Correction to last week's lecture: We were proving.

\textbf{Prop} Assume \( h^2(A,A) = 0 \) and let
\[
 f_t(a,b) = ab + tF_1(a,b) + t^2F_2(a,b) + \ldots
\]
Then \( f_t \) is equivalent to the trivial deformation
\[
 f(a,b) = ab
\]

\textbf{Pf}
We saw last week that we have equivalences
\[
 f_t \sim f_1 \sim f_2 \sim f_3 \sim \ldots
\]
where \( f_i \) is defined by
\[
 f_i(a,b) = ab + t^{i+1}F_{i+1}(a,b) + t^{i+2}F_{i+2}(a,b) + \ldots
\]
\[i.e \ f_i \ \text{has no terms in degree} \ +1 \leq i \]
As was pointed out last week, this does not suffice to finish the proof.
However, we can use the fact that infinite power series are allowed in the definition of equivalences.
Namely, each \( \phi_j \) has the form
\[
 \phi_j = id + t^{\lambda_j} \psi_j \quad \text{for some} \quad \psi_j \in C^1(A,A)
\]
and set $\phi_c := \phi_c \circ \phi_{c_{i-1}} \circ \ldots \circ \phi_1$; this defines an equivalence $f_t \sim f_{c_t}$

Now define $\Phi_0 := \bigoplus B_j \cdot \epsilon_j$

where $B_j$ is the sum of all terms $p_{i_1} p_{i_2} \circ \ldots \circ p_{i_k}$

for $e_l := i_1 > i_2 > \ldots > i_k$

(we set $B_0 := \text{id}$).

$\Phi_0$ agrees with $\Phi_1$ for functors of order less than $c_t$.

Thus $\Phi_0$ defines an equivalence from $f_t$ to the trivial deformation $f_t^{\sim}$

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**Functors of Artin Rings**

$\text{SI}$ Yoneda's Lemma

Let $C$ be a category. **Yoneda's Lemma** lets you view an object $X \in C$ as "the same thing" as a contravariant functor $h_X : C \to \text{(Sets)}$

(contravariant means $A \to B \Rightarrow F(B) \to F(A)$ "arrow-reversing" functor)
Indeed, $h_x$ is defined by

$$h_x(y) = \text{Hom}_E(y, x)$$

If $f : x \to y$ is a morphism and $U \in E$ an object, we can define a function

$$h_f : h_x(U) \to h_y(U)$$

via composition with $f$.

In fact, $f$ defines a morphism of functors.

The weak version of Yoneda's Lemma now states: $x, y \in E$. Then

$$\text{Hom}_E(x, y) \to \text{Hom}(h_x, h_y)$$

is bijective.

**Exercise:** Prove Yoneda's lemma.

Define that $F$ be a contravariant functor $F : E \to \text{(Sets)}$. We say $F$ is representable if $F$ is isomorphic to $h_x$ for some $x \in E$.

Now suppose one wants to study families of some object, such as schemes or line bundles. One starts by defining an appropriate contravariant functor.
$F : \text{Sch}(k) \rightarrow \text{Sets}$

**Example: The Picard Functor**

Let $X$ be a smooth, projective variety over a field $k$.
Let $S$ be a scheme over $k$.
We define $\text{Pic}(X_S) := \{\text{isomorphism classes of line bundles on } X_S := X \times_k S \}$.

$\text{Pic}(X_S)$ is an abelian group.
Pullback gives $\text{Pic}(S) \cong \text{Pic}(X_S)$ and we set
$\text{Pic} : \text{CSch}(k) \rightarrow \text{Sets}$
$S \mapsto \text{Pic}(X_S)/\text{Pic}(S)$

One considers, in a loose sense, $F$ as parametrizing the objects in the set $\text{FCSpec}(k)$.
For instance, $\text{Pic}(\text{Spec}(k)) = \{\text{line bundles on } X \}$ up to isomorphism.

Q: When does there exist a scheme $Z$ parametrizing the objects in the set $\text{FCSpec}(k)$, in an appropriate sense. The strongest way to interpret the above question is:
Q: Is $F$ representable by a scheme $Z$?

