SS 2020 Marc Kegel

Topology of 3-Manifolds

Exercise sheet 4

Exercise 1.
Let M be a connected closed orientable 3-manifold presented by a Heegaard diagram.

(a) Conduct a presentation of the first homology group $H_1(M;\mathbb{Z})$ only depending on the homological information of the Heegaard diagram.

(b) Describe a presentation of the fundamental group of M.

(c) Compute the fundamental group and homology groups of the lens spaces $L(p,q)$ from their Heegaard diagrams. What are the higher homotopy groups of lens spaces?

Exercise 2.
Let M and N be two connected, smooth, oriented, closed n-manifolds. The **connected sum** $M\#N$ is the closed, oriented n-manifold defined as follows. Choose embeddings $i_M\colon D^n \to M$ and $i_N\colon D^n \to N$, where i_M preserves the orientation and i_N reverses the orientation. The connected sum is obtained from

$$\big(M \setminus i_M(0)\big) + \big(N \setminus i_N(0)\big)$$

by identifying points $i_M(tp)$ with points $i_N((1-t)p)$ for $p \in S^{n-1}$ and $0 < t < 1$.

(a) It is possible to show that this is a well-defined operation. (This uses methods from differential topology and is not your task.) What would you have to show for it?

(b) Let M and N be two connected, smooth, compact, oriented n-manifolds with non-empty connected boundary. The **boundary connected sum** $M\natural N$ is obtained from M and N by attaching an 1-handle to the boundary of M and N such that the resulting manifold is oriented and connected. Show that this is well-defined and that we have $\partial(M\natural N) = \partial M \# \partial N$.

(c) Show that the Heegaard genus is sub-additive under connected sum, i.e. show that

$$g(M\#N) \leq g(M) + g(N)$$

holds. To do this, figure out how to get a Heegaard diagram of $M\#N$ from Heegaard diagrams of M and N.

Remark: From Haken's lemma it follows even that $g(M\#N) = g(M) + g(N)$ holds. Another conclusion from Haken's lemmma is the existence of the **prim decomposition** of 3-manifolds, i.e. every closed orientable 3-manifold can (uniquely up to reordering and addition of S^3-factors) be written as

$$M = M_1 \# \cdots \# M_k$$

where the M_i cannot be further decomposed in non-trivial connected sums. For a closed discussion of this see for example L. Struth: Hakens Lemma, available online at
`https://www2.mathematik.hu-berlin.de/~kegemarc/Kirby/Hausarbeit%20Lennart%20Struth.pdf`

p.t.o.

Exercise 3.

(a) The Heegaard genus of T^3 is 3.
Hint: Consider the first homology or the fundamental group of T^3.

(b) A bit more general, construct for any natural number g a 3-manifold with Heegaard genus g

(c) The Heegaard genus of $\Sigma_g \times S^1$ is equal to $2g+1$.

Bonus: The Heegaard genus of a surface bundle of a surface Σ_g of genus g over S^1 is equal to $2g+1$. Where a surface bundle over S^1 is defined as follows. We start with a surface Σ_g of genus g and a diffeomorphism $\phi\colon \Sigma_g \to \Sigma_g$. Then the **surface bundle** over S^1 with **monodromy** ϕ is defined as the quotient space $\Sigma \times I/\sim$ where $(p,1) \sim (\phi(p), 0)$.

Exercise 4.
Which 3-manifold is presented by the following planar Heegaard diagram?

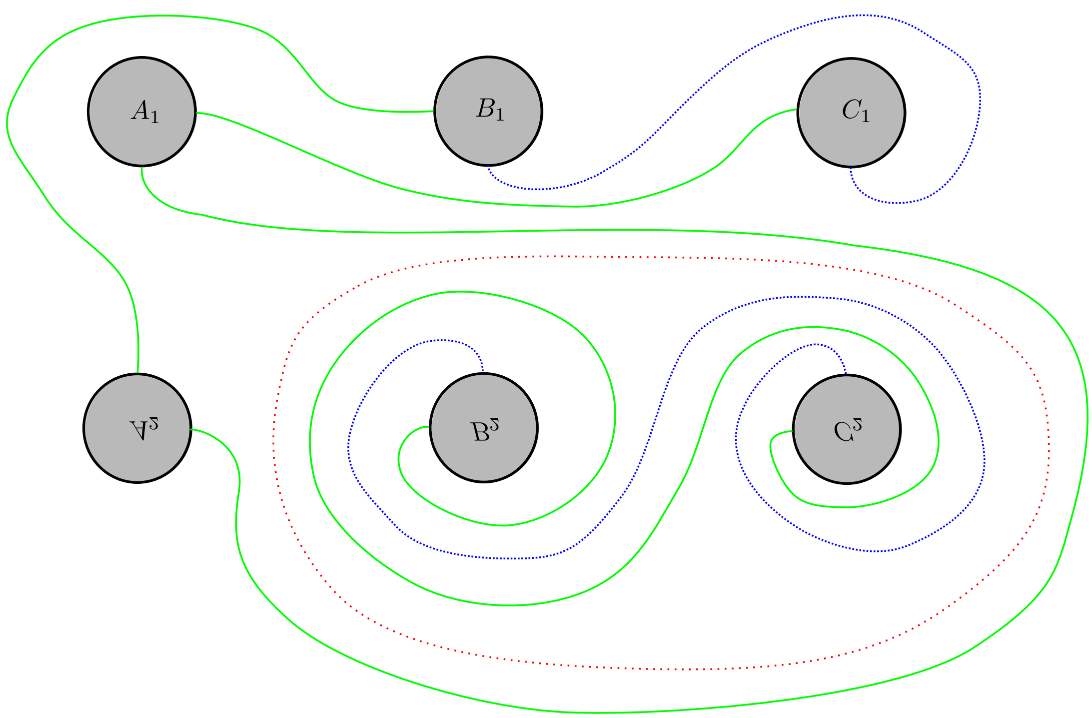

Abbildung 1: The attaching disks of the 1-handles are pairwise identified via a reflection along the horizontal middle line in this planar Heegaard diagram.

Bonus exercise.
Which conditions does a system of simple closed curves on Σ_g has to fulfill to arise as a Heegaard diagram of a closed 3-manifold?

This sheet will be discussed in the exercise session on 12.6. and should be solved by then.