb{R}$

Theorem 3.5 *For $F : M \subseteq X \to \mathbb{R} \cup \{+\infty\}$ the two statements are equivalent*

1° *F is sequ. w-slc (slc) on M;*

2° $\operatorname{epi}(F)$ *is weakly sequentially closed (closed) in $M \times \mathbb{R}$.*

Proof $1^\circ \Rightarrow 2^\circ$. Let $(u_k,t_k) \in \operatorname{epi}(F), (u,t) \in M \times \mathbb{R}$ such that

$$(u_k,t_k) \rightharpoonup (u,t) \quad \text{in} \quad X \times \mathbb{R} \quad \text{as} \quad k \to +\infty.$$

Then $u_k \rightharpoonup u$ in X and $t_k \to t$ in $\mathbb{R}$. Hence $F(u) \le \liminf F(u_k) \le \liminf t_k = t$, which proves 2°.

$2^\circ \Rightarrow 1^\circ$. Let $(u_k) \subset M, u \in X$ with

$$u_k \rightharpoonup u \quad \text{in} \quad X \quad \text{as} \quad k \to +\infty.$$

Set $t_k := \max\{F(u_k), -m\}$ $(m \in \mathbb{N})$. Without loss of generality we may assume that $\liminf t_k < +\infty$. Thus, there exists a subsequence (t_{k_l}) and $t \in \mathbb{R}$, such that $t_{k_l} \to t = \liminf t_k = \max\{\liminf F(u_k), -m\}$. Consequently,

$$(u_k,t_k) \rightharpoonup (u,t) \quad \text{in} \quad X \times \mathbb{R} \quad \text{as} \quad k \to +\infty.$$

From 2° it follows that $(u,t) \in \operatorname{epi}(F)$, i.e. $F(u) \le t = \max\{\liminf F(u_k), -m\}$. Letting tend $-m \to -\infty$ gives $F(u) \le \liminf F(u_k)$.
Analogously, one proves F is lsc $\Leftrightarrow \operatorname{epi}(F)$ is closed.

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3.4 Properties of convex functionals

Definition 3.6 Let $M \subseteq X$ be convex. $F : M \subseteq X \to \mathbb{R} \cup \{+\infty\}$ is called *convex* if

$$F((1-\lambda)u + \lambda v) \leq (1-\lambda)F(u) + \lambda F(v) \quad \forall \lambda \in [0,1],\, \forall u, v \in M.$$

Theorem 3.7 *Let $M \subseteq X$ be convex. For $F : M \to \mathbb{R} \cup \{+\infty\}$ the two statements are equivalent*

$$1^\circ \quad F \quad \textit{is convex}; \qquad 2^\circ \quad \mathrm{epi}(F) \quad \textit{is convex}.$$

Proof $1^\circ \Rightarrow 2^\circ$. Let $(u,t), (v,s) \in \mathrm{epi}(F)$. For $\lambda \in [0,1]$ we have

$$F((1-\lambda)u + \lambda v) \leq (1-\lambda)F(u) + \lambda F(v) \leq (1-\lambda)t + \lambda s.$$

Thus, $(1-\lambda)(u,t) + \lambda(v,s) \in \mathrm{epi}(F)$.
$2^\circ \Rightarrow 1^\circ$ Let $u, v \in M, \lambda \in [0,1]$. Set $t := F(u)$ and $s := F(v)$. Then $(u,t), (v,s) \in \mathrm{epi}(F)$ and hence $(1-\lambda)(u,t) + \lambda(v,s) \in \mathrm{epi}(F)$, which is equivalent to

$$F((1-\lambda)u + \lambda v) \leq (1-\lambda)t + \lambda s = (1-\lambda)F(u) + \lambda F(v).$$

∎

Corollary 3.8 *Let $M \subseteq X$ be convex. If $F : M \to \mathbb{R} \cup \{+\infty\}$ is convex and lsc, then F is sequ. w-lsc.*

Proof From Th. 3.5 we see that $\mathrm{epi}(F)$ is closed. In addition, by Th. 3.7 $\mathrm{epi}(F)$ is convex. Thus, $\mathrm{epi}(F)$ is sequ. weakly closed. Once more applying Th. 3.5 implies that F is sequ. w-lsc.

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Theorem 3.9 *Let $M \subseteq X$ be convex. Let $F : M \to \mathbb{R} \cup \{+\infty\}$ be convex with $\mathrm{int}(\mathrm{epi}(F)) \neq \emptyset$. Then the restriction of F on $U := \mathrm{int}(\mathrm{dom}(F))$ is locally Lipschitz continuous, i.e. for every $u \in U$ there exists an open set $V \subseteq U$ containing u such that $F|V$ is Lipschitz continuous.*

Before turning to the proof of Th. 3.9 we state the following

Lemma 3.10 *Let $U \subseteq X$ be open and convex. Let $F : U \to \mathbb{R}$ be convex with $\mathrm{int}(\mathrm{epi}(F)) \neq \emptyset$. Given $(u,t) \in \partial(\mathrm{epi}(F))$, there exists $w^* \in X^*$, such that*

$$\langle w^*, u - v \rangle \leq s - t \quad \forall (v,s) \in \mathrm{epi}(F). \tag{3.2}$$

Moreover there holds $t \leq F(u)$.

Proof From Th. 3.7 we deduce that $\mathrm{epi}(F)$ is convex. Let $(u,t) \in \partial\,\mathrm{epi}(F)$. Since $\mathrm{int}(\mathrm{epi}(F)) \neq \emptyset$ by the aid of a theorem of Mazur (cf. Theorem 7 below) there exists $g \in (X \times \mathbb{R})^*$ with $g \neq 0$ such that

$$g((u - v, t - s)) \leq 0 \quad \forall (v,s) \in \mathrm{epi}(F).$$

Define $f(v) := g(v,0)$ for $v \in X$ and $\alpha := g(0,1)$. Obviously, $f \in X^*$ and from the above inequality it follows

$$f(u-v) \leq \alpha(s-t) \quad \forall (v,s) \in \mathrm{epi}(F).$$

For $v = u$ and $s = |F(u)| + |t| + 1$ one deducec $\alpha \geq 0$. If we assume $\alpha = 0$ then

$$f(u-v) \leq 0 \quad \forall v \in U.$$

Since U is open, there exists $r > 0$ such that $B_r(u) \subseteq U$. Hence, $f(h) \leq 0$ for all $h \in B_r(0)$, which implies $f \equiv 0$. This together with $\alpha = 0$ contradicts to $g \neq 0$. Thus, $\alpha > 0$. Setting $w^* := \alpha^{-1} f$ gives (3.2). In addition, in (3.2) setting $v = u$ and $s = F(u)$ yields $t \leq F(u)$.

