# LECTURE 8

## last week

We were proving Schlessinger's thm.

We saw

(i) $F$ has a hull $\iff$ $F$ satisfies $(H_1), (H_2), (H_3)$

It remains to prove:

(i) Assume $F$ has a hull. Then $F$ is prorepresentable $\iff$ $F$ satisfies conditions $(H_1), (H_2), (H_3)$.

Suppose $F$ is prorepresentable; say $F \simeq h_R$, $R \in \hat{Art}_k$

Let $A' \to A$, $A'' \to A$ be any morphisms in $Art_k$.

Then $h_R(A' \times_A A'') = Hom(R, A' \times_A A'')$

$$\cong Hom(R, A') \times_{Hom(R,A)} Hom(R, A'') = h_R(A') \times_{h_R(A)} h_R(A'')$$

By the universal property of fibre products

$\Rightarrow (H_1), (H_2), (H_4)$ hold.

Next, suppose $F \in Fun$ s.t. $(H_1), (H_2), (H_3), (H_4)$ hold. In particular, $F$ has a hull, i.e. a complete, local $k$-algebra $R \in \hat{Art}_k$ and some $\xi \in F(R)$ s.t. the induced morph

$$h_R \to F$$

is smooth + bijective on Zariski tangent spaces.

We will prove that the morphism $h_R \to F$ is an isomorphism of functors.

So for any $A \in \mathrm{Art}_{k}$, we need to prove that we have an iso

$$h_R(A) \xrightarrow{\sim} F(A)$$

We will do this by induction on the length of $A$ (i.e. the maximal length $n$ of a chain of ideals $0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = A$)

If $A$ has length $1 \Rightarrow (0)$ is the maximal ideal. $\Rightarrow A \simeq k$. In this case the claim is obvious (as $F(k) = \{p^*\}$).

Let $\pi : A' \to A$ be a principal small extension. Then $\mathrm{length}(A') = \mathrm{length}(A) + 1$.

By the inductive hypothesis

$$h_R(A) \xrightarrow{\sim} F(A) \text{ is bijective.}$$

By smoothness, the map

$$\gamma : h_R(A') \to h_R(A) \times_{F(A)} F(A')$$

$$\wr\wr \text{ as } h_R(A) \simeq F(A)$$

$$F(A')$$

is surjective.

We need to show $\gamma$ is bijective.

For each $\eta \in h_R(A) \simeq F(A)$ consider the fibres $h_R(\pi)^{-1}(\eta) \subseteq h_R(A')$ and $F(\pi)^{-1}(\eta)$

$\gamma$ induces a surjection

$$\gamma : h_p(\Pi)^{-1}(\eta) \twoheadrightarrow F(\Pi)^{-1}(\eta)$$

we need to show that this is injective.

If $I = \ker \Pi$, then we know that $t_p \otimes I$ gives a transitive action of $h_p(\Pi)^{-1}(\eta)$ and $F(\Pi)^{-1}(\eta)$ by conditions $(H_1)$, $(H_2)$.

<u>Exercise</u> Show that condition $(H_4)$, i.e. the fact that $F(A' \times_A A') \xrightarrow{\sim} F(A') \times_{F(A)} F(A')$

$\Rightarrow$ the action of $t_p \otimes I$ on the fibres is <u>free</u>.

(See Somesi pg 57 if you need help).

Suppose $x, y \in h_p(\Pi)^{-1}(\eta)$ are s.t. $\gamma(x) = \gamma(y)$.

Then $\exists\, g \in t_p \otimes I$ with $g \cdot x = y$.

$\gamma(y) = \gamma(g \cdot x) = g \cdot \gamma(x)$ (the action of $t_p \otimes I$ on the 2 fibres is compatible)

$= g \cdot \gamma(y)$

$\Rightarrow g = 1$, as $t_p \otimes I$ acts freely on $F(\Pi)^{-1}(\eta)$.

