Lecture 10

If X is a smooth variety over k we know $\mathrm{Def}_X(k[\varepsilon]) \simeq H^1(X, T_X)$.

What if X is an arbitrary (integral) variety?

Background on Ext Groups

Reference: Hartshorne III.6

Let X be a scheme$/k$ and $\mathrm{Mod}(X)$ the category of sheaves of $\mathcal{O}_X$-modules. Recall that $\mathrm{Mod}(X)$ has enough injectives, so that, for each object $\mathcal{F}$ we can find an injective resolution

$$0 \to \mathcal{F} \to I^0 \to I^1 \to \cdots$$

(Recall: I^n injective means $\mathrm{Hom}_X(-, I^n)$ is exact).

We define $\mathrm{Ext}^i_X(\mathcal{G}, -)$ as the i-th right derived functor of $\mathrm{Hom}(\mathcal{G}, -)$, for $\mathcal{G} \in \mathrm{Mod}(X)$, where $\mathrm{Hom}_X(\mathcal{G}, \mathcal{F})$ is the group of $\mathcal{O}_X$-module homomorphisms.

In terms of the resolution above, $\mathrm{Ext}^i(\mathcal{G}, -)$ is given by the ith cohomology of the complex

$$0 \to \underset{\substack{\uparrow \\ \text{degree } 0}}{\mathrm{Hom}_X(\mathcal{G}, I^0)} \to \mathrm{Hom}_X(\mathcal{G}, I^1) \to \mathrm{Hom}_X(\mathcal{G}, I^2) \to \cdots$$

$$\to \mathrm{Hom}_X(\mathcal{G}, I^{i-1}) \to \mathrm{Hom}_X(\mathcal{G}, I^i) \to \mathrm{Hom}_X(\mathcal{G}, I^{i+1}) \to \cdots$$

Notice $\mathrm{Ext}^0_X(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_X(\mathcal{G}, \mathcal{F})$

For $\mathcal{G}, \mathcal{F}$ coherent, X a variety, $\mathrm{Ext}^i(\mathcal{G}, \mathcal{F})$

has the natural structure of a k-vector space.
Hartshorne exercise 6.1, p.g 237 gives a concrete description of $\mathrm{Ext}^1_X(\mathcal{G},\mathcal{F})$ which is useful.
Let $A, B \in \mathrm{Mod}(X)$. Recall that an "extension of B by A" is a short exact sequence ξ given by $0 \to A \to C \to B \to 0 \in \mathrm{Mod}(X)$, and that there is a natural notion of isomorphism of extensions. From the short exact sequence ξ as above, we get a long exact sequence

$$0 \to \mathrm{Hom}_X(B, A) \to \mathrm{Hom}_X(B, C) \to \mathrm{Hom}_X(B, B) \xrightarrow{\delta} \mathrm{Ext}^1_X(B, A) \to \cdots$$

Define $[\xi] := \delta(\mathrm{id}) \in \mathrm{Ext}^1_X(B, A)$.
This gives a bijection (even k-vector space iso) between isomorphism classes of B by A and $\mathrm{Ext}^1_X(B, A)$.

Ext sheaves

Let $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ denote the $\mathcal{H}om$-sheaf.
DO NOT CONFUSE THIS WITH THE GROUP $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$!
(The sheaf is written in curly font!)
$\mathcal{H}om(\mathcal{F}, -)$ gives a left-exact functor from $\mathrm{Mod}(X)$ to itself and the ith-right derived functor is the $\mathcal{E}xt^i$-sheaf.
If X is a variety $/k$, there is a relation between the

cohomology of the sheaf $\mathcal{E}xt^i$ and the group Ext^i, called the local-to-global Ext-spectral sequence. In particular, it gives the following exact sequence:

$$0 \to H^1(X, \mathcal{H}om(\mathcal{F}, \mathcal{G})) \to \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{G})$$
$$\to H^0(X, \mathcal{E}xt^1(\mathcal{F}, \mathcal{G})) \to H^2(X, \mathcal{H}om(\mathcal{F}, \mathcal{G})) \to \mathrm{Ext}^2_X(\mathcal{F}, \mathcal{G})$$

For this see: Huybrechts "Fourier-Mukai transforms in algebraic geometry" §2.3, or the Wikipedia page on the "Grothendieck spectral sequence" and the references therein (also see page on "Five term exact sequence").

We can now state the main theorem for today.

<u>Thm</u> Let X be a variety (i.e. $\Rightarrow X$ integral). Then $\mathrm{Def}_X(k[\varepsilon]) \cong \mathrm{Ext}^1_X(\Omega^1_X, \mathcal{O}_X)$

<u>Remark</u>

We have the exact sequence

$$0 \to H^1(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) \to \mathrm{Ext}^1_X(\Omega^1_X, \mathcal{O}_X)$$
$$\to H^0(X, \mathcal{E}xt^1(\mathcal{F}, \mathcal{G})) \to H^2(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) \to \mathrm{Ext}^2_X(\mathcal{F}, \mathcal{G})$$

The image of the first map is the set of locally trivial, first-order deformations of X, by last week's theorem.

Before proving the theorem, we need a lemma which gives a criterion for flatness over certain local rings.

<u>Lemma</u> Let $A \in \mathrm{Art}_k$, and let M be an A module. Then M is a flat A-module

$$\iff \mathrm{Tor}_1^A(M, k) = 0.$$

<u>Pf</u> M flat $\iff \mathrm{Tor}_1^A(M, A/I)$ for every ideal $I \subset A$.