∎

Proof of Th. 3.9 Since M is convex we have $\mathrm{dom}(F)$ is convex. Thus, $U = \mathrm{int}(\mathrm{dom}(M))$ is convex [1)] . Let $(u,t) \in \mathrm{int}(\mathrm{epi}(F))$. Then there exists $r > 0$, such that $B_r(u) \times (t-r, t+r) \subseteq \mathrm{epi}(F)$. This, shows that $B_r(u) \subseteq U$. Therefore we may apply Lemma 3.10 to the restriction of F on U, which will be denoted again by F.

Let $u \in U$. First, we claim $(u, F(u)) \in \partial\,\mathrm{epi}(F)$. Indeed, every ball $B_r(u) \times (F(u) - r, F(u) + r)$ contains $(u, F(u)) \in \mathrm{epi}(F)$ and $(u, F(u) - r/2) \notin \mathrm{epi}(F)$, which shows that $(u, F(u)) \in \partial\,\mathrm{epi}(F)$. Hence, by the aid of Lemma 3.10 setting $t = F(u)$ therein there exist $w^* \in X^*$:

$$\langle w^*, u-v \rangle \leq s - F(u) \quad \forall (v,s) \in \mathrm{epi}(F). \tag{3.2}$$

In particular, setting $s = F(v)$ therein yields

$$F(v) \geq \langle w^*, u-v \rangle + F(u) \quad \forall v \in U. \tag{3.3}$$

Next, we claim $(u, F(u)+1) \in \mathrm{int}(\mathrm{epi}(F))$. Otherwise, observing $(u, F(u)+1) \in \partial\,\mathrm{epi}(F)$ by Lemma 3.10 one deduces $F(u)+1 \leq F(u)$, which cannot be true. Hence there exists $r > 0$ such that

$$\overline{B_r(u)} \times (F(u)+1-r, F(u)+1+r) \subseteq \mathrm{epi}(F),$$

which implies together with (3.3)

$$C \leq F(v) \leq F(u) + 1 \quad \forall v \in \overline{B_r(u)}, \tag{3.4}$$

where $C = -r\|w^*\| + F(u)$.

Now, let $v, w \in B_{r/2}(u)$ with $F(w) > F(v)$. Set $h := \frac{w-v}{\|w-v\|}$ and $a := \|w-v\|$. Thus, $w = v + ah$. Since $\phi : \tau \mapsto \|v + \tau h - u\|$ is continuous there exists $b > 0$, such that

$$\|v + bh - u\| = r.$$

[1)] Let K be convex. Let $x, y \in \mathrm{int}(K)$. For $\lambda \in (0,1)$ we set $z_0 := (1-\lambda)x + \lambda y$. There exists $r > 0$ such that $B_r(x) \subseteq K$ and $B_r(y) \subseteq K$. Set $R := \min\{\lambda r, (1-\lambda)r\}$. Then for $z \in B_R(z_0)$ we write $z = (1-\lambda)x' + \lambda y'$ with $x' := \frac{\frac{z+z_0}{2} - \lambda y}{1-\lambda}$ and $y' := \frac{\frac{z+z_0}{2} - (1-\lambda)y}{\lambda}$. One easily verifies $\|x - x'\| < \frac{R}{2(1-\lambda)} \leq r$ and $\|y - y'\| < \frac{R}{2\lambda} \leq r$. Thus, $x', y' \in K$ and therefore $z \in K$. Hence, $z_0 \in \mathrm{int}(K)$.

Using the triangular inequality gives

$$\begin{aligned} b-a &= \|v+bh-u-(v+ah-u)\| \geq \|v+bh-u\| - \|v+ah-u\| \\ &= r - \|w-u\| \geq \frac{r}{2}. \end{aligned}$$

Observing $w = v + ah = \left(1 - \frac{a}{b}\right)v + \frac{a}{b}(v+bh)$ by means of the convexity of F one obtains

$$F(w) \leq \left(1-\frac{a}{b}\right)F(v) + \frac{a}{b}F(v+bh).$$

Observing (3.4) gives

$$F(w) - F(v) \leq \frac{a}{b}(F(v+bh) - F(v)) \leq \frac{2}{r}(F(u)+1-C)\|w-v\|.$$

■

Theorem 3.11 *Let $M \subseteq X$ be convex. Let $F : M \to \mathbb{R} \cup \{+\infty\}$ be convex. Then the following statements are equivalent*

1° $\operatorname{int}(\operatorname{epi}(F)) \neq \emptyset$*;*

2° $\exists u \in \operatorname{int}(\operatorname{dom}(F))$ *such that F is continuous in u;*

3° $\exists u \in \operatorname{int} M$ *and* $r > 0$ *such that* $B_r(u) \subseteq M$ *and* $\sup\limits_{v \in B_r(u)} F(v) < +\infty$.

Proof 1° $\Rightarrow$ 2° follows from $Th.$3.9 and 2° $\Rightarrow$ 3° follows from the definition of the continuity. It remains to show that 3° $\Rightarrow$ 1°. Indeed, from 3° it follows that $t \geq C \geq F(v)$ for all $v \in B_r(u)$ and $t \in (C, C+2)$. Thus,

$$B_r(u) \times (C, C+2) \subset \operatorname{epi}(F),$$

which shows that $(u, C+1)$ is an inner point of $\operatorname{epi}(F)$.

■

Remark According to Th. 3.11 the assumption $\operatorname{int}(\operatorname{dom}(F))$ in Th. 3.9 can be replaced by the assumption that there exists $u \in \operatorname{int}(\operatorname{dom}(F))$, such that F is continuous in u. On the other hand, if $\dim(X) = +\infty$ this condition is necessary since there exists a convex functional being everywhere discontinuous. Namely, if $\{a_i\}_{i \in I}$ denotes a Hamel basis of X, the set I is infinite. Thus, there exists a countable set $A = \{i_1, i_2, \ldots\} \subseteq I$. Then we define

$$f(a_{i_n}) = n\|a_{i_n}\|, \quad n \in \mathbb{N}, \quad f(a_i) = 0, \quad i \in I \setminus A.$$

For every $x \in X$ there exists $\lambda_i \in \mathbb{R}\,(i \in I)$ with $\lambda_i = 0$ except finite $i \in I$, such that $x = \sum\limits_{i \in I} \lambda_i a_i$. In addition, the numbers λ_i are unique. Then we define

$$f(x) = \sum_{i \in I} \lambda_i f(a_i).$$

Then f is an unbounded linear functional, which is also convex and is discontinuous in every point.