If $F$ is represented by a scheme $Z$, then there is a universal object $g \in F(Z)$, i.e., for any object $Y \in E$ and any $x \in F(X)$, there is a morphism $f : Y \to Z$ such that $F(f)(g) = x$.

$$F(f) : F(Z) \to F(X)$$

Exercise (Important!): Verify the above, i.e., show $F$ is represented by $Z \Rightarrow$ there is a universal object $g \in F(Z)$.

For instance, if $Pic$ is representable (for some $X$), then there exists a scheme $T$ over $k$, together with a line bundle $L \in Pic(X \times T)$ such that for any scheme $S$ over $k$ and line bundle $M \in Pic(X_S)$, there is a morphism $\phi : S \to T$ with $\phi^*L \cong M$ modulo $Pic(S)$.

It is often very difficult to verify representability of some functor of families of geometric objects. The first step is often to restrict the functor to Artin rings.
§2 Arrow Rings

From the previous week, we saw that the deformation theory of associative algebras was done in a step-by-step fashion. To be precise: we started with a differential $F_i \in \mathfrak{h}^2(\text{A}, \text{A})$ and we wanted to find a formal family of deformations $A$ of the form $F_c(ab) = ab + tF_f(ab) + t^2F_2(ab) + \ldots$.

The procedure was: using $F_i$ produce an appropriate $F_{i+1}$ under some condition on $F_i$ (namely the condition is that the associator $F_i$ has $[F_i, i] = 0 \in t^3 \mathfrak{h}^3(\text{A}, \text{A})$).

1. Using $F_1, \ldots, F_n$ produce an appropriate $F_{n+1}$ under some condition (namely that $[G_n, i] = 0 \in t^3 \mathfrak{h}^3(\text{A}, \text{A})$).

2. By induction get all forms of $F_i$.

We wish to formalise this procedure.

Let $k$ be a fixed field. Let $A$ be a commutative algebra $A/k$ (with unit). We say $A$ isartinian if it satisfies the descending chain condition on ideals, i.e. if any descending sequence

$$\ldots \subseteq I_4 \subseteq I_3 \subseteq I_2 \subseteq I_1 \subseteq A$$

eventually stabilises (i.e. $I_n = I_{n+1}$).

Therefore $A$ isartinian.
Example \( k \subset \mathbb{E} \) is Artinian.

More generally, an integral domain is Artinian \( \iff \) if it is a field.

Recall that a commutative ring \( R \) is local if it has a unique maximal ideal \( \mathfrak{m} \); then the field \( R/\mathfrak{m} \) is the residue field.

Let \( \text{Art}_k \) be the category of local Artinian \( k \)-algebras with residue field \( k \).

By definition, morphisms in this category are local homomorphisms; i.e., \( \phi : A \to B \) has \( \phi(\mathfrak{m}A) \subseteq \mathfrak{m}B \).

(If \( A \in \text{Art}_k, \mathfrak{m}A \) = max. ideal)

For the next few weeks, the main objects of study are (covariant/ordinary) functors \( F : \text{Art}_k \to (\text{Sets}) \) satisfying the condition \( F(k) \) has one element.

Example

Let \( \text{Pic} : \text{Coh}(X) \to (\text{Sets}) \) be the (contravariant) functor defined previously. Pick \( M \in \text{Pic}(\text{Spec} k) \), i.e., \( M \) is a \( k \)-module on \( X \).
we define a **covariant** functor 

\[ \text{Pic}_m: \mathbb{A}^{op} \rightarrow \text{Sets} \]

by 

\[ (\text{Pic}_m C_A) = \{ N \in \text{Pic}(\text{Spec} A) \mid \tau_A^N N \cong M \} \]

\[ \tau_A: \text{Spec} k \rightarrow \text{Spec} A, \quad \tau_A^* X_k \rightarrow X_{\text{Spec} A} \]

\[ A \rightarrow k = A/m \]

Obviously \( \text{Pic}_m(k) = \text{EM}_k \) *contravariant*

In a similar fashion from \( F: \text{Sch} / k \rightarrow \text{Sets} \) and an object \( a \in F(k) \) we can define a **covariant** functor \( F_a: \text{Art} / k \rightarrow \text{Sets} \) by restriction.

