Deformation Theory

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Lecture 10

If X is a smooth variety over k then we know that $\mathrm{Def}_X(k[\varepsilon]) \simeq H^1(X, \mathcal{T}_X)$. We will give a similar expression for the tangent space of the deformation functor for any (integral) variety.

1 Background on extensions

The main reference for this section is Section III.6 of [Har77].

Let X be a scheme over k and $\mathrm{Mod}(X)$ be the category of $\mathcal{O}_X$-modules. An object $\mathcal{I}$ of $\mathrm{Mod}(X)$ is called *injective* if the functor $\mathrm{Hom}_X(-, \mathcal{I})$ is exact. Recall that $\mathrm{Mod}(X)$ has enough injectives. This means that for any object $\mathcal{F} \in \mathrm{Mod}(X)$ we can find an exact sequence (called an injective resolution of $\mathcal{F}$):

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \dots$$

where $\mathcal{I}^n \in \mathrm{Mod}(X)$ is injective for all $n \geq 0$.

Fix $\mathcal{G} \in \mathrm{Mod}(X)$. We will define $\mathrm{Ext}^i_X(\mathcal{G}, -)$ to be the i^{th} right derived functor of $\mathrm{Hom}_X(\mathcal{G}, -)$.

Using the resolution above $\mathrm{Ext}^i_X(\mathcal{G}, \mathcal{F})$ is, by definition, the i^{th} cohomology of the following complex:

$$0 \longrightarrow \mathrm{Hom}_X(\mathcal{G}, \mathcal{I}^0) \longrightarrow \mathrm{Hom}_X(\mathcal{G}, \mathcal{I}^1) \longrightarrow \mathrm{Hom}_X(\mathcal{G}, \mathcal{I}^2) \longrightarrow \dots$$

where $\mathrm{Hom}_X(\mathcal{G}, \mathcal{I}^n)$ is considered to be the n^{th} term of the complex. Observe that $\mathrm{Ext}^0_X(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_X(\mathcal{G}, \mathcal{F})$.

If X is a scheme over k then $\mathrm{Ext}^i_X(\mathcal{G}, \mathcal{F})$ has a natural k-vector space structure. Indeed, in general the set of homomorphisms between two $\mathcal{O}_X$-modules has the natural structure of a $\Gamma(X, \mathcal{O}_X)$-module. Thus, so will their Ext.

1.1 Useful exercise

Exercise III.6.1 in [Har77] gives a concrete description of $\mathrm{Ext}^1_X(\mathcal{G}, \mathcal{F})$. An exact sequence

$$\xi : \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow \mathcal{G} \longrightarrow 0$$

is called *an extension of* $\mathcal{G}$ *by* $\mathcal{F}$. Two extensions are considered isomorphic if there is an isomorphism of short exact sequences that is an equality on $\mathcal{F}$ and $\mathcal{G}$. There is a natural k-vector space structure on the set of isomorphism classes of extensions. To get an idea on how this structure is defined see p.12-13 of [Ser06].

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Using the extension ξ we get a long exact sequence:

$$0 \to \mathrm{Hom}_X(\mathcal{G},\mathcal{F}) \to \mathrm{Hom}_X(\mathcal{G},\mathcal{H}) \to \mathrm{Hom}_X(\mathcal{G},\mathcal{G}) \xrightarrow{\delta} \mathrm{Ext}^1_X(\mathcal{G},\mathcal{F}) \to \dots$$

Define $[\xi] := \delta(\mathrm{Id}_\mathcal{G}) \in \mathrm{Ext}^1_X(\mathcal{G},\mathcal{F})$. This gives a bijection (and even a k-vector space isomorphism) between isomorphism classes of extensions of $\mathcal{G}$ by $\mathcal{F}$ and elements of $\mathrm{Ext}^1_X(\mathcal{G},\mathcal{F})$.

1.2 Ext sheaves

Let $\mathscr{H}\mathrm{om}(\mathcal{G},\mathcal{F})$ denote the hom-sheaf which is a $\mathcal{O}_X$-module. Do not confuse this with the $\Gamma(X,\mathcal{O}_X)$-module $\mathrm{Hom}_X(\mathcal{G},\mathcal{F})$.