Theorem 3.12 *Let X be reflexive. Let $M \subseteq X$ be closed and convex. Let $F : M \to \mathbb{R} \cup \{+\infty\}$ be convex, such that*

$$\begin{cases} \forall\, (u_k) \subset \mathrm{dom}(F) \quad \textit{with} \quad \|u_k\| \to +\infty \quad \textit{or} \quad \mathrm{dist}(u_k, \partial(\mathrm{dom}(F)) \to 0 \\ \textit{there holds} \quad F(u_k) \to +\infty. \end{cases}$$

If $\mathrm{int}\,(\mathrm{epi}(F)) \neq \emptyset$*, then there exists $u \in M$, such that $F(u) = \min_{v \in M} F(v)$. In addition, if F is strictly convex, then the minimal element u is unique.*

Proof Without loss of generality we may assume that $F \not\equiv +\infty$. Then let $(u_k) \subset \mathrm{dom}(F)$, such that $\lim_{k \to \infty} F(u_k) = \inf_{v \in M} F(v)$. From our assumption it follows that (u_k) is bounded. By virtue of the reflexivity eventually passing to a subsequence there exists $u \in M$ such that

$$u_k \rightharpoonup u \quad \text{in} \quad X \quad \text{as} \quad k \to +\infty.$$

Set $K := \mathrm{conv}\,\{u_1, u_2, \ldots\}$. Then $u \in \overline{K}$. Since $K \subset \mathrm{dom}(F)$ we have $u \in \overline{\mathrm{dom}(F)}$. We claim $\overline{K} \subseteq \mathrm{int}\,(\mathrm{dom}(F))$. Assume this is not true. Let $w \in \overline{K} \setminus \mathrm{int}\,(\mathrm{dom}(F))$. Then $w \in \partial\, \mathrm{dom}(F)$ and there exists a sequence $(v_n) \subset K$ such that $v_n \to w$ in X. By the assumption of the theorem this implies $F(v_n) \to +\infty$, which contradicts to $F|K$ is bounded from above [2].

Hence, by Th. 3.9 $G := F|\overline{K}$ is continuous. Thus $\mathrm{epi}(G)$ is closed and convex and by Th. 3.7 $\mathrm{epi}(G)$ is sequ. weakly closed. Therefore, G is sequ. w-lsc on $\overline{K}$. This proves $F(u) \leq \liminf F(u_k) = \inf_{v \in M} F(v) \leq F(u)$.

Let F be strictly convex. Assume there are $u_1, u_2 \in M, u_1 \neq u_2$ with $F(u_1) = F(u_2) = \inf_{v \in M} F(v)$. Then $F\left(\frac{u_1 + u_2}{2}\right) < \frac{F(u_1) + F(u_2)}{2} = \inf_{v \in M} F(v)$ which is a contradiction.

■

3.5 Differentiability

Definition 3.13 $F : X \to \mathbb{R}$ is called *in $u \in X$ directional differentiable* if for all $h \in X$ there exists

$$DF(u; h) := \lim_{t \to 0} \frac{F(u + th) - F(u)}{t} \quad \text{in} \quad \mathbb{R}.$$

Remark Clearly, $D(u; \lambda h) = \lambda D(u; h)\ \forall \lambda \in \mathbb{R}$, however it is possible that

$$D(u; h_1 + h_2) \neq D(u; h_1) + D(u; h_2).$$

[2] Notice, $c_0 := \sup F(u_k) < +\infty$. Let $v \in K$ with $v = \sum_{i=1}^{m} \lambda_i u_{k_i}$ ($\lambda_i \geq 0$, $\sum_{i=1}^{m} \lambda_i = 1$). By the convexity of F one estimates $F(v) \leq \sum_{i=1}^{m} \lambda_i F(u_{k_i}) \leq c_0$.

Definition 3.14 $F : X \to \mathbb{R}$ is called GATEAUX *differentiable* in $u \in X$ if F is in u directional differentiable and $DF(u;\cdot)$ is linear and bounded, i.e. $DF(u;\cdot) \in X^*$. Then we write $\operatorname{grad} F(u)$ instead of $DF(u;\cdot)$.

$F : X \to \mathbb{R}$ is called GATEAUX *differentiable* on $M \subseteq X$ if F is GATEAUX differentiable in every $u \in M$. Then $\operatorname{grad} F : u \in M \mapsto \operatorname{grad} F(u)$ is called the GATEAUX *derivative* or the *gradient* of F on M.

Example 4 Let $X = H$ be a Hilbert space with scalar product $(\cdot,\cdot)$. Let $T \in \mathscr{L}(H,H)$. Consider, $F(u) := (Tu,u), u \in H$. Then F is Gateaux differentiable on X. Indeed, for $u,h \in H$ one gets

$$\begin{aligned}\frac{F(u+th)-F(u)}{t} &= \frac{(T(u+th),u+th)-(Tu,u)}{t}\\ &= \frac{t(Tu,h)+t(Th,u)+t^2(Th,h)}{t}\\ &= (Tu,h)+(Th,u)+t(Th,h)\\ &\to (Tu,h)+(Th,u) \quad \text{as} \quad t\to 0.\end{aligned}$$

Thus, $DF(u;h) = (Tu,h)+(Th,u) = (Tu+T^*u,h)$. Since $h \mapsto (Tu+T^*u,h)$ belongs to X^* we have $\langle \operatorname{grad} F(u),h\rangle = (Tu+T^*u,h)$ for all $u,h \in H$.

Especially, for $T = \text{id}$ we have $F(u) = \|u\|^2$ and $\operatorname{grad} F(u) = 2u$.
Next, Consider $G(u) := \|u\|, u \in H$. Then, $\|\cdot\|^2 = G^2$ and

$$2(u,h) = \lim_{t\to 0}\frac{G^2(u+th)-G^2(u)}{t} = \lim_{t\to 0}(G(u+th)+G(u))\frac{G(u+th)-G(u)}{t}$$

For $\|u\| \neq 0$ the limit $DG(u;\cdot)$ exists, and there holds

$$2(u,h) = 2\|u\|DG(u;\cdot) \quad \Longleftrightarrow \quad DG(u;h) = \left(\frac{u}{\|u\|},h\right) \quad \forall h \in H.$$

Example 5 Let Ω be an open set. Let $X = L^p(\Omega)\,(1<p<\infty)$. We consider $F(f) = \|f\|_p^p, f \in L^p(\Omega)$. Then there holds

$$\langle \operatorname{grad} F(f),h\rangle = p\int_\Omega \operatorname{sign}(f)|f|^{p-1}h\,dx, \quad u,h \in L^p(\Omega).$$

Proof Set $\phi(s) = |s|^p, s \in \mathbb{R}$. Then $\phi \in C^1(\mathbb{R})$ with

$$\phi'(s) = p\operatorname{sign}(s)|s|^{p-1} \quad \forall s \in \mathbb{R}.$$

Let $f,h \in L^p(\Omega)$. For a. e. $x \in \Omega$ we calculate

$$\begin{aligned}\lim_{t\to 0}\frac{|f(x)+th(x)|^p-|f(x)|^p}{t} &= \phi'(f(x))h(x)\\ &= p\operatorname{sign}(f(x))|f(x)|^{p-1}h(x).\end{aligned}$$

