SS 2020 Marc Kegel

Topology of 3-Manifolds

Exercise sheet 1

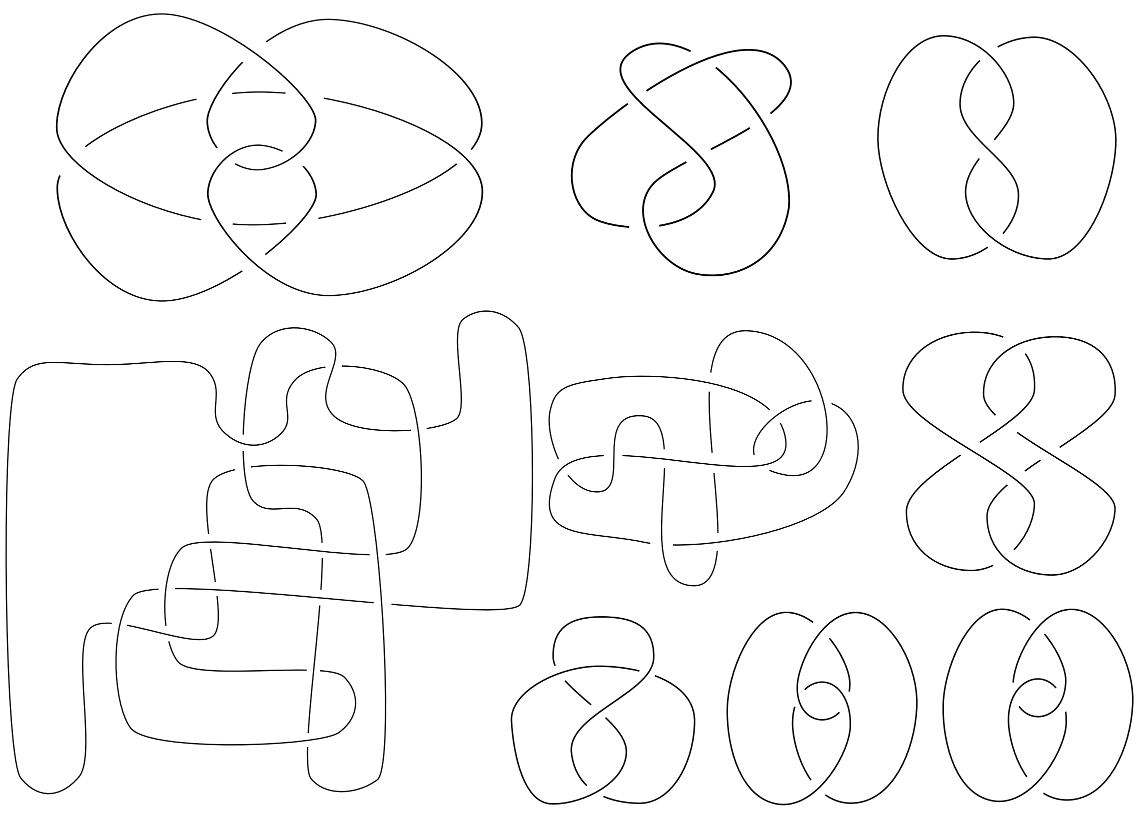
Exercise 1.
A knot diagram D_K of a knot K is called **3-colorable** if one can color each arc in exactly one of three colors such that we use every color and at each crossing all three colors or only on color meet.

(a) Show that 3-colorability is a property of the knot K.

(b) Deduce that the trefoil is non-trivial (i.e. not isotopic to the unknot).

(c) Which other knots can you distinguish from each other via 3-colorability?

Exercise 2.
Determine the isotopy type of the following knots and links.
Hint: The diagram in in the middle is called culprit. The reason is that you first have to make the diagram more complicated (in therms of number of crossing) before you can simplify it. The diagram on the lower left is called Thistlethwaite knot. For many people it turned out to be complicated to determine its isotopy type.

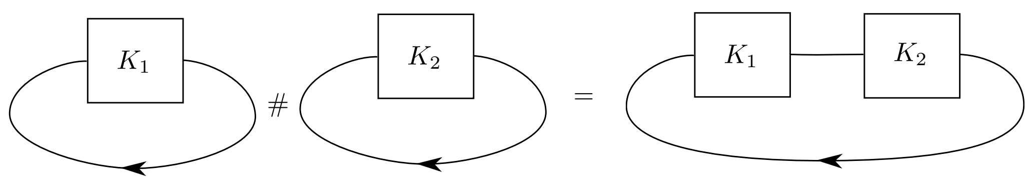


Exercise 3.

(a) Any knot admits a regular projection (i.e. prove Lemma 1.2).
Bonus: Show that a generic projection of a given knot is regular.
Hint: First, you should make the word 'generic' precise.

(b) Two knot diagrams D_K and $D_{K'}$ represent isotopic knots K and K' if and only if D_K can be transformed into $D_{K'}$ via a finite sequence of Reidemeister moves and planar isotopies (i.e. prove Theorem 1.3).

Exercise 4.
The **connected sum** of two *oriented* knots K_1 and K_2 is defined in the following picture.



(a) Show that the connected sum is well-defined. Given an example showing that this is not true anymore if we work with unoriented knots.

(b) $K_1 \# K_2$ is isotopic to $K_2 \# K_1$.

(c) For which knots K_1 and K_2 is $K_1 \# K_2$ isotopic to the unknot?

Bonus exercise (For listeners with background in algebraic and differential topology)**.**

(a) Two isotopic links L_1, L_2 in S^3 have homeomorphic exteriors $S^3 \setminus \nu \mathring{L}_i$, where $\mathring{\nu} L_i$ denotes an open tubular neighborhood of L_i. (Thus, any invariant of the exterior is also an invariant of the link.) Is the reverse implication true?
Hint: You will need to use the isotopy extension theorem.

(b) The homology and cohomology groups of a knot exterior are independent of the knot.

(c) Describe an algorithm to compute the **knot group** (the fundamental group of the knot exterior) starting from a knot diagram. Distinguish the unknot, the trefoil and the figure eight knot from each other via their knot groups.

Challenge: Describe the knot group of Fox's wild arc and deduce that it is not isotopic to a tame knot.

Challenge.
Can you determine the isotopy type of Haken's knot?
Hint: I needed 647 Reidemeister moves for transforming the diagram of Hakens knot into a diagram that I know. (Although, I am sure that it is possible to do it with fewer moves.) At a first glance, this exercise may seem like an unnecessary diligence task. But trying to simplify that complicated diagram will reveal how Haken has build it and how to construct way more complicated examples. We will come back to this exercise at a later point of the lecture.

This sheet will be discussed in the week 27.4.–1.5. and should be solved by then.