The functor $\mathscr{H}\mathrm{om}(\mathcal{G},-)$ gives a left-exact functor from $\mathrm{Mod}(X)$ to itself and its i^{th}-right derived functor is denoted by $\mathscr{E}\mathrm{xt}^i_X(\mathcal{G},-)$.

If X is a variety then there is a relation between the cohomology groups of the sheaves $\mathscr{E}\mathrm{xt}^i_X$ with the modules Ext^i_X. This relationship is given by a spectral sequence called *local-to-global* Ext-*sequence*. In particular, it gives the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(X,\mathscr{H}\mathrm{om}(\mathcal{G},\mathcal{F})) \arrow[r] & \mathrm{Ext}^1_X(\mathcal{G},\mathcal{F}) \arrow[r] & \mathrm{H}^0(X,\mathscr{E}\mathrm{xt}^1_X(\mathcal{G},\mathcal{F})) \arrow[dll, rounded corners] \\ & \mathrm{H}^2(X,\mathscr{H}\mathrm{om}(\mathcal{G},\mathcal{F})) \arrow[r] & \mathrm{Ext}^2_X(\mathcal{G},\mathcal{F}) & \end{tikzcd}$$

See Huybrecths "Fourier-Mukari transforms in algebraic geometry" section 2.3 or the Wikipedia page on the "Grothendieck spectral sequence" and the references therein (also see the page on "Five term exact sequence").

2 Main theorem

We are now in a position to state the main theorem we are aiming for.

Theorem 2.1. *Let X be a variety over k (in particular X is integral). Then*

$$\mathrm{Def}_X(k[\varepsilon]) \simeq \mathrm{Ext}_X(\Omega^1_X,\mathcal{O}_X).$$

Remember that $\mathcal{T}_X = \mathscr{H}\mathrm{om}(\Omega^1_X,\mathcal{O}_X)$, therefore the first part of the local-to-global Ext-sequence gives us

$$0 \longrightarrow \mathrm{H}^1(X,\mathcal{T}_X) \longrightarrow \mathrm{Ext}^1_X(\Omega^1_X,\mathcal{O}_X)$$

We proved that the first term parametrizes the first order *locally trivial* deformations of X. This injection corresponds to the inclusion of locally trivial deformation functor into the deformation functor.

2.1 Preliminaries for the main theorem

We need two lemmas before we start proving the theorem next week. The first gives a criterion for flatness over artinian local rings.

Remark. It is well known that a *finite* module over a local ring is flat iff it is free. The following is a stronger result, eliminating the finiteness condition, for Artinian rings. Furthermore, we give a simple criterion to check for flatness.

Lemma 2.2. *Let $A \in \mathrm{Art}_k$ and let M be an A-module. Then M is flat iff*

$$\mathrm{Tor}_1^A(M, k) = 0.$$

Moreover, if M is flat then it is free.

Proof.
($\Longrightarrow$) If M is flat then for any ideal $I \subset A$ we have $\mathrm{Tor}_1^A(M, A/I) = 0$. In particular for $I = \mathfrak{m}_A$.

($\Longleftarrow$) We are going to show that M is free and hence flat.

Consider the following exact sequence:

$$0 \longrightarrow \mathfrak{m}_A \longrightarrow A \longrightarrow k \longrightarrow 0 \tag{1}$$

Let $\{x_i\} \subset M$ be such that the images of this subset forms a basis in $M \otimes_A k$. Using Nakayama's lemma for nilpotent maximal ideals we conclude that $\{x_i\}$ generate M.

Suppose there is a relation amongst $\{x_i\}$, i.e. there exists $\{f_i\} \subset A$ such that $\sum f_i x_i = 0$ (note that almost all terms have to be zero for the sum to make sense). We wish to show that in fact all f_i are zero. Since x_i restrict to a basis on $M \otimes k$, the f_i's must vanish under the map $A \to k$. Therefore, for any relation $\{f_i\}$, each f_i belongs to $\mathfrak{m}_A$.