On the other hand, using the mean value theorem there exists $\theta \in [0,1]$, such that

$$\begin{aligned}\Big|\frac{|f(x)+th(x)|^p - |f(x)|^p}{t}\Big| &\le |\phi'(f(x)+\theta t h(x))||h(x)| \\ &= p|f(x)+\theta t h(x)|^{p-1}|h(x)| \\ &\le p2^{p-1}(|f(x)|^{p-1}|h(x)| + |h(x)|^p)\end{aligned}$$

for a. e. $x \in \Omega$, for all $t \in (0,1)$. Since $|f|^{p-1}|h| + |h|^p \in L^1(\Omega)$ we are in a position to apply Lebegue's theorem of dominated convergence to get

$$\lim_{t\to 0} \frac{F(f+th)-F(f)}{t} = \lim_{t\to 0} \frac{1}{t} \int_\Omega |f+th|^p - |f|^p dx = p \int_\Omega \operatorname{sign}(f)|f|^{p-1} h\, dx.$$

We consider $G(f) := \|f\|_p$. Let $f, h \in L^p(\Omega)$ with $f \neq 0$. Then having $F(f) = G^p(f)$ yields

$$\begin{aligned}\langle \operatorname{grad}(f), h\rangle &= \lim_{t\to 0} \frac{G^p(f+th)-G^p(f)}{t} \\ &= \lim_{t\to 0} \frac{G^p(f+th)-G^p(f)}{G(f+th)-G(f)} \cdot \frac{G(f+th)-G(f)}{t}.\end{aligned}$$

Verifying that

$$\lim_{t\to 0} \frac{G^p(f+th)-G^p(f)}{G(f+th)-G(f)} = p\|f\|_p^{p-1} \neq 0$$

it follows that $\lim_{t\to 0} \frac{G(f+th)-G(f)}{t} = DG(f;h)$ exists and there holds

$$\langle \operatorname{grad}(f), h\rangle = p\|f\|_p^{p-1} DG(u;h).$$

This shows that

$$DG(f;h) = \langle \operatorname{grad} G(f), h\rangle = \|f\|^{1-p} \int_\Omega \operatorname{sign}(f)|f|^{p-1} h\, dx.$$

Analogously, one proves

$$\langle J(f), h\rangle = \langle \operatorname{grad} \frac{1}{2}\|f\|^2, h\rangle = \|f\|_p^{2-p} \int_\Omega \operatorname{sign}(f)|f|^{p-1} h\, dx.$$

Theorem 3.15 *Let $F : X \to \mathbb{R}$ be directional differentiable on X. Let $U \subseteq X$ be open and $u \in U$ such that*

$$F(u) = \min_{v\in U} F(v).$$

Then $DF(u;\cdot) = 0$.

Proof Let $r > 0$ such that $B_r(u) \subseteq U$. Let $h \in B_r(0)$. Set $\phi(t) := F(u+th), t \in (-1,1)$. Clearly, $F(u) = \varphi(0) = \min_{t\in(-1,1)} \varphi(t)$. Thus, $DF(u;h) = \varphi'(0) = 0$. Whence, $DF(u;\cdot) = 0$.

■

Theorem 3.16 *Let $F : X \to \mathbb{R}$ be Gateaux differentiable on X. Then the following statements are equivalent.*

1° *F is convex;*

2° $\operatorname{grad} F : X \to X^*$ *is monotone;*

3° $F(v) \geq F(u) + \langle \operatorname{grad} F(u), v-u\rangle \quad \forall u, v \in X$.

Proof 1° $\Rightarrow$ 2°. Let $u, v \in X$. Set $h := u - v$. Define $\varphi(t) := F(v+th), t \in \mathbb{R}$. Since F is convex φ is differentiable and convex on $\mathbb{R}$. This implies φ' is non decreasing. Consequently,

$$\varphi'(1) = \langle \operatorname{grad} F(u), h\rangle \geq \varphi'(0) = \langle \operatorname{grad} F(v), h\rangle$$

which implies

$$\langle \operatorname{grad} F(u) - \operatorname{grad} F(v), u-v\rangle \geq 0 \quad \forall u, v \in X.$$

2° $\Rightarrow$ 3°. Let $u, v \in X$. Set $h := v-u$ and $\varphi(t) := F(u+th), t \in \mathbb{R}$. Then we have from $\operatorname{grad} F$ being monotone one gets

$$\begin{aligned}
\varphi(1) &= \varphi(0) + \int_0^1 \varphi'(t)dt = F(v) + \int_0^1 \langle \operatorname{grad} F(u+th), h\rangle dt \\
&= F(u) + \langle \operatorname{grad} F(u), v-u\rangle + \int_0^1 \langle \operatorname{grad} F(u+th) - \operatorname{grad} F(u), h\rangle dt \\
&\geq F(u) + \langle \operatorname{grad} F(u), v-u\rangle.
\end{aligned}$$

Whence 3°.
3° $\Rightarrow$ 1°. Let $w, v \in X$ and $\lambda \in [0,1]$. Set $h := v - w$. Then applying 3° with $u = w + \lambda h = v + (\lambda - 1)h$, first setting $v = w$ and then $v = v$ yields

$$\begin{aligned}
&F(w) \geq F(w+\lambda h) - \lambda \langle \operatorname{grad} F(w+\lambda h), h\rangle, \\
&F(v) \geq F(w+\lambda h) + (1-\lambda)\langle \operatorname{grad} F(w+\lambda h), h\rangle.
\end{aligned}$$

Multiplying the first inequality by $1-\lambda$, the second by λ and taking the sum of both yields

$$(1-\lambda)F(w) + \lambda F(v) \geq F(w+\lambda h) = F((1-\lambda)w + \lambda v).$$

■

Definiiton 3.17 Let $F : X \to \mathbb{R} \cup \{+\infty\}$. Let $u \in \operatorname{dom}(F)$. Then $w^* \in X^*$ is called *subgradient of F* in u, if

$$F(v) \geq F(u) + \langle w^*, v-u\rangle \quad \forall v \in X;$$

By $\partial F(u)$ we denote the set of all subgradients of F in u and the mapping $u \mapsto \partial F(u)$ is called the *subdifferential of* F.

F is called *subdifferentiable in* $u \in \mathrm{dom}(F)$ if $\partial F(u) \neq \emptyset$.

Example 6 $X = \mathbb{R}$, $F(x) = |x|, x \in \mathbb{R}$. Then

$$\partial F(x) = \begin{cases} \{1^*\} & \text{for} \quad x > 0 \\ \{\alpha^* \mid -1 \leq \alpha \leq 1\} & \text{for} \quad x = 0 \\ \{-1^*\} & \text{for} \quad x < 0 \end{cases}$$

where $\langle \alpha^*, x \rangle := \alpha \cdot x, \alpha, x \in \mathbb{R}$.