Define \( \mathbb{E}E_1 \) := \( \mathbb{E}E_1 \times k \) the {

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recalling \( X \) variety \( /k \) with a chosen point
\( x \in X \) (or \( x \in \text{Hom}(\text{Spec} k, X) \)) then the 
Zariski tangent space could be identified with the set of morphisms
\( Y \in \text{Hom}(\text{Spec} \mathbb{E}E_1, X) \)
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such that the composition 
\( \text{Spec} k \rightarrow \text{Spec} \mathbb{E}E_1 \) \( \times \) \( x \) is \( x: \text{Spec} k \rightarrow X \)

\( \mathbb{E}E_1 \rightarrow k \)
Generalising this definition we get

Def. (Schlessinger): Let $F: \mathbf{A}^1_{\mathbb{K}} \to \text{Sets}$ be a functor s.t. $F(1)_{\mathbb{K}}$ is a one-element set. Then $F(1_{\mathbb{K}})_{\mathbb{K}}$ is called the tangent space of $F$.

**Completions of local rings**

Recall the definition of inverse limits

Let $\{A_n\} \subseteq \text{Groups}$ be a sequence of groups together with homomorphisms $\Theta_{n+1}: A_{n+1} \to A_n$.

The inverse limit $\lim\leftarrow A_n$ is the group of all sequences $\{a_n\}$ s.t. $a_n \in A_n$, $\Theta_{n+1}(a_{n+1}) = a_n$.

The inverse limit comes with obvious projections $\pi_i: \lim\leftarrow A_n \to A_i$ which are universal in the following sense: suppose $\exists X$ with projections $\psi_i: X \to A_i$ s.t. $\exists_! \psi_i$ have

\[
\begin{array}{c}
\psi_i \\
\downarrow \downarrow \\
A_i \\
\downarrow \downarrow \\
\lim\leftarrow A_n
\end{array}
\]

Then $\exists! X \to \lim\leftarrow A_n$.
If $R$ is a commutative ring, $I \subseteq R$ ideal, we set $\hat{R}_I := \varprojlim \left( R / I^n \right)$. This is a comm. ring.

If $R$ is a local commutative ring with max. ideal $M$, we set $\hat{R} := \hat{R}_M$

**Example**

(i) $R = \mathbb{k}[x_1, \ldots, x_n]$, $M = (x_1, \ldots, x_n)
R_m = \mathbb{k}[\mathbb{C}[x_1, \ldots, x_n]]$ ring of formal power series

(ii) $R = \mathbb{Z}$, $I = (p)$
$\hat{R}_I$ ring of p-adic integers
$\mathbb{Z} \ni a \equiv a \pmod{p^n}$, $0 \leq a < p^n$

More generally, if $M$ is an $R$-module, $I \subseteq R$ an ideal, we define $\hat{M} := \varprojlim \left( M / I^n M \right)$; this is an $\hat{R}_I$-module.
Define: Let \( R \) be a comm. ring, \( I \subseteq R \) an ideal. We say \( R \) is complete wrt \( I \) if the natural morphism \( \phi: R \to R/I \)
\[ a \mapsto (a + I) \]
\( \) is an iso.

Exercise: Let \( R \) be a local ring, \( I \subseteq R \) an ideal. Then \( R/I \) is complete wrt. ideal generated by \( \langle CI \rangle \).
We let \( \text{Art}_k \) denote the category of complete, noetherian local \( k \)-algebras \( R \) st.
\[ R_n := R/M^n_R \]
is in \( \text{Art}_k \) for all \( n \in \mathbb{N} \).

Exercise
1. Show \( \text{Art}_k \) is a subcategory of \( \text{Art}_k^\text{op} \).
2. Let \( A \in \text{Art}_k \), \( M_A \in \text{Ob} \).
\[ R \in \text{Art}_k \] \( R_n := R/M^n_R \in \text{Ob} \).
Show \( \text{Hom}(R, A) = \text{Hom}(R_n, A) \).

Main Definition
A functor \( F: \text{Art}_k \to \text{Sets} \) is called pro-representable if it has the form
\[ F(A) \cong \text{Hom}(R, A) \uplus A \in \text{Art}_k \]
for some \( R \in \text{Art}_k \).