$\Rightarrow x = y$ ($g = 1$ is actually $0$, the unit of the abelian gp $t_p \otimes I$)

$\square$

# Tangent-Obstruction Theories

Let $F \in \mathrm{Fun}$ be a functor from $\mathrm{Art}_k$ to Sets with $F(k) = \{pt\}$.

We say that $F$ admits a tangent-obstruction theory if there exists finite dimensional $k$-vector spaces $T_1$ and $T_2$ s.t. the following holds:

(1) For any small extension $0 \to M \to B \to A \to 0$ there exists an "exact sequence of sets"

$$T_1 \otimes_k M \rightsquigarrow F(B) \xrightarrow{g} F(A) \xrightarrow{ob} T_2 \otimes_k M$$

This is defined to mean the following: (a) $g, ob$ are maps & if $x \in F(A)$, then $\exists y \in F(B)$ with $g(y) = x$

$\iff ob(x) = 0$ ("exactness at $F(A)$")

(b) we are given an action of $T_1 \otimes_k M$ on the fibres of $F(B) \to F(A)$ and, furthermore, this action is transitive. ("exactness at $F(B)$")

(2) In case $A = k$, the action of $T_1 \otimes_k M$ on fibres is free (i.e. $g.x = x \iff g = 0 \in T_1 \otimes_k M$)

(3) The exact sequence in (1) is functorial.

This means the following:

Let
$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\varphi_M"] & B \arrow[r] \arrow[d, "\varphi_B"] & A \arrow[r] \arrow[d, "\varphi_A"] & 0 \\ 0 \arrow[r] & M' \arrow[r] & B' \arrow[r] & A' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram, with rows small

extensions. Then the following diagram is commutative

$$\begin{tikzcd}
T_1 \otimes_k M \arrow[r, squiggly] \arrow[d, "id \otimes \varphi_M"] & F(B) \arrow[r] \arrow[d, "F(\varphi_B)"] & F(A) \arrow[r, "obs"] \arrow[d, "F(\varphi_A)"] & T_2 \otimes_k M \arrow[d, "id \otimes \varphi_M"] \\
T_1 \otimes M' \arrow[r, squiggly] & F(B') \arrow[r] & F(A') \arrow[r, "obs"] & T_2 \otimes_k M'
\end{tikzcd}$$

Commutativity on the leftmost square is defined to mean that the action of $T_1 \otimes_k M$ resp. $T_1 \otimes M'$ are compatible (i.e. $F(\varphi_B)(g \cdot x) = (id \otimes \varphi_M)(g) \cdot F(\varphi_B)(x)$)

We will prove in a few weeks the following.

<u>Thm</u>

Let $F$ be a prorepresentable functor with $F \simeq h_R$, $R \in \hat{Art}_k$. Then

(i) $F$ admits a tangent obstruction theory with

$$T_1 \simeq F(k[\varepsilon]) \simeq \mathrm{Hom}_{k\text{-alg}}(R, k[\varepsilon]).$$

(ii) Let $d = \dim_k T_1$. Then we have an isomorphism $R \simeq k[[x_1, \dots, x_d]]/J$, with

$J \subseteq M^2$, $M = (x_1, \dots, x_d)$.

We may choose $T_2 = (J/MJ)^\vee$

The reason for the terminology "we may choose" is that, in general, the obstruction space is not unique.

Note that the theorem above shows that finding a tangent-obstruction theory for a prorepresentable functor gives a lot of information about the complete local ring $R$ representing $F$.

<u>Exercise</u> Suppose $F \in \mathrm{Fun}$ admits a tangent-obstruction theory with tangent space $T_1$ and obstruction space $T_2$. Let $T_2'$ be any $k$-vector space containing $T_2$. Then $F$ admits a t.-o. theory with tangent space $T_1$ and obstruction space $T_2'$.

In contrast to the obstruction space, the tangent space of a functor admitting a t.-o. theory IS unique.