Example 7 Let $C \subset X$ be a convex set with $\mathrm{int}(C) \neq \emptyset$. For the indicator functional I_C we have $\mathrm{dom}(F) = C$. For $u \in C$ there holds

$$\partial I_C(u) = \{w^* \in X^* \mid \langle w^*, v - u \rangle \leq 0 \, \forall v \in C\}.$$

In particular $\partial I_C(u) = \{0\}$ for every $u \in \mathrm{int}(C)$. If $u \in \partial C$ by a well known theorem of Mazur (cf. Th. 7 below) there exists $w^* \in X^*$ such that $\langle w^*, u - v \rangle \leq 0$ for all $v \in C$. Thus, $w^* \in \partial I_C(u)$. This shows that I_C is subdifferentiable on C.

Example 8 Let $F : X \to \mathbb{R}$ be convex and GATEAUX differentiable in $u \in X$. Then

$$\partial F(u) = \{\mathrm{grad}\, F(u)\}.$$

Clearly, by Th. 3.16; 3° we have $\mathrm{grad}\, F(u) \in \partial F(u)$. Let $w^* \in \partial F(u)$. Let $h \in X$. Then we have

$$F(u + th) - F(u) \geq t \langle w^*, h \rangle \quad \forall t \in \mathbb{R}.$$

Now, dividing both sides by $t > 0$ and letting tend $t \to 0$ gives

$$\langle \mathrm{grad}\, F(u), h \rangle \geq \langle w^*, h \rangle.$$

Replacing h by $-h$ one gets the opposite inequality, which proves the assertion.

Lemma 3.18 (**Properties of** ∂F) *Let* $F : X \to \mathbb{R} \cup \{+\infty\}$. *Then*

1. $\partial F(u)$ *is convex and sequ. weakly*- closed,*
2. F *subdifferential in* $u \in X \quad \Rightarrow \quad F$ *is sequ. w-lsc. in* u*,*
3. F *subdifferential in* $u \in X \quad \Rightarrow \quad F$ *has global minimum in* u *iff* $0 \in \partial F(u)$*,*
4. $\langle w_1^* - w_2^*, u_1 - u_2 \rangle \geq 0 \quad \forall w_i^* \in \partial F(u_i)$ $(i = 1, 2)$,
   i.e. $\partial F : X \to 2^{X^*}$ *is a monotone map.*

Proof 1. The convexity of $\partial F(u)$ is obvious. Let $(w_k^*) \subset \partial F(u)$ such that

$$w_k^* \overset{*}{\to} w^* \quad \text{in} \quad X^* \quad \text{as} \quad k \to +\infty.$$

Then for $v \in X$ we have

$$\langle w^*, v-u \rangle = \lim_{k\to\infty} \langle w_k^*, v-u \rangle \leq F(v) - F(u).$$

Consequently, $w^* \in \partial F(u)$.

2. Let $w^* \in \partial F(u)$. Let $(u_k) \subset X$ with $u_k \rightharpoonup u$.

$$F(u) \leq F(u_k) + \langle w^*, u - u_k \rangle \quad \forall k \in \mathbb{N}.$$

Since $\liminf \langle w^*, u-u_k \rangle = 0$ one gets $F(u) \leq \liminf F(u_k)$.

3. Let $w^* \in \partial F(u)$. Clearly, $F(u) = \min F$ implies $0 \in \partial F(u)$. Assume $0 \in \partial F(u)$. Then by the definition of $\partial F(u)$ it follows $F(v) \geq F(u)$ for all $v \in X$ which proves $F(u) = \min F$.

4. Let $w_i^* \in \partial F(u_i)\,(i = 1,2)$. Then from the definition of ∂F one gets

$$F(u_2) - F(u_1) \geq \langle w_1^*, u_2 - u_1 \rangle$$
$$F(u_1) - F(u_2) \geq \langle w_2^*, u_1 - u_2 \rangle.$$

Taking the sum of both inequalities implies $0 \geq \langle w_1^* - w_2^*, u_2 - u_1 \rangle$, which is equivalent to the assertion.

■

Theorem 3.19 *Let* $F : X \to \mathbb{R} \cup \{+\infty\}$ *be convex with* $\mathrm{int}(\mathrm{epi}(F)) \neq \emptyset$. *Then*

$$\partial F(u) \neq \emptyset \quad \forall u \in \mathrm{int}(\mathrm{dom}(F)).$$

Proof Let $u \in \mathrm{int}(\mathrm{dom}(F))$. As we have seen in the proof of Th. 3.9, $(u, F(u)) \in \partial\,\mathrm{epi}(F)$. Applying Lemma 3.10 with $(u,t) = (u, F(u))$ one gets a functional $w^* \in X^*$ such that

$$F(v) \geq F(u) + \langle w^*, v-u \rangle \quad \forall v \in \mathrm{int}(\mathrm{dom}(F)).$$

Since $u \in \mathrm{int}(\mathrm{dom}(F))$ there exists $r > 0$ such that $B_r(u) \subseteq \mathrm{int}(\mathrm{dom}(F))$, which implies

$$F(u+h) \geq F(u) + \langle w^*, h \rangle \quad \forall h \in B_r(0).$$

For $h \in B_r(0)$ we define, $\Phi(t) := F(u+th) - F(u) - t\langle w^*, h \rangle, t \in \mathbb{R}$. Clearly, $\Phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is convex with

$$\Phi(t) \geq 0 \quad \forall t \in (-1,1), \quad \Phi(0) = 0.$$

Assume $\Phi(t) < 0$ for $t \in \mathbb{R}$. Then by the convexity of Φ we have for every $\lambda \in (0,1)$

$$\Phi(\lambda t) = \Phi((1-\lambda)0 + \lambda t) \leq (1-\lambda)\Phi(0) + \lambda\phi(t) < 0,$$

which is a contradiction. Thus, $\Phi(t) \geq 0$ for all $t \in \mathbb{R}$ and therefore

$$F(u+th) \geq F(u) + \langle w^*, t\, h\rangle \quad \forall t \in \mathbb{R}.$$

Whence, $w^* \in \partial F(u)$.

■

Definition 3.20 (**Duality map**) Let $u \in X$. We define the *duality map* $J : X \to 2^{X^*}$ by

$$J(u) := \{w^* \in X^* \,|\, \|w^*\|_*^2 = \|u\|^2 = \langle w^*, u\rangle\}.$$

Remark Thanks to Hahn-Banach's theorem we have $J(u) \neq \emptyset$. It is also readily seen that

$$\langle w_1^* - w_2^*, u_1 - u_2\rangle \;=\; \|u_1\|^2 + \|u_2\|^2 - \langle w_1^*, u_2\rangle - \langle w_2^*, u_1\rangle \geq (\|u_1\| - \|u_2\|)^2$$

for all $u_i \in X, w_i^* \in J(u_i)\,(i = 1,2)$. Hence, J is strictly monotone.

