SS 2021 Marc Kegel

4-Manifolds and Kirby calculus

Exercise sheet 7

The Kirby diagram in Figure 1 shows the Akbulut-Kirby sphere W. Before Gompf showed that the Akbulut-Kirby sphere is diffeomorphic to S^4, it was long considered a potential counterexample to the smooth 4-dimensional Poincaré conjecture. The goal of this sheet is to understand this.

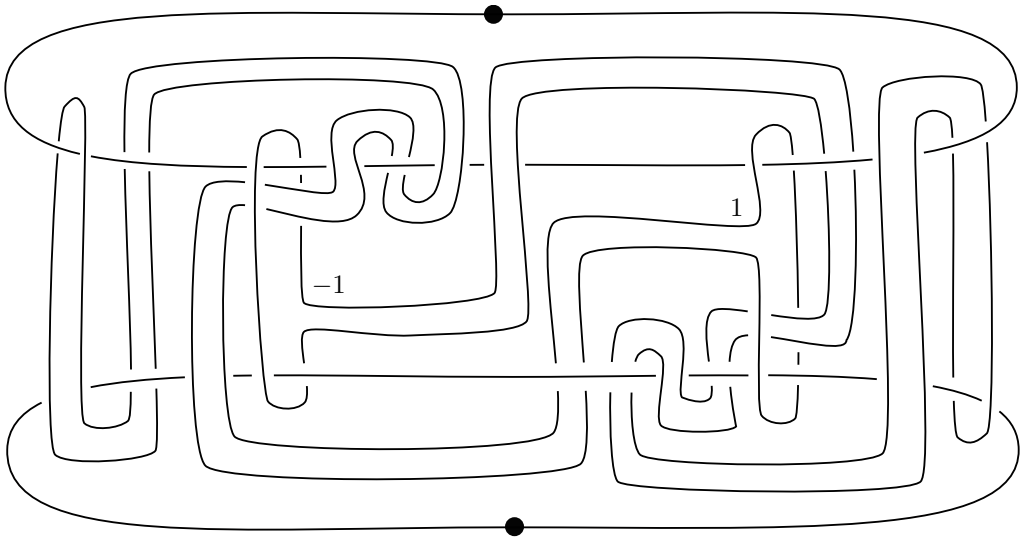

Abbildung 1: The Akbulut-Kirby sphere W.

Exercise 1.

(a) Show, by reading the Kirby diagram of the 2-handlebody W_2 as a a surgery diagram of ∂W_2, that ∂W_2 is diffeomeorphic to S^3. So $W = W_2 \cup h_4$ represents a smooth closed 4-manifold.

(b) Show that W is homeomorphic to S^4. To do this, show that W is simply connected and use Freedmann's theorem.

Exercise 2.
Next, consider for $n, k \in \mathbb{Z}$ the handlebodies $H_{n,k}$ given by the Kirby diagram in Figure 2. Analogously to the first exercise, show that $H_{n,k} \cup h_4$ represents a smooth closed 4-manifold which is homeomorphic to S^4.

Exercise 3.

(a) Show that $H_{n,k}$ is diffeomorphic to $H_{-n-1,k}$. Thus, without restriction, we can assume $n \geq 0$.

(b) Show that $H_{0,k}$ is diffeomorphic to D^4.

(b) Perform a 2-handle slide of the two parallel strands of the 0-framed 2-handle parallel over the (-1)-framed 2-handle, see Exercise 3 (a) on Sheet 5.

p.t.o.

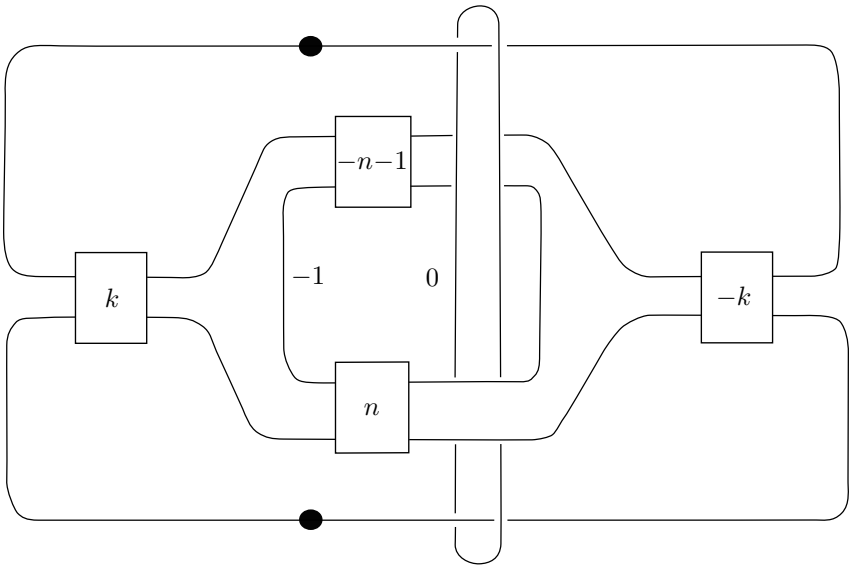

Abbildung 2: The handlebodies $H_{n,k}$.

Exercise 4.

(a) Show the equivalence of the Kirby diagrams in Figure 3.

(b) Show that adding a (+1)-framed meridian to the top 1-handle in Figure 2 is equivalent to inserting a canceling 2-/3-handle pair.
Hint: Use Lemma 5.8 from the lecture for this.

(c) Use (b) and several times (a) to show that $H_{n,k}$ is diffeomorphic to $H_{n-1,k}$.

(d) Conclude that $H_{n,k} \cup h_4$ is diffeomorphic to S^4.

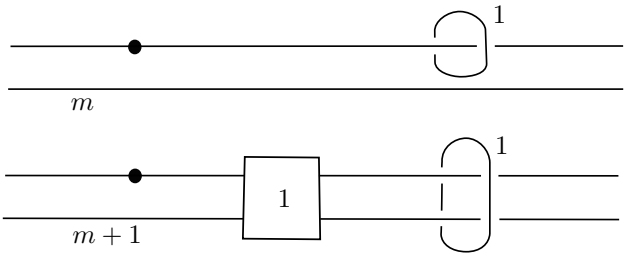

Abbildung 3: Two equivalent Kirby diagrams.

Challenge.
Show by 2-handle slides that W is diffeomorphic to $H_{4,1} \cup h_4$ and conclude that the Akbulut-Kirby sphere is diffeomorphic to S^4.
Hint 1: It might be very helpful to download and using the Kirby calculator at
`https://community.middlebury.edu/~mathanimations/klo/`.
Hint 2: You can also have a look at R. Gompf, Killing the Akbulut-Kirby 4-sphere, with relevance to the Andrews-Curtis and Schoenflies problems, *Topology* **30** (1991), 97–115.

This sheet will be discussed on Friday 16.7. and should be solved by then.