We are now going to show that if all relations belong to $\mathfrak{m}_A^j$ then in fact all relations belong to $\mathfrak{m}_A^{j+1}$. Since $\mathfrak{m}_A$ is a nilpotent ideal, this means that there are no non-zero relations amongst $\{x_i\}$ and that M is free with generators $\{x_i\}$.

We established the base case $j = 1$ above. (If we take the obvious $j = 0$ to be the base case then the induction step below will not work. Why?) Observe that if all relations belong to $\mathfrak{m}_A^j$ then $\{x_i\}$ restricts to a free basis of $M/\mathfrak{m}_A^j M$.

Take a relation $\{f_i\} \subset \mathfrak{m}_A^j$ and consider the natural map

$$\mathfrak{m}_A \otimes_A M \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^{j+1} \otimes_{A/\mathfrak{m}_A^j} M/\mathfrak{m}_A^j$$

If $\sum_i f_i \otimes x_i = 0$ then its image under this map must also be zero. But x_i maps to a free basis on $M/\mathfrak{m}_A^j$ so this implies that the image of each f_i in $\mathfrak{m}_A/\mathfrak{m}_A^{j+1}$ must be zero, hence $f_i \in \mathfrak{m}_A^{j+1}$. So we need only show that for any relation $\sum_i f_i x_i = 0$ the tensor $\sum_i f_i \otimes x_i$ is also zero.

To see this, tensor the equation (1) with M over A. Using the fact that $\mathrm{Tor}_1^A(M, k) = 0$ we conclude $\mathfrak{m}_A \otimes_A M \to \mathfrak{m}_A M$ is an isomorphism. □

The following lemma may be considered as a flatness criterion for "thickenings" of deformations via principal small extensions.

Lemma 2.3. *Suppose $A \to B \in \mathrm{Art}_k$ is a principal small extension. Let $f : A \to C$ be a morphism of k-algebras. The morphism f is flat iff the following two conditions are satisfied:*

- *$f' : B \to B \otimes_A C$ is flat*
- *$\ker(C \to B \otimes_A C) \simeq k \otimes_A C$.*

Proof.

($\Longrightarrow$) Flatness is preserved under base change. Hence, if f is flat so is f'. Tensor the exact sequence $0 \to k \to A \to B \to 0$ (treated as a sequence of A-modules) with the flat A-module C to get

$$0 \longrightarrow k \otimes_A C \longrightarrow C \longrightarrow B \otimes_A C \longrightarrow 0$$

This gives us the second condition.

($\Longleftarrow$) Now assume the two listed conditions are satisfied. The second of the conditions imply that $\mathrm{Tor}_1^A(B, C) = 0$ as the first map in the following exact sequence is injective:

$$k \otimes_A C \longrightarrow C \longrightarrow B \otimes_A C \longrightarrow 0$$

Now tensor the exact sequence $0 \to \mathfrak{m}_B \to B \to k \to 0$ seen as A-modules with C:

$$\mathfrak{m}_B \otimes_A C \longrightarrow B \otimes_A C \longrightarrow k \otimes_A C \longrightarrow 0$$

We could have viewed the exact sequence as B-modules and tensored with $B \otimes_A C$ instead, to get the same exact sequence. Since $B \otimes_A C$ is flat over B, the map $\mathfrak{m}_B \otimes_A C \to B \otimes_A C$ is injective. Using the Tor-exact sequence we conclude that $\mathrm{Tor}_1^A(C, B) \to \mathrm{Tor}_1^A(C, k)$ is surjective. But $\mathrm{Tor}_1^A(C, B) \simeq \mathrm{Tor}_1^A(B, C) = 0$ and hence $\mathrm{Tor}_1^A(C, k) = 0$. Applying the previous lemma finishes the proof. $\square$

References

[Har77] R. Hartshorne, *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496, Graduate Texts in Mathematics, No. 52.

[Ser06] E. Sernesi, *Deformations of algebraic schemes*, ser. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006, vol. 334, pp. xii+339.