Theorem 3.21 *For every $u \in X$ the set $J(u)$ equals to the subgradient of the Functional $F : v \mapsto \frac{1}{2}\|v\|^2$ in u.*

Proof Let $w^* \in \partial F(u)$, i.e.

$$(1) \qquad \frac{1}{2}\|v\|^2 \geq \frac{1}{2}\|u\|^2 + \langle w^*, v-u\rangle, \quad \forall v \in X.$$

Let $h \in X, \|h\| = 1$. Into (1) inserting $v = u+th, t > 0$ one obtains

$$\begin{aligned} \langle w^*, th\rangle \;&\leq\; \frac{1}{2}(\|u+th\|^2 - \|u\|^2) \;=\; \frac{1}{2}(\|u+th\| + \|u\|)(\|u+th\| - \|u\|) \\ (2) \qquad &\leq\; \frac{t}{2}(\|u+th\| + \|u\|). \end{aligned}$$

Dividing both sides by t and letting tend $t \to 0$ gives

$$\langle w^*, h\rangle \;\leq\; \|u\| \quad \Longrightarrow \quad \|w^*\|_* = \sup_{h\in X, \|h=1\|} \langle w^*, h\rangle \leq \|u\|.$$

Next, in (1) setting $v = tu\,(t \in \mathbb{R})$ one gets

$$(t-1)\langle w^*, u\rangle \;\leq\; \frac{t^2-1}{2}\|u\|^2.$$

For $t < 1$ dividing both sides by $t-1$ gives $\langle w^*, u\rangle \;\geq\; \dfrac{t+1}{2}\|u\|^2$. Letting tend $t \to 1$ shows that

$$\langle w^*, u\rangle \;\geq\; \|u\|^2 \quad \Longrightarrow \quad \|w^*\|_* \geq \|u\|.$$

This implies, that $\|w^*\|_* = \|u\|$. Furthermore, estimating $\langle w^*, u\rangle \;\leq\; \|w^*\|_*\|u\| = \|u\|^2$ yields $\langle w^*, u\rangle = \|u\|^2 = \|w^*\|_*^2$, i.e. $w^* \in J(u)$.

On the other hand, let $w^* \in J(u)$. Then for every $v \in X$

$$\langle w^*, v-u\rangle \leq \|u\|\|v\| - \|u\|^2 = -\frac{1}{2}(\|v\| - \|u\|)^2 + \frac{\|v\|^2}{2} - \frac{\|u\|^2}{2}$$
$$\leq \frac{\|v\|^2}{2} - \frac{\|u\|^2}{2}.$$

Thus, $w^* \in \partial F(u)$.

■

3.6 Functionals on $W^{1,p}(\Omega)\,(1 < p < \infty)$

Let $\Omega \subset \mathbb{R}^n$ open, bounded.

Theorem (SCORZA-DRAGONI) *The following statements are equivalent*

1° $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *is a Carathéodory function;*

2° $\forall K \subset \Omega$ *compact,* $\forall \varepsilon > 0\ \exists K_\varepsilon \subset K$, *such that*

$$|K \setminus K_\varepsilon| \leq \varepsilon, \quad f|_{K_\varepsilon \times \mathbb{R}^m \times \mathbb{R}^N} \quad \textit{is continuous}.$$

(Ekeland/Temam; S. 218-219)

Let $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ be a Carathéodory function. Set

$$F(u) := \begin{cases} \int\limits_\Omega f(x,u,\nabla u)dx & \text{if} \quad f(\cdot,u,\nabla u) \in L^1(\Omega) \\ +\infty & \text{else.} \end{cases}$$

For each $\alpha \in \mathbb{R}$ the set

$$\mathcal{F}_\alpha = \left\{u \in W^{1,p}(\Omega)\,|\,F(u) \leq \alpha\right\}$$

is closed.

Theorem 2.1 (SERRIN) *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be convex, continuous,* $f(\xi) \geq 0$ *for all* $\xi \in \mathbb{R}^n$. *Let* $u, u_k \in W^{1,1}(\Omega)\,(k \in \mathbb{N})$ *with*

$$u_k \rightharpoonup u \quad in \quad L^1(\Omega) \quad as \quad k \to \infty.$$

Then there holds

$$I(u;\Omega) := \int\limits_\Omega f(\nabla u)dx \leq \liminf I(u_k;\Omega).$$

Proof Let $\omega_\rho \in C^\infty(\mathbb{R}^n)\,(\rho > 0)$ denote the usual mollifying kernel satisfying

(i) $\operatorname{supp}(\omega_\rho) \subseteq B_\rho$,

(ii) $0 \le \omega_\rho(x) \le c_0 \rho^{-n} \quad (c_0 = \text{const} > 0)$,

(iii) $\int\limits_{\mathbb{R}^n} \omega_\rho(x)dx = 1$.

Fix $\Omega' \subset\subset \Omega$. Set $d_0 = d(\Omega', \partial\Omega) > 0$. Then for $0 < \rho < d_0$ we define

$$I_\rho(v;\Omega') := \int\limits_{\Omega'} f(\nabla v_\rho)dx, \quad v \in L^1(\Omega).$$

Since

$$|f((\nabla v)_\rho(x)| = \left| \int\limits_{\mathbb{R}^n} \omega_\rho(x-y)\nabla v(y)dy \right| \le c_1 \rho^{-n-1} \|v\|_{L^1}$$

the functional $I_\rho : L^1(\Omega) \to \mathbb{R}$ is convex and bounded. By Th. 3.9 and Th. 3.7 this functional is sequ. w-lsc, which shows that

$$I_\rho(u;\Omega') \le \liminf I_\rho(u_k;\Omega').$$

On the other hand, by Jensen's inequality one finds

$$\begin{aligned} f((\nabla u_k)_\rho(x)) &= f\Bigg(\int\limits_{\mathbb{R}^n} \omega_\rho(x-y)\nabla u_k(y)dy \Bigg) \\ &= f\Bigg(\frac{1}{\mu_{\rho,x}(\mathbb{R}^n)} \int\limits_{\mathbb{R}^n} \nabla u_k(y) d\mu_{\rho,x} \Bigg) \\ &\le \frac{1}{\mu_{\rho,x}(\mathbb{R}^n)} \int\limits_{\mathbb{R}^n} f(\nabla u_k) d\mu_{\rho,x} = \int\limits_{\Omega} \omega_\rho(x-y) f(\nabla u_k)dy, \end{aligned}$$

for a. e. $x \in \Omega'$, where

$$\mu_{\rho,x}(A) := \int\limits_A \omega_\rho(x-y)dy, \quad A \subseteq \mathbb{R}^n \quad (A \text{ Lebesgue mesurable}).$$

