SS 2020 Marc Kegel

Topology of 3-Manifolds

Exercise sheet 2

Exercise 1.
Compute the Jones polynomial of the figure eight knot in two ways:

(a) via the Kauffman polynomial, and

(b) by directly using the Skein relation.

Deduce that the figure eight knot is non-trivial.

Exercise 2.
A knot K is called **amphicheiral** if it is isotopic to its mirror $\overline{K}$. An oriented knot K is called **invertible** if its is isotopic to itself with the reversed orientation $-K$.
Are the trefoil and the figure eight knot amphicheiral or invertible?

Exercise 3.
Let L be an oriented link with an odd (respectively even) number of components. Then its Jones polynomial $V(L)$ consists only of terms of the form q^k (respectively $q^{k+1/2}$) for integers $k \in \mathbb{Z}$.
Hint: Use the skein relation and an induction argument.

Exercise 4.

(a) For oriented knots K_1 and K_2 we have $V(K_1 \# K_2) = V(K_1)V(K_2)$. Can you prove something similar for oriented links?

(b) For the disjoint union $L_1 \sqcup L_2$ of oriented links L_1 and L_2 we have

$$V(L_1 \sqcup L_2) = -(q^{-1/2} + q^{1/2})V(L_1)V(L_2).$$

(c) Construct non-isotopic links with the same Jones polynomial.

Challenge: Can you construct non-isotopic knots with the same Jones polynomial?
Hint: The idea of the construction is similar as for links. But at the moment it will be hard to show that the constructed knots with equal Jones polynomial are really non-isotopic.

Bonus exercise.
A **Seifert surface** of an oriented link L is an oriented surface embedded surface F in $\mathbb{R}^3$ which intersects the link exactly as its oriented boundary.

(a) Describe an algorithm to produce a Seifert surface of an oriented link from one of its diagrams.
Hint: First resolve the crossings appropriately and fill the remaining circles by disks. Then try to glue the disks by drilled bands to obtain a Seifert surface of the original link.

(b) The **genus** $g(L)$ of an oriented link is defined to be the minimal genus among all its Seifert surfaces. How does the genus depend on the orientation of the link? Compute the genus for the trefoil and the figure eight knot.

(c) Let K_1 and K_2 be oriented knots. Then $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$.
Remark: In fact, equality is true. But this is harder to show.

Challenge.
A Brunnian n-link is a non-trivial n-component link consisting of n-unknots, such that removing any of its components yields a trivial $(n-1)$-component link.

(a) Construct for every $n \in \mathbb{N}$ a Brunnian n-link.

(b) Construct infinitely many different 3-component Brunnian links.

This sheet will be discussed in the exercise session on 15.5. and should be solved by then.