SS 2020 Marc Kegel

Topology of 3-Manifolds

Exercise sheet 6

Exercise 1.

(a) Construct two linked oriented knots with vanishing linking numbers.

(b) Let K_1 and K_2 oriented knots in S^3. Let Σ_2 be a Seifert surface of K_2, see the bonus exercise from Sheet 2. Then the linking number of K_1 and K_2 can be computed as

$$\operatorname{lk}(K_1, K_2) = K_1 \bullet \Sigma_2$$

where $K_1 \bullet \Sigma_2$ denotes the oriented count of transverse intersections of K_1 and Σ_2.

Exercise 2.

(a) (-1)-surgery along the right-handed trefoil yields the same manifold as $(+1)$-surgery along the figure eight.

(b) Show that all three surgery descriptions in Figure 1 represent the Poincaré homology sphere.

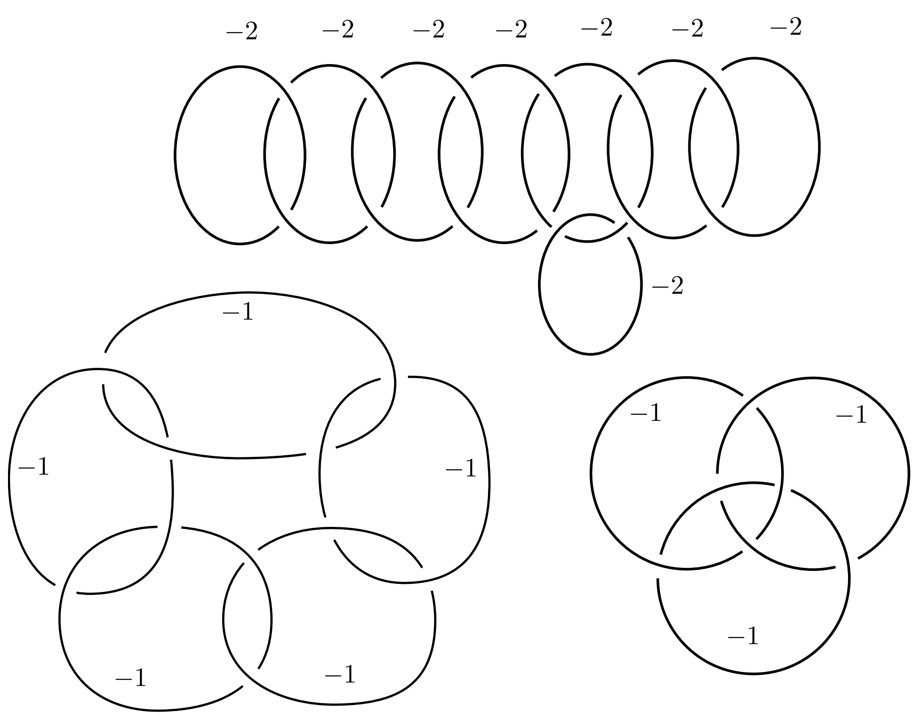

Abbildung 1: Three more surgery presentations of the Poincaré homology sphere.

Exercise 3.

(a) The lens spaces $L(p,q)$ and $L(p,q+np)$ are homeomorphic for every integer $n \in \mathbb{Z}$.

(b) If $qq' \equiv 1 \bmod(p)$, then the lens spaces $L(p,q)$ and $L(p,q')$ are homeomorphic.

(c) Moreover, are $L(-p,q)$, $L(p,-q)$ and $-L(p,q)$ orientation preserving homeomorphic.

Remark: The relations from (a), (b) and (c) give the complete classification of lens spaces up to orientation preserving homeomorphisms. However, the classification of lens spaces up to homotopy equivalence differs. Two lens spaces $L(p,q)$ and $L(p,q')$ are orientation preserving homotopy equivalent if and only if qq' is a square $\bmod(p)$. For example $L(7,1)$ and $L(7,2)$ are homotopy equivalent but not homeomorphic.

(d) (+5)-surgery along the right-handed trefoil yields yields a lens space.

(e) Describe a surgery presentation of the connected sum of any two lens spaces.

(f) (+6)-surgery along the right-handed trefoil yields the connected sum of two lens spaces.

Exercise 4.

(a) Compute the homology groups of a 3-manifold from one of its surgery presentations, i.e. prove Lemma 5.8 from the lecture.

(b) Show that, we cannot get the 3-torus T^3 by surgery along a link with less than 3 components. Describe a surgery diagram of the 3-torus along a 3-component link.

(c) For every natural number $k \in \mathbb{N}$ there exists a 3-manifold that can be obtained by surgery along k-component link but not along a link with less than k components.

Bonus exercise.

(a) Describe a Heegaard splitting of the Poincaré homolgy sphere P.

Bonus: How can we get a Heegaard splitting of a 3-manifold M from one of its surgery presentations?
Hint: Use Exercise 1(a) from Sheet 5 and try to reverse the proof of Theorem 5.8.

(b) Use the Heegaard splitting of P from (a) and Exercise 1(a) from Sheet 4 to verify that the fundamental group of P is isomorphic to the binary icosahedral group

$$I^* := \langle a, b | a^5 = b^3 = (ba)^2 \rangle.$$

(c) Describe a way to compute the fundamental group of a 3-manifold directly from one of its surgery descriptions and compute the fundamental group of P directly from its surgery diagram.
Hint: Use part (c) from the bonus exercise from Sheet 1.

This sheet will be discussed in the exercise session on 10.7. and should be solved by then.