Thus, applying Fubini's theorem gives

$$\begin{aligned} \int\limits_{\Omega'} f((\nabla u_k)_\rho(x))dx &\le \int\limits_{\Omega'} \Bigg(\int\limits_{\Omega} \omega_\rho(x-y) f(\nabla u_k(y))dy \Bigg) dx \\ &= \int\limits_{\Omega} \int\limits_{\Omega'} \omega_\rho(x-y)dx f(\nabla u_k(y))dy \\ &\le \int\limits_{\Omega} \int\limits_{\mathbb{R}^n} \omega_\rho(x)dx f(\nabla u_k(y))dy = \int\limits_{\Omega} f(\nabla u_k)dy. \end{aligned}$$

Consequently,

$$
\begin{aligned}
I_\rho(u;\Omega') &\leq \liminf_{k\to\infty} \int_{\Omega'} f((\nabla u_k)_\rho(x))dx \\
&\leq \liminf_{k\to\infty} \int_{\Omega} f(\nabla u_k)dx = \liminf_{k\to\infty} I(u_k;\Omega).
\end{aligned}
$$

By Fatou's Lemma one gets

$$
\begin{aligned}
I(u;\Omega') &= \int_{\Omega'} \liminf_{\rho\to 0} f((\nabla u)_\rho)dx \\
&\leq \liminf_{\rho\to 0} \int_{\Omega'} f((\nabla u)_\rho)dx \leq \liminf_{k\to\infty} I(u_k;\Omega).
\end{aligned}
$$

Finally, by virtue of monotone convergence ($\Omega' \to \Omega$) one arrives at

$$
I(u;\Omega) \leq \liminf_{k\to\infty} I(u_k;\Omega).
$$

■

Theorem 2.2 *Let $f : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function, such that*

(1) $-c_0 \leq f(x,u,\xi) \leq c_1(1+|\xi|^m)$ *for all* $(x,u,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$
$(m>1; c_0, c_1 = \text{const} \geq 0)$*,*

(2) $|f(x,u,\xi)-f(x,v,\xi)| \leq \omega(|u-v|)(1+|\xi|)^m$ *for all* $x\in\Omega, u,v\in\mathbb{R}, \xi\in\mathbb{R}^n$*,*
where $\omega : [0,\infty)\to[0,\infty)$ *with* $\omega(\tau)\to 0$ *as* $\tau\to 0$

(3) $\xi\mapsto f(x,u,\xi)$ *is convex for all* $(x,u)\in\Omega\times\mathbb{R}$*,*

(4) $I(u;\Omega)\to+\infty$ *as* $\|u\|_{W^{1,m}(\Omega)}\to+\infty$.

Let $W_0^{1,m}(\Omega)\subseteq V\subseteq W^{1,m}(\Omega)$ be a closed subspace. Then there exists $u\in V$ such that

$$
I(u;\Omega) = \min_{v\in V} I(v;\Omega),
$$

where

$$
I(v;\Omega) = \int_\Omega f(x,v,\nabla v)dx, \quad v\in W_0^{1,m}(\Omega).
$$

Proof Let $\alpha := \inf_{v\in V} F(v)$. By (1) we have $\alpha\in\mathbb{R}$. Let $u_k\in V$ such that

$$
F(u_k)\to\alpha \quad .
$$

From (4) one deduces that (u_k) is bounded in $W^{1,m}(\Omega)$. By means of the reflexivity of V and Riesz-Fischer's theorem there exists $u\in V$ such that (eventually passing to a subsequence)

$$
\begin{aligned}
&u_k \rightharpoonup u \quad \text{in} \quad V, \\
&u_k \to u \quad \text{a. e. in} \quad \Omega \quad \text{as} \quad k\to+\infty.
\end{aligned}
$$

Let $\varepsilon > 0$ be arbitrarily chosen. By Egorov's theorem, there exists a measurable set $E_\varepsilon \subset \Omega$ with $|\Omega \setminus E_\varepsilon| \leq \varepsilon$ and

$$u_k|E_\varepsilon \to u|E_\varepsilon \quad \text{uniformly in} \quad \text{as} \quad k \to +\infty.$$

We define

$$F(v) := \int_{E_\varepsilon} f(x,u,\nabla v)dx, \quad v \in V.$$

Then F is convex and bounded. In particular, F is sequ. w-slc. Hence,

$$\begin{aligned} I(u;E_\varepsilon) = F(u;E_\varepsilon) &\leq \liminf_{k\to\infty} F(u_k;E_\varepsilon) = \liminf_{k\to\infty} \int_{E_\varepsilon} f(x,u,\nabla u_k)dx \\ &\leq \liminf_{k\to\infty} \int_{E_\varepsilon} f(x,u_k,(\nabla u_k)_\rho)dx. \end{aligned}$$

By (3) we get

$$\begin{aligned} f(x,u_k,\nabla u_k) &= f(x,u_k,\nabla u_k) - f(x,u,\nabla u_k) + f(x,u,\nabla u_k) \\ &\geq f(x,u,\nabla u_k) - \omega(|u_k - u|)(1 + |\nabla u_k|^m). \end{aligned}$$

In addition,

$$\lim_{k\to\infty} \int_{E_\varepsilon} \omega(|u_k - u|)(1 + |\nabla u_k|^m)dx = 0$$

Thus,

$$\alpha \geq \liminf_{k\to\infty} \int_{E_\varepsilon} f(x,u_k,\nabla u_k)dx \geq \liminf_{k\to\infty} \int_{E_\varepsilon} f(x,u,\nabla u_k)dx \geq I(u;E_\varepsilon).$$

Hence

$$I(u;\Omega) = I(u;E_\varepsilon) + I(u;\Omega \setminus E_\varepsilon) \leq \alpha + I(u;\Omega \setminus E_\varepsilon) \to \alpha \text{ as } \quad \varepsilon \to 0.$$

■

Geometric version of Hahn-Banachs theorem

Theorem 1 *Let* $p :\to [0,\infty)$ *be a functional with*

(i) $p(x+y) \le p(x)+p(y) \quad \forall x,y \in X$,

(ii) $p(tx) = tp(x) \qquad \forall x \in X, \forall t \ge 0$.

Let $X_0 \subset X$ *a linear subspace of* X *and* $f_0 : X_0 \to \mathbb{R}$ *a linear functional with* $f_0(x) \le p(x)$ *for all* $x \in X_0$*. Then there exists a linear functional* $f : X \to \mathbb{R}$ *such that*

$$f|X_0 = f_0, \qquad f(x) \le p(x) \quad \forall x \in X.$$

(For the proof cf. K. Yosida; Functional Analysis).

Definition 2 Let $C \subseteq X$ be a convex, open set with $0 \in C$. Define

$$p_C(x) := \inf\left\{\frac{1}{t}\,\middle|\, tx \in C, t > 0\right\}$$

The functional $p_C : X \to [0,\infty)$ is called the *Minkowski functional* for C.