<u>Prop</u> Let $F \in \mathrm{Fun}$ be a functor admitting a tangent obstruction theory $(T_1, T_2)$. Then there is canonical isomorphism $T_1 \simeq t_F$.

Pf/ Consider the small extension

$$\begin{array}{ccccccccc} 0 & \to & (x) & \to & k[x]/x^2 & \to & k & \to & 0 \\ & & \wr\wr & & \wr\wr & & & & \\ & & k & & k[\varepsilon] & & & & \end{array}$$

By definition of a t.-o. theory, this sequence defines a free and transitive action of

$T_1$ on $F(k[\varepsilon]) \simeq t_F$

Now there is a canonical element $\Theta_F \in t_F$ defined as the image of the unique element in $F(k)$ under the morphism $k \to k[\varepsilon]$
$1 \mapsto 1$.

We have a canonical bijection
$$\iota : T_1 \to t_F$$
$$v \mapsto v \cdot \Theta_F$$

We will show that we can define a $k$-vector space structure on $t_F$ s.t. $\iota$ is $k$-linear (note: we do not know if $F$ has a good deformation theory!)
For $\lambda \in k$, we have a morph
$$\varphi_\lambda : k[\varepsilon] \to k[\varepsilon]$$
$$\varepsilon \mapsto \lambda \varepsilon$$

This gives $F(\varphi_\lambda) : t_F \to t_F$ and defines multiplication by $\lambda$. We now define addition:
We have small extensions
$$p_i : k[\varepsilon_1, \varepsilon_2] \twoheadrightarrow k[\varepsilon] \quad \text{for } i = 1, 2$$
where $k[\varepsilon_1, \varepsilon_2] := k[x, y]/(x, y)^2$

and $p_i(\varepsilon_j) = \delta_{ij} \varepsilon$

And $(F(p_1), F(p_2))$ defines a map
$$\phi : F(k[\varepsilon_1, \varepsilon_2]) \to t_F \times t_F \simeq T_1 \times T_1$$
from above.

We claim that this is a bijection.
The inverse is defined by the action of $T_1$ on fibres of $p_i : F(k[\varepsilon_1, \varepsilon_2]) \to F(k[\varepsilon])$.
Namely $p_1$ & $p_2$ define two different actions, say $\bullet_1$ & $\bullet_2$, of $T_1$ on $F(k[\varepsilon_1, \varepsilon_2])$.
Let $\Theta \in F(k[\varepsilon_1, \varepsilon_2])$ be the image of

$$p \in F(k) \text{ under } k \to k[\varepsilon_1, \varepsilon_2]$$
$$1 \mapsto 1$$

We define $T_1 \times T_1 \to F(k[\varepsilon_1, \varepsilon_2])$

$$(v_1, v_2) \mapsto v_2 \bullet_1 (v_1 \bullet_2 \Theta)$$

This defines an inverse of $\phi$ (use functoriality)

Exercise

We have a morphism $k[\varepsilon_1, \varepsilon_2] \xrightarrow{g} k[\varepsilon]$

$$\varepsilon_i \mapsto \varepsilon$$

so

$$t_F \times t_F \xrightarrow{\phi^{-1}} F(k[\varepsilon_1, \varepsilon_2]) \xrightarrow{F(g)} t_F$$

defines the addition.

Exercise: Show the bijection $\phi : T_1 \to t_F$ is linear with this vector space structure on $t_F$.

In practice, the main reason to want a tangent-obstruction theory for a prorepresentable functor is that it gives dimension estimate for the complete

ring $R$ representing the functor. These dimension estimates are extremely useful in practice.

Indeed, we will show next week that if $F$ is a prorepresentable functor and $d = \dim(T_1)$, $r = \dim(T_2)$ for a t.-o. theory $T_1, T_2$ and $F = h_R$ then

$$\boxed{d \geq \dim R \geq d - r}$$

$\square$