So obviously M flat $\Rightarrow \mathrm{Tor}_1^A(M, \overset{= A/m_A}{k}) = 0$.

Next, assume $\mathrm{Tor}_1^A(M, k) = 0$.

Consider $M/m_A M$. This is a finite dim. vector space over $k = A/m_A$, so let $x_1, \dots, x_r \in M$ be a set s.t. $\overline{x_1}, \dots, \overline{x_r} \in M/m_A M$ give a basis for $M/m_A M$ over k.

We claim that M is free, with basis $x_1, \dots, x_r$. This will imply the result as free $\Rightarrow$ flat.

From $0 \to m_A \to A \to A/m_A \to 0$

Get

$$\begin{array}{ccccccccc}
\mathrm{Tor}_1^A(A/m_A, M) & \to & m_A \otimes_A M & \to & \overset{M \otimes_A A}{M} & \to & \overset{(=A)}{M/m_A M} & \to & 0 \\
\wr\wr & & & & \wr\wr & & \wr\wr & & \\
\mathrm{Tor}_1^A(M, k) = 0 & & & & & & A/m_A \otimes_A M & &
\end{array}$$

By Nakayama's lemma for nilpotent ideals $\{x_i\}$ generate M over A. Suppose $\sum f_i x_i = 0$

for $f_i \in A$ is a relation.

As $\sum \bar{f}_i \bar{x}_i = 0$, $\bar{f}_i \in k$, we have $\bar{f}_i = 0$ (as $\{\bar{x}_i\}$ a basis) and so $f_i \in m$.

Since (*) $\Rightarrow$ $m \otimes_A M \simeq mM$ we get

$$\sum f_i \otimes x_i = 0 \quad \text{in } m \otimes_A M$$

We have a natural map

$$m \otimes_A M \to m/m^2 \otimes_{A/m} M/mM$$

k vector spaces

and the $\{x_i\}$ are a basis for M/mA.

So we have $\sum \bar{f}_i \otimes \bar{x}_i = 0 \in m/m^2 \otimes_{A/m} M/mM$

$\Rightarrow f_i \in m^2 \ \forall j$. (as $\bar{f}_i = 0$)

$\Rightarrow \{\bar{x}_i\}$ give a basis for M/m^2M

(Exercise: Check this!)

Now using $m \otimes_A M \to m/m^3 \otimes_{A/m^2} M/m^2M$

we see $f_i \in m^3$. Repeating this process, we see $f_i \in m^n \ \forall n \Rightarrow f_i = 0 \Rightarrow \{x_i\}$ are a basis.

<u>Remark</u> we actually proved that flat modules over Artinian rings ~~AG~~ Art_k are free.

We now easily deduce the following lemma which will be important in the proof of the theorem

<u>Lemma</u> Let $A \twoheadrightarrow B$ be a principal small extension in Art_k. Let $f: A \to C$ be a morphism of k-algebras and $C_0 = C \otimes_A k$. Then f is flat

$$\Longleftrightarrow \left.\begin{array}{l} \ker(C \to C \otimes_A B) \simeq C_0 \\ \text{and } f': B \to C \otimes_A B \text{ is flat.} \end{array}\right\} (*)$$

Pf/ Assume f is flat.

Since flatness is preserved by base change f' is flat. To see the second property, as $A \twoheadrightarrow B$ is a principal small extension, we have

$$0 \to k \to A \twoheadrightarrow B \to 0$$

$$\underset{\substack{\| \\ \mathrm{Tor}_1^A(B, C)}}{0} \to \underset{\substack{\wr\wr \\ C \otimes_A k}}{C_0} \to C \twoheadrightarrow C \otimes_A B \to 0$$

as C is flat $\Rightarrow$ second condition holds.

Next assume $(*)$ holds. Have

$$\mathrm{Tor}_1^A(k, C) \to \underset{\substack{\| \\ 0 \\ \text{as } A \text{ flat}}}{\mathrm{Tor}_1^A(A, C)} \to \mathrm{Tor}_1^A(B, C) \to \underset{\substack{\wr\wr \\ C_0}}{C \otimes_A k} \to C \twoheadrightarrow C \otimes_A B \to 0$$

So 1st condition $\Rightarrow \mathrm{Tor}_1^A(B, C) = 0$.

From $0 \to M_B \to B \to k \to 0$ get $\underset{\wr\wr}{M_B \otimes_B (B \otimes_A C)}$

$$0 = \mathrm{Tor}_1^A(B, C) \to \mathrm{Tor}_1^A(k, C) \to M_B \otimes_A C \to B \otimes_A C \to k \otimes_A C \to 0$$

Flatness of $B \otimes_A C$ over B

$\Rightarrow M \otimes_B (B \otimes_A C) \longrightarrow B \otimes_A C$ injective

As we have $\underset{\substack{\uparrow \\ \mathrm{Tor}_1^B(k, B \otimes_A C)}}{0} \to M \otimes_B (B \otimes_A C) \to B \otimes_A C \to k \otimes_B (B \otimes_A C) \to 0)$

$\Rightarrow \mathrm{Tor}_1^A(C, k) = \mathrm{Tor}_1^A(k, C) = 0$ by precision lemma. $\square$

Next time we will finish the theorem.

Enjoy the break!