SS 2021 Marc Kegel

4-Manifolds and Kirby calculus

Exercise sheet 6

Exercise 1.

(a) The lens spaces $L(p, q)$ and $L(p, q + np)$ are orientation-preserving diffeomorphic for any integer n.

(b) The lens spaces $L(p, q)$ and $L(p, q')$ are orientation-preserving diffeomorphic if $qq' \equiv 1 \bmod(p)$ holds.

(c) Moreover, $L(-p, q)$, $L(p, -q)$ and $-L(p, q)$ are orientation-preserving diffeomorphic.
Remark: The relations of (a), (b) and (c) provide the complete classification of lens spaces up to orientation-preserving diffeomorphism.

(d) Show that (+5)-surgery along the right-handed trefoil knot yields a lens space.

(e) Show that (+6)-surgery along the right-handed trefoil knot yields the connected sum of two lens spaces.

Exercise 2.
A 3-manifold $M(g, n; r_1, \dots r_k)$ with $n \in \mathbb{Z}$, $g \in \mathbb{N}_0$ and $r_i \in \mathbb{Q}$ with a surgery diagram of the form from Figure 1 is called **Seifert fibered** 3-manifold with **Seifert invariants** $(g, n; r_1, \dots r_k)$.

(a) Show that $M(g, n; 0, \dots 0)$ is an S^1-bundle over Σ_g with Euler number n.

(b) Show that one can assume that $r_i \geq 1$ holds.

(c) Construct a Kirby diagram of a compact 4-manifold W with $\partial W = M(g, n; r_1, \dots r_k)$.

(d) Show that lens spaces are Seifert fibered. What are the Seifert invariants?

(e) Construct an integer surgery diagram with only even coefficients of the lens space $L(8, 3)$.

(f) Show that r-surgery along the right-handed trefoil knot is a Seifert fibered space. What are the Seifert invariants?

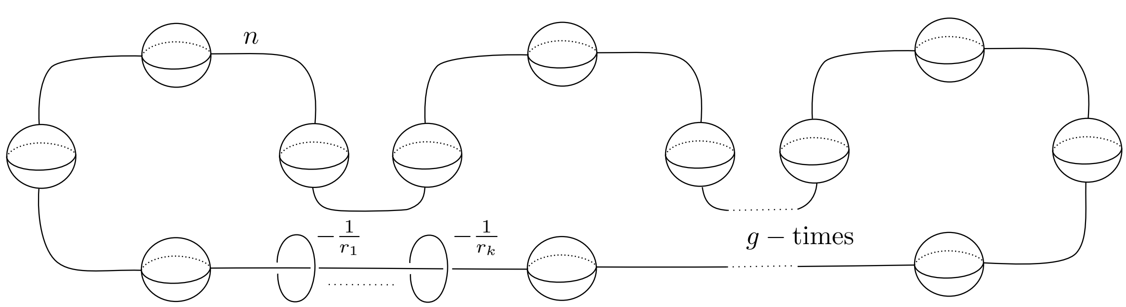

Abbildung 1: A surgery diagram of a Seifert-fibered 3-manifold.

Exercise 3.

The **property** R **theorem** (proved by David Gabai) states that if $S^1 \times S^2$ can be obtained as 0-surgery along a knot K in S^3, then K is the unknot.

(a) Use the property R theorem to show that any 4-dimensional homology sphere with a handle decomposition with exactly one 2-handle and no 3-handle must already be diffeomorphic to the 4-sphere S^4.

The **generalized property** R **conjecture** (which is unknown to be true) states that any surgery diagram for $\#_n S^1 \times S^2$ along an n-component link L in S^3 can be transformed into the 0-framed n-component unlink by 2-handle slides.

(b) Show that if the generalized property R conjecture is true, then any 4-dimensional homology sphere with a handle decomposition without 3-handles is already diffeomorphic to S^4.

(c) Show that the surgery diagram from Figure 2 can be transformed into the standard surgery diagram of $\#_2 S^1 \times S^2$ by 2-handle slides.

(d) Show that all components of a framed n-component link representing a surgery diagram of $\#_n S^1 \times S^2$ must be 0-framed and algebraically unlinked.

(e) Describe a completely 3-dimensional statement equivalent to the smooth 4-dimensional Poincaré conjecture.

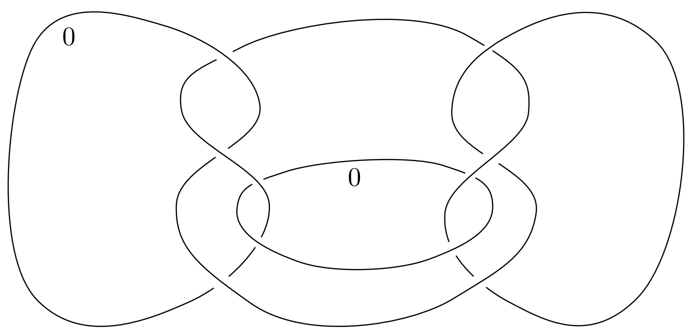

Abbildung 2: A surgery diagram of $\#_2 S^1 \times S^2$.

This sheet will be discussed on Friday 2.7. and should be solved by then.