Lemma 3 *Let* $C \subset X$ *be convex and open. Then the functional* p_C *satisfies the properties (i) and (ii) of Theorem 1.*

Proof First we prove that $p_C(x) < +\infty$ for all $x \in X$. Indeed, letting $x \in X$ because $0 \in C$ and C is open there exists $t > 0$ such that $tx \in C$, which shows that $p_C(x) \le \frac{1}{t}$. Next, let $\varepsilon > 0$ and let $x,y \in X$. Take $s > 0$ such that $sx \in C$, $\frac{1}{s} \le p_C(x)+\varepsilon$ and $t > 0$ such that $ty \in C$, $\frac{1}{t} \le p_C(y)+\varepsilon$. Then by means of the convexity of C there holds

$$\frac{1}{\frac{1}{s}+\frac{1}{t}}(x+y) = \frac{t}{s+t}sx + \frac{s}{s+t}ty \in C.$$

Thus, $p_C(x+y) \le \frac{1}{s}+\frac{1}{t} \le p_C(x)+p_C(y)+2\varepsilon$. Whence (i). Next, let $\varepsilon > 0$ and let $x \in X$ and $t > 0$. Let $s > 0$ such that $stx \in C$ and $p_C(tx)\frac{1}{s} \le p_C(tx)+\varepsilon$. Then

$$p(x) \le \frac{1}{st} = (p_C(tx)+\varepsilon)\frac{1}{t}.$$

which implies $tp(x) \le p(tx)$. Replacing x by $y = tx$ and t by $\tau = \frac{1}{t}$ yields

$$p(tx) = p(y) \le \frac{1}{\tau}p(\tau y) = tp(x).$$

This implies (ii).

■

Lemma 4 *Let* $C \subset X$ *be convex and open. Let* $f : X \to \mathbb{R}$ *be linear with* $f(x) \le p_C(x)$ *for all* $x \in X$*. Then* $f \in X^*$*.*

Proof Since $0 \in C$ and $C$is open there exists $r > 0$ such that $B_r(0) \subseteq C$. Therefore for every $x \in X$ with $\|x\| \leq 1$ there holds $\frac{r}{2}x \in C$. Thus,

$$f(x) \leq p_C(x) \leq \frac{2}{r} \quad \forall x \in X \quad \text{with} \quad \|x\| \leq 1.$$

This shows that f is bounded and therefore $f \in X^*$.

Theorem 5 *Let $A, B \subseteq X$ convex with $A \cap B = \emptyset$, where A is open. Then there exists $f \in X^*$ such that*

$$f(a) < f(b) \quad \forall a \in A, \quad \forall b \in B.$$

Proof Let $a_0 \in A$ and $b_0 \in B$. Set $x_0 := b_0 - a_0$. Define

$$C := A - B + x_0 = \{a - b + x_0 \,|\, a \in A, b \in B\}.$$

Clearly, C is open and convex with $0 \in C$. Set $X_0 = lin\{x_0\}$ and $f_0 : X_0 \to \mathbb{R}$ by

$$f_0(tx_0) := t, \quad t \in \mathbb{R}.$$

It is easy to see that $f_0(tx_0) \leq p_C(tx_0)$ for all $t \in \mathbb{R}$. Indeed, for $t \leq 0$ the inequality is trivially fulfilled since $p_C \geq 0$. On the other hand, observing $x_0 \notin C$ we have $p_C(x_0) > 1$. Thus, for all $t > 0$

$$f_0(tx_0) = t < t p_C(x_0) = p_C(tx_0).$$

According to Theorem 1 along with Lemma 4 there exists $f \in X^*$ such that

$$f(x) \leq p_C(x) \quad \forall x \in X.$$

Thus,

$$f(a - b + x_0) = f(a) - f(b) + 1 \leq p(a - b + x_0) \leq 1 \quad \forall a \in A, \forall b \in B.$$

which proves the assertion.

Theorem 6 *Let $C \subset X$ be convex set. Let $u \in X \setminus \overline{C}$. Then there exists $f \in X^*$ with $f \neq 0$ such that*

$$\langle f, u - v \rangle \leq 0 \quad \forall v \in C.$$

Proof There exists an open ball $B = B_r(u) \subseteq X \setminus \overline{C}$. Let $v_0 \in C$. Set $U := B - C + v_0 - u$. Clearly, U is an open and convex set with $0 \in U$. Set $X_0 = \{t(v_0 - u) \,|\, t \in \mathbb{R}\}$. Define

$$\langle f_0, t(v_0 - u) \rangle = t, \quad t \in \mathbb{R}.$$

According to Theorem 1 along with Lemma 4 there exists $f \in X^*$, such that $\langle f, x \rangle \le p_U(x)$ for all $x \in X$. Thus,

$$\langle f, b - v + v_0 - u \rangle = \langle f, b - v \rangle + 1 \le p_U(b - v + v_0 - u) \le 1 \quad \forall v \in C, b \in B.$$

In particular, setting $b = u$ therein one gets

$$\langle f, u - v \rangle \le 0 \quad \forall v \in C.$$

■

Theorem 7 *Let $C \subset X$ be a closed convex set with $\operatorname{int}(C) \neq \emptyset$. Then for every $u \in \partial C$ there exists $f \in X^*, f \neq 0$, such that*

$$\langle f, u - v \rangle \le 0 \quad \forall v \in C.$$

Proof Let $u \in \partial C$. There exists a sequence $(u_k) \subset X \setminus \overline{C}$ with $u_k \to u$ as $k \to \infty$. Applying Theorem 6 for every $k \in \mathbb{N}$ one gets a functional $f_k \in X^*$ with $\|f_k\|_* = 1$, such that

$$\langle f_k, u - v \rangle \le 0 \quad \forall v \in C.$$

By Tychonov's theorem the ball $\{w^* | \, \|w^*\|_* \le 1\}$ is weakly* compact in X^*. Thus, there exists a subsequence $(f_{k_j}) \subset X^*, f \in X^*$, such that

$$f_{k_j} \stackrel{*}{\rightharpoonup} f \quad \text{in} \quad X^* \quad \text{as} \quad j \to +\infty.$$

This shows that for every $v \in C$ there holds

$$\langle f, u - v \rangle = \lim_{j \to \infty} \langle f_{k_j}, u_{k_j} - v \rangle \le 0.$$

Thus, it only remains to verify that $f \neq 0$. By our assumption there exists $w \in C$ and $r > 0$ such that $\overline{B_r(w)} \subset C$. Let $h \in X$ with $\|h\| = 1$. Then $w + rh \in C$. For every $j \in \mathbb{N}$ we have

$$\langle f_{k_j}, w - u_{k_j} \rangle \ge -r \langle f_{k_j}, h \rangle \quad \Longrightarrow \quad \langle f_{k_j}, w - u_{k_j} \rangle \ge r \quad \forall j \in \mathbb{N}.$$

This yields $\langle f, w - u \rangle \ge r$ and therefore $f \neq 0$.

■