Recall

Last week we saw Schlessinger's conditions \((H_1, \ldots, H_4)\) on a functor \(F: \text{Art}_k \to \text{Sets}\) with \(F(a)\) a point.

Schlessinger's Theorem (which we will start proving next week), says \(F\) is pro-representable when \((H_1, \ldots, H_4)\) are satisfied.

The goal of this week is to get a feel for Schlessinger's Conditions. We are first going to prove that the Picard functor

\[
\text{Pic}_M: \text{Art}_k \to \text{Sets}
\]

satisfies the Schlessinger conditions.

**Picard Functor**

Let \(X\) be a scheme, \(M \in \text{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)\). For \(A \in \text{Art}_k\), the Picard functor \(\text{Pic}_M \in \text{Fun}\) is defined by

\[
\text{Pic}_M(A) := \{ \xi \in \text{Pic}(X_A) : L \otimes k = M \}
\]

Recall that \(X_A := X \times \text{Spec} A\)

For any morphism \(A \to B\) in \(\text{Art}_k\), there is a pullback induced

\[
\text{Pic}(X_A) = H^1(X_A, \mathcal{O}_A^*) \to \text{Pic}(X_B) = H^1(X_B, \mathcal{O}_B^*)
\]
For \( L \in \text{Pic}(X_A) \), we denote the pull-back of \( L \) as \( L \otimes_A B \). In the definition of \( \text{Pic}_m(A) \) above, \( k \) is the residue field \( A/\mathfrak{m}_A \).

We will make throughout the following 2 assumptions on \( X \)

**Assumptions:**

1. \( H^0(X, \mathcal{O}_X) \cong k \)
2. \( \dim_k H^1(X, \mathcal{O}_X) < \infty \)

For instance, if \( X \) is a proper algebraic variety then the assumptions are satisfied.

We will first show that \( \text{Pic}_m \) satisfies conditions \( CH_1 \) and \( CH_2 \). Recall their definition:

- Let \( A' \to A \), \( A'' \to A \in \text{Art}_k \) and \( F \in \text{Fun} \).

- There is a natural map \( F(A' \times_A A'') \to F(A') \times F(A'') \).

\( F \) satisfies \( CH_1 \) if:

- \( F: A' \to A \in \text{Art}_k \) and any "principal small extension" \( A'' \to A \) (Case) is surjective.

\( F \) satisfies \( CH_2 \) if:

- For \( A = k \), \( A'' = k[[E]] \) the dual numbers \( CE \) is bijective.

Recall: \( A'' \to A \) principal small extension if

\[
\mathfrak{m}_{A''}, (\text{Ker} f) = 0 \quad \text{and} \quad \dim_{A''/\mathfrak{m}_{A''}} \text{Ker} f = 1
\]
We will firstly show that Picm satisfies (C1), (C2).

We start with a lemma.

Lemma 1

Let \( A \) be a ring, \( J \subseteq A \) a nilpotent ideal, and \( \psi : M \rightarrow N \) a morphism of \( A \)-modules such that \( N \) is flat over \( A \).

Assume \( \overline{\psi} : M/\overline{J}M \rightarrow N/\overline{J}N \) is an isomorphism.

Then \( \psi : M \rightarrow N \) is also an isomorphism.

Proof (PF)

Let \( K = \ker \psi \); this is an \( A \)-module.

We have \( M \rightarrow N \rightarrow K \rightarrow 0 \).

Tensor this sequence of \( A \)-modules by \( A/\overline{J} \); as tensoring is right exact, there is an exact sequence \( M/\overline{J}M \rightarrow N/\overline{J}N \rightarrow K/\overline{J}K \rightarrow 0 \).

By assumption \( \overline{\psi} \) is an isomorphism, so \( K/\overline{J}K = 0 \).

Thus \( K = \overline{J}K = J^nK \cap N \).

\( \Rightarrow K = 0 \) as \( J \) is nilpotent.

Hence \( \psi \) is surjective, let \( H = \ker (\psi) \).
We have a short exact sequence of $A$-modules:
$$0 \to H \to M \xrightarrow{u} N \to 0.$$ Since $N$ is flat over $A$, after tensoring by $A/\mathfrak{m}$ we get a short exact sequence:
$$0 \to H/\mathfrak{m}H \to M/\mathfrak{m}M \xrightarrow{\bar{u}} N/\mathfrak{m}N \to 0.$$ Thus $H/\mathfrak{m}H = 0 \Rightarrow H = \mathfrak{m}H = 0 \Rightarrow u$ is an isomorphism.

