Lecture 10

If $X$ is a smooth variety over $k$, we know $\text{Det} (\mathcal{E} \otimes \mathcal{F}) \cong H^1(X, \mathcal{T}_X)$.

What if $X$ is an arbitrary (integral) variety?

Background on Ext Groups


Let $X$ be a scheme and $\text{Mod} (X)$ the category of sheaves of $O_X$-modules. Recall that $\text{Mod}(X)$ has enough injectives, so that for each object $F$ we can find an injective resolution

$$0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$

(Recall: $I^n$ injective means $\text{Hom}_X (\mathbb{G}_m, I^n)$ is exact).

We define $\text{Ext}^i_x (G, -)$ as the $i$-th right derived functor of $\text{Hom}_X (G, -)$ for $G \in \text{Mod}(X)$, where $\text{Hom}_X (G, F)$ is the group of $O_X$-module homomorphisms.

In terms of the resolution above $\text{Ext}^i_x (G, -)$ is given by the $i$-th cohomology of the complex

$$0 \rightarrow \text{Hom}_X (G, I^0) \rightarrow \text{Hom}_X (G, I^1) \rightarrow \text{Hom}_X (G, I^2) \rightarrow \cdots$$

and $\text{Ext}^0_x (G, F) = \text{Hom}_X (G, F)$.

Notate $\text{Ext}^0_x (G, F) = \text{Hom}_X (G, F)$.

For $G,F$ coherent, $X$ a variety, $\text{Ext}^1 (G, F)$
For the natural structure of a k-vector space, Hartshorne exercise 6.1, p. 237 gives a concrete description of \( \text{Ext}^i_k(B, \mathcal{F}) \) which is useful.

Let \( A, B \in \text{Mod}(\mathcal{X}) \). Recall that an "extension of \( B \) by \( A \)" is a short exact sequence \( 0 \to A \to C \to B \to 0 \) in \( \text{Mod}(\mathcal{X}) \), and that there is a natural notion of isomorphism of extensions. From the short exact sequence, as above, we get a long exact sequence

\[
0 \to \text{Hom}(C, A) \to \text{Hom}(C, B) \to \text{Ext}^i_k(B, A) \to \ldots
\]

Define \( \text{Ext}^i_k(C, D) \equiv \text{Ext}^i_k(D, C) \).

This gives a bijection (even k-vector space iso) between isomorphism classes of \( B \) by \( A \) and \( \text{Ext}^i_k(B, A) \).

Ext Sheaves

Let \( \text{Hom}^i_k(F, G) \) denote the \( i \)-th \( \text{Hom} \)-sheaf.

Do not confuse this with the group \( \text{Hom}_k(F, G) \).

(The sheaf is written in curly font!)

\( \text{Hom}^i_k(F, -) \) gives a left-exact functor from \( \text{Mod}(\mathcal{X}) \) to itself and the \( i \)-th right derived functor is the \( \text{Ext}^i_k \)-sheaf.

If \( \mathcal{X} \) is a variety, the there is a relation between the
Coextension of the sheaf $\mathcal{O}_X^i$ and the group $\text{Ext}_i^1$ called the local-to-global $\text{Ext}_1$ spectral sequence. In particular, it gives the following exact sequence:

$$0 \to \text{H}^1(X, \text{Hom}(\mathcal{F}, \mathcal{G})) \to \text{Ext}_X^1(\mathcal{F}, \mathcal{G}) \to \text{H}^0(X, \text{Ext}_X^1(\mathcal{F}, \mathcal{G})) \to \text{H}^2(X, \text{Hom}^1(\mathcal{F}, \mathcal{G})) \to \text{Ext}_X^2(\mathcal{F}, \mathcal{G})$$

For this see: Forster-Weber, "Fourier-Mukai transforms in algebraic geometry," §2.3, or the Wikipedia page on the "Grothendieck spectral sequence" and the reference therein (also see page on "Five term exact sequence").

We can now state the main theorem for today:

Then let $X$ be a variety (be $\mathbb{C}$ integral). Then $\text{Det}_X(c_1(\mathcal{E})) \cong \text{Ext}_X^1(c_1(\mathcal{E}), \mathcal{O}_X)$.

**Remark**

We have the exact sequence

$$0 \to \text{H}^1(X, \text{Hom}(\mathcal{F}, \mathcal{O}_X)) \to \text{Ext}_X^1(\mathcal{F}, \mathcal{O}_X) \to \text{H}^0(X, \text{Ext}_X^1(\mathcal{F}, \mathcal{O}_X)) \to \text{H}^2(X, \text{Hom}(\mathcal{F}, \mathcal{O}_X)) \to \text{Ext}_X^2(\mathcal{F}, \mathcal{O}_X)$$

The image of the first map is the set of locally trivial, first-order deformations of $X$. By last week's theorem.

Before proving the theorem, we need a lemma which gives a criterion for flatness over certain local rings.
Lemma \[ \text{Let } A \text{ be a ring, and let } M \text{ be an } A \text{-module.} \]

Then \( M \) is a flat \( A \)-module if and only if \( \text{Tor}_1^A(M, I) = 0 \) for every ideal \( I \) of \( A \).

