SS 2021 Marc Kegel

4-Manifolds and Kirby calculus

Exercise sheet 2

Exercise 1.
The **complex projective space** $\mathbb{C}P^n$ is defined as the quotient of $S^{2n+1} \subset \mathbb{C}^{n+1}$ under the diagonal group action of $S^1 \subset \mathbb{C}$, i.e.

$$\mathbb{C}P^n := \big\{[z_0 : \ldots : z_n] \,\big|\, (z_0, \ldots, z_n) \in S^{2n+1} \subset \mathbb{C}^{n+1}\big\},$$

where $[z_0 : \ldots : z_n] = [w_0 : \ldots : w_n]$ if and only if there exist a $\lambda \in S^1 \subset \mathbb{C}$ with

$$(z_0, \cdots, z_n) = \lambda(w_0, \cdots, w_n).$$

It is not hard to show that $\mathbb{C}P^n$ is a connected smooth oriented closed manifold of real dimension $2n$ and that $\mathbb{C}P^1$ is S^2. (You do not have to show this here.)

(a) Describe a Morse function on $\mathbb{C}P^n$ with exactly $n+1$ critical points. Which indices do the critical points have?

(b) Describe an explicit handle decomposition of $\mathbb{C}P^2$. In particular, write down the attaching maps of the handles.

Exercise 2.

(a) Find a formula for computing the Euler characteristic from a handle decomposition and deduce that a closed oriented 3-manifold has vanishing Euler characteristic. Show that your formula for the Euler characteristic is independent of the chosen handle decomposition and thus an invariant of the manifold by using Cerf's theorem.

(c) Describe a way to compute the homology groups of a compact manifold M with a given handle decomposition.

(d) Conduct a presentation of the integeral first homology group of a 3-manifold only depending on the homological information of one of its Heegaard diagrams.

Exercise 3.
We consider the 3-torus $T^3 := S^1 \times S^1 \times S^1$.

(a) Show that we can obtain T^3 from the cube $I \times I \times I$ by identifying opposite sides.

(b) Describe a handle decomposition of T^3.

(c) Draw a planar Heegaard diagram of T^3.

(d) The Heegaard genus of T^3 is 3.
Hint: Consider the first homology or the fundamental group of T^3.

(e) Show that the Heegaard genus is sub-additive under connected sum, i.e. show that

$$g(M\#N) \leq g(M) + g(N)$$

holds. To do this, figure out how to get a Heegaard diagram of $M\#N$ from Heegaard diagrams of M and N.

(f) Construct for any natural number g a 3-manifold with Heegaard genus g.

Exercise 4.
Which 3-manifold is presented by the following planar Heegaard diagram?

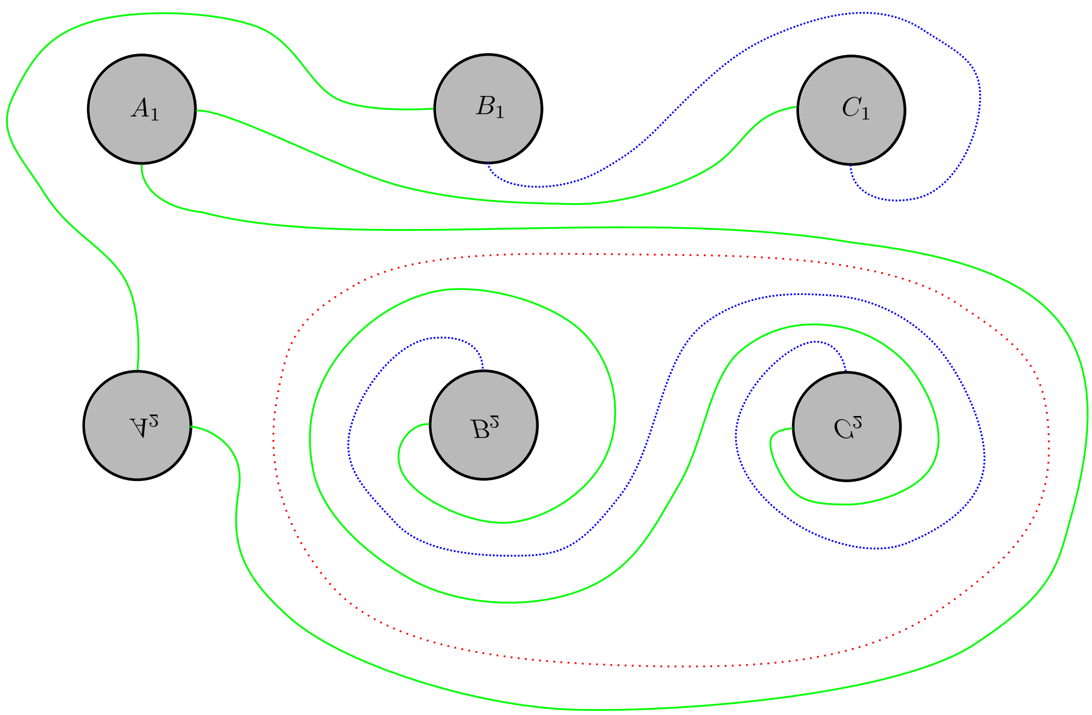

Abbildung 1: The attaching disks of the 1-handles are pairwise identified via a reflection along the horizontal middle line in this planar Heegaard diagram.

Bonus exercise.
Which conditions does a system of simple closed curves on Σ_g has to fulfill to arise as a Heegaard diagram of a closed 3-manifold?

This sheet will be discussed on Friday 7.5. and should be solved by then.