**Exercise**
Using the lemma above show that if $A, A', A''$ are complete local Noetherian rings, and $M$ is flat over $A$ then $M$ is in fact free. We also need

**Lemma 2**
Support we have a Cartesian diagram of rings $A' \leftarrow A \rightarrow A''$

![Diagram](image)

as well as a Cartesian diagram of $B$ modules:

![Diagram](image)
Assume in addition $M'$ resp. $M''$ resp $M$ are also $A'$ resp. $A''$ resp. $A$ modules, and that the ring and module morphisms above are compatible. Assume $M'$ is free over $A'$ and $M''$ is flat over $A''$.

Further suppose:

(i) There is a nilpotent ideal $S \subseteq A''$ s.t.
\[ A''/S \cong A. \]

(ii) $u'$ resp. $u''$ induces
\[ M' \otimes_A A' \cong M \text{ (iso of $A$ modules)} \]
\[ M'' \otimes_A A'' \cong M. \]

Then, $N$ is flat over $B$ and $p'$ resp. $p''$ induces $N \otimes_B A' \cong M'$ resp. $N \otimes_B A'' \cong M''$.

pf Choose a basis $(X_i)$ for $M'$ as an $A'$ module.

By assumption (i) $(u'(X_i))$, $i \in I$, is a basis for $M'$ as an $A$ module.

By assumptions (i) and (ii)
\[ N \otimes_B A' \cong M' \otimes_A A' \cong M. \]

Choose $X_i \in M''$ s.t.
\[ u''(X_i) = u'(X_i). \]

This gives a morphism $\otimes_{A''} A' X_i \twoheadrightarrow M''$ of $A''$ module.
which is an iso. after tensoring by $A^\times 3$
($A = A^\times 3$ module).
As $S$ is nilpotent, Lemma 1 $\Rightarrow$ $(x_1')$ is a
basis for $M'$ as an $A'$ module.
From the property $u''(x_1) = u'(x_1')$, we have that $M' \times M''$ is free with basis
$X_1' \times X_2''$ as a $B$-module.
(Exercise! Convince yourself that this is true.)
Thus $N = M' \times M''$ is flat over $B$.
The morphism $N \otimes_A A' \rightarrow M'$ induced by $p'$ is
just the iso $X_1' \times X_2'' \rightarrow X_1'$. Hence, the
morphism $N \otimes_A A'' \rightarrow M''$ induced by $p''$ is an isomorphism.

Corollary: Let

$$
\begin{array}{ccc}
M' \times M'' & \rightarrow & M'' \\
p' & \downarrow & u'' \\
M' & \rightarrow & M
\end{array}
$$

be a commutative diagram as above and assume we have a $B$-module $L$ and a commutative
diagram

$$
\begin{array}{ccc}
L & \rightarrow & M'' \\
q' & \downarrow & u'' \\
M' & \rightarrow & M
\end{array}
$$
By the universal property of fibre product, there is an induced morphism \( u: L \to N \) giving a commutative diagram:

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\[ \begin{array}{ccc}
L & \xrightarrow{u} & N \\
\downarrow{q} & \nearrow{p} & \downarrow{e} \\
M' & \to & M'' \\
\downarrow{m} & & \downarrow{m''} \\
M & & M
\end{array} \]
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Assume \( q' \) induces an isomorphism \( L \otimes_B A' \cong M' \).

Then \( u: L \to N \) is an iso.