Proof: Suppose \( M \) is flat. Then \( \text{Tor}_1^A(M, I) = 0 \) for every ideal \( I \) of \( A \).

Next, assume \( \text{Tor}_1^A(M, I) = 0 \).

Consider \( M/M_A \). This is a finite-dimensional vector space over \( k = \mathbb{F}_p \). Let \( x_1, \ldots, x_r \in M \) be a set such that \( \{ x_1, \ldots, x_r \} \) is a basis for \( M/M_A \) over \( k \).

We claim that \( M \) is free with basis \( x_1, \ldots, x_r \).

Thus, we will apply the result on free \( A \)-modules.

From \[ 0 \rightarrow M_A \rightarrow A \rightarrow A/M_A \rightarrow 0 \]

we get \[ \text{Tor}_1^A(A/M_A, M) \rightarrow M \otimes_A A/M_A \rightarrow 0 \]

Thus, \[ \text{Tor}_1^A(M, I) = 0 \] for every ideal \( I \) of \( A \).

Therefore, \( M \) is flat.

Noting that \( M \) is a free \( A \)-module, \( M \) is also flat.

Suppose \( \mathbb{F}_p x_i = 0 \) for all \( i \) and \( x_i \in M \) over \( A \).
for \( f: x \mapsto \) is a relation.

As \( \Sigma f_i x_i = 0 \), \( f_i \in x \), we have \( f_i = 0 \) (as \( \exists x_i \) a basis, and so \( f_i \) is \( k \).

Since \( \# ) \Rightarrow \mathfrak{m}_f \mathfrak{m} = \mathfrak{n} \mathfrak{m} \) we get

\[
\sum f_i \circ x_i = 0 \quad \text{in} \quad \mathfrak{m}_f \mathfrak{m}
\]

we have a natural map

\[
\mathfrak{m}_f \mathfrak{m} \rightarrow \frac{\mathfrak{n}}{\mathfrak{m}} \otimes \frac{\mathfrak{n}}{\mathfrak{m}}
\]

\( \rightarrow \)

\( \rightarrow \)

\( k \) vector spaces

and the \( \Sigma f_i x_i \) are a basis for \( \mathfrak{n} \mathfrak{m} \).

So we have \( \sum f_i \circ x_i = 0 \subset \frac{\mathfrak{n}}{\mathfrak{m}} \otimes \frac{\mathfrak{n}}{\mathfrak{m}}
\]

\( \Rightarrow f_i \in \mathfrak{m} \). (As \( f_i = 0 \))

\( \Rightarrow \exists x_i \) give a basis for \( \mathfrak{n} \mathfrak{m} \).

(Exercise! Check this!)

Now using \( \mathfrak{m}_f \mathfrak{m} = \mathfrak{n} \mathfrak{m} \otimes \mathfrak{n} \mathfrak{m} \)

we see \( f_i \in \mathfrak{m} \). Repeating this process, we see \( f_i \in \mathfrak{m} \) but \( \Rightarrow f_i = 0 \) as \( \exists x_i \) are a basis.

Remark: we actually proved that all modules over Artinian rings \( A \), \( \text{Art}_k \) are free.
We now easily deduce the following lemma which will be important in the proof of the theorem.

**Lemma** let \( A \to B \) be a principal small extension in \( \mathfrak{A} \mathfrak{B} \). Let \( f : A \to C \) be a morphism of \( \mathfrak{B} \)-algebras and \( C_0 = C \otimes_A k \). Then \( f \) is flat.

\[
\Rightarrow \text{ ker } (C \to C_0 \otimes_B) \cong C_0 \otimes_B \mathfrak{m}
\]

and \( f' : B \to C_0 \otimes_B \) is flat.

**Proof** Assume \( f \) is flat.

Since flatness is preserved by base change \( f' \) is flat. To see the second proof, as \( A \to B \) is a principal small extension, we have

\[
0 \to k \to A \to B \to 0
\]

and

\[
0 \to C_0 \to C \to C_0 \otimes_B \to 0
\]

Thus \( C \) is flat.

Next assume \( C_0 \) is flat. Have

\[
\text{Tor}_1^A(B, C) \cong k, \quad \text{Tor}_1^A(B, C) \to C \otimes_A k \to C \to C_0 \otimes_B \to 0
\]

so 1st condition \( \Rightarrow \text{Tor}_1^A(B, C) = 0 \).

From \( 0 \to M_B \to B \to k \to 0 \) get

\[
0 = \text{Tor}_1^A(B, C) \to \text{Tor}_1^A(B, C) \to M_B \otimes_A (B \otimes_A C) \to 0
\]
Facts of \( A \otimes_C B \) over \( B \)

\[
\Rightarrow M \otimes_B (B \otimes_C A) \rightarrow B \otimes_C A \text{ injective}
\]

(As we have \( 0 \rightarrow M \otimes_B B \otimes_A C \rightarrow B \otimes_A C \rightarrow B \otimes_B (B \otimes_C A) \rightarrow 0 \))

\[
\Rightarrow \text{Tor}^B_1(B, B \otimes_C A)
\]

\[
\Rightarrow \text{Tor}^A_i(C, B) = \text{Tor}^B_i(B \otimes_C A, B) = 0 \text{ by previous lemma.}
\]

Next time we will finish the theorem.

Enjoy the break!