**Proof:** We have a Cartesian diagram:

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\begin{array}{ccc}
B & \to & A'' \\
\downarrow{L} & \nearrow{J} & \downarrow{\text{blue } J \subseteq A'' \text{ nilpotent}} \\
A' & \to & A' = A''/J
\end{array}
```

**Exercise (very easy):**
Using the explicit construction of fibre products of rings, show \( A' \cong B/J' \), where \( J' \) is nilpotent.

By Lemma 2, \( p' \) induces an iso \( \overline{N} \otimes_B A' \cong M' \).

By the assumption on \( q' \), we see that \( u \) induces an iso \( L \otimes_B A' \cong \overline{N} \otimes_B A' \).

Feeding \( N \) is flat over \( B \) by Lemma 2.
\[
A' \cong B'/j' \text{ for } j' \text{ nilpotent, by the exercise, so } u \text{ is an iso by lemma 1.}
\]

We are finally ready to prove our first result.

**Proposition 1**

Let \( A'' \to A \) be any surjection in \( \mathcal{A}^f_k \).

Then the natural morphism

\[
\text{Pic}_m (A' \times_k A'') \to \text{Pic}(A') \times \text{Pic}(CA'')
\]

is an isomorphism, for any \( A' \to A \) in \( \mathcal{A}^f_k \).

In particular, \( \text{Pic}_m \) satisfies Schlessinger's conditions \((C_1)\) and \((C_2)\).

To ease notation, write \( S \) for \( \text{Pic}_m \mathcal{E}_{\text{Fun.}} \).

Let \((L', L'') \in \text{Pic}(A') \times \text{Pic}(CA'')\).

and let \( L = L' \circ_A A = L'' \circ_{A'} A \) be the pull-back of \( L'' \).

Let \( B = A' \times_{A''} A \).

For any \( A' \to A \), the underlying topological space of \( A \) is a point (as \( \text{Spec } A \) is nilpotent by the Artinian hypothesis \( \implies \text{Spec } A = \text{Spec } A_{\text{nil}} \)).

Let \( L_X \) denote the underlying topological space of \( X \).
We have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O} & \xrightarrow{X_B} & \mathcal{O} \\
\downarrow \mathcal{O} & & \downarrow \mathcal{O} \\
\mathcal{O} & \xrightarrow{X_{A'}} & \mathcal{O} \\
\downarrow \mathcal{O} & & \downarrow \mathcal{O} \\
\mathcal{O} & \xrightarrow{X_{A''}} & \mathcal{O}
\end{array}
\]

of sheaves on \( X \), which induce a canonical morphism \( \mathcal{O} \to \mathcal{O} \) of sheaves on \( X \), by definition, for any open \( U \subseteq X \)

\[
\mathcal{O}(U) = \mathcal{O}(U) \otimes \mathcal{O}(U)
\]

As \( A'' \to A \) is surjective, \( A \cong A'' / S \), where \( S = \ker(A'' \to A) \) is nilpotent.

Thus Cor 3.6 applies and shows that we have a canonical \( \mathcal{O} \to \mathcal{O} \) of sheaves (or \( B \) modules) on \( X \), other varying from sheaves to modules.

Thus \( \mathcal{N} := \mathcal{L} \times \mathcal{L}'' \) is an immediate sheaf on \( X \).

Lemma 2 immediately implies that we have isomorphisms

\[
\mathcal{O} \otimes_B A' \cong \mathcal{L}' \quad \text{and} \quad \mathcal{O} \otimes_B A'' \cong \mathcal{L}''
\]
Thus $N \in \mathcal{P}(\mathcal{B})$ is mapped to $\mathcal{P}(\mathcal{A}^f) \times \mathcal{P}(\mathcal{A})$

and $\mathcal{P}(\mathcal{B}) \to \mathcal{P}(\mathcal{A}^f) \times \mathcal{P}(\mathcal{A})$ is surjective.

It remains to verify injectivity. So, let $M$ be an invertible sheaf on $X$ and assume that there are isomorphisms

$$M \cong A^f \otimes L', \quad M \cong A^f \otimes L''.$$

Precomposing with the morphisms $M \to M \otimes A^f$ and $M \to M \otimes A^f''$ (induced from $B \otimes A^f$ resp. $B \otimes A^f''$) we get morphisms $\varphi : M \to L'$, $\varphi' : M \to L''$ fitting into a diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\varphi} & L' \\
\downarrow{\varphi'} & & \downarrow{a'} \\
L & \xrightarrow{a} & L''
\end{array}
$$

where $\Theta$ is the automorphism of $L$ given by $L \xrightarrow{\Theta} L \otimes_A A \xrightarrow{\varphi} L'$.

Now recall that we have assumed that the natural morphism $L \to H^0(C, \Theta)$ is an isomorphism.
This implies that the natural morphism
\[ A \to \text{ho}(X_A, \Phi_A, \Theta_A) = \text{ho}(X \otimes A) \]
is an isomorphism by the following exercise.

**Exercise!**
Prove that for any \( k \)-module of finite length \( M \) (i.e. finite-dim vector space) the morphism
\[ M \to \text{ho}(X, \Theta \otimes M) \]
is an isomorphism, under the assumptions \( \Theta \neq \text{ho}(\Phi \otimes) \).

Third: use induction on length and the S-lemma.

Thus \( \Theta \) is multiplication by a unit \( \alpha \in A \).

Since \( A^* \to A \) is a surjection, we may lift \( \alpha \) to \( \alpha^* \) in \( A^* \) and then replace \( \alpha^* \) with \( \alpha^* \alpha^* \alpha \). This allows us to assume \( \Theta \) is the identity \( \alpha^* = \alpha^* \).

But then Lemma 2 applies to show
\[ M = N, \]
\[ P(A^*) \to P(A^*) \] is surjective.

Next week: show Pic\(_m\) also satisfies
\( (C_2), (C_4) \). Begin the proof of Schlesinger's Thm.