Last week we were proving Schlessinger's Theorem. We saw:

(i) \( F \) has a hull \( \iff \) \( F \) satisfies \((H_1), (H_2), (H_3)\).

It remains to prove:

(ii) Assume \( F \) has a hull. Then \( F \) is prorepresentable \( \iff \) \( F \) satisfies conditions \((H_1), (H_2), (H_3)\).

Suppose \( F \) is prorepresentable; say \( F \equiv \text{Hom}_{\mathcal{A}}(A, -) \). Let \( A' \rightarrow A, A'' \rightarrow A \) be any morphisms in \( \mathcal{A} \).

Then \( \text{Hom}_F(A' \times_A A'') = \text{Hom}_{\mathcal{A}}\left(KA' \times_A KA''\right) \)

\[ \cong \text{Hom}_{\mathcal{A}}(KA', \text{Hom}_{\mathcal{A}}(A, -)) \times \text{Hom}_{\mathcal{A}}(KA'', \text{Hom}_{\mathcal{A}}(A, -)) \]

By the universal property of the product,\( \implies (H_1), (H_2), (H_3) \) hold.

Next, suppose \( F \in \text{Fun} \) s.t. \((H_1), (H_2), (H_3), (H_4)\) hold. In particular, \( F \) has a hull, i.e., a complete, local \( k \)-algebra \( F \in \mathcal{A}_k \), and some \( g \in F(k) \)

s.t. the induced morphism \( h_g : F \rightarrow \mathcal{A}_k \)

is smooth and bijective on Zariski tangent spaces.

We will prove that the morphism \( h_g : F \rightarrow \mathcal{A}_k \) is an isomorphism of functors.
So for any $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ we need to prove that we have an iso

$$h_r(A) \cong FCA$$

We will do this by induction on the length of $A$ (i.e., the maximal length $n$ of a chain of ideals $0 = I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = A$).

If $A$ has length 1, i.e., $(0)$ is the maximal ideal, $A = I_1$. In this case the claim is obvious ($A = FCA \cong \mathbb{P}^1$).

Let $11: A' \rightarrow A$ be a principal small extension. Then $\text{length}(A') = \text{length}(A) + 1$.

By the inductive hypothesis

$$h_r(A') \cong FCA$$

is bijective.

By smoothness, the map

$$\gamma: h_r(A') \rightarrow h_r(A) \times FCA'$$

is surjective.

We need to show $\gamma$ is bijective.

For each $N \in h_r(A) \cong FCA$ consider the fiber

$$h_r(A')^{-1}(N) \subset h_r(A')$$

and $FCA'^{-1}(N)$.
\( f : h(\mathbb{N}) \times (\mathbb{N}^* \times \mathbb{N}) \rightarrow \mathbb{F}(\mathbb{N}) \times (\mathbb{N}) \)

We need to show that this is injective.

If \( \mathbf{T} \in \text{ker } I \), then we know that \( \mathbf{T} \circ \mathbf{I} \)
give a transitive action of \( h(\mathbb{N})^{-1}(\mathbb{N}) \) and \( \mathbb{F}(\mathbb{N})^{-1}(\mathbb{N}) \) by conditions \((H_1),(H_2)\).

Exercise shows that condition \((H_4)\), i.e., the fact that \( \mathbb{F}(\mathbb{A}) \times_A \mathbb{A}' \cong \mathbb{F}(\mathbb{A}) \times \mathbb{F}(\mathbb{A}) \)

\( \mathbf{t} \) the action of \( \mathbf{T} \circ \mathbf{I} \) on the fibres is free.
(See Somesi pg 57 if you need help).

Suppose \( x, y \in h(\mathbb{N})^{-1}(\mathbb{N}) \) are st. \( \phi(x) = \phi(y) \)

Then \( \exists g \in \mathbb{F}(\mathbb{A}) \) with \( g \cdot x = y \).

\( (g) \cdot \phi(x) = \phi(y) \) (the action of \( \mathbf{T} \circ \mathbf{I} \) on the \( \mathbb{F}(\mathbb{A}) \) fibres is compatible)

\( \Rightarrow g = 1 \) as \( \mathbf{T} \circ \mathbf{I} \) acts freely on \( \mathbb{F}(\mathbb{N})^{-1}(\mathbb{N}) \).

\( \Rightarrow x = y \). (\( g \cdot 1 \) is actually \( 0 \), the kernel of the abelian \( \mathbf{g} \circ \mathbf{t} \circ \mathbf{I} \))
Let $F: R \to \text{Fun}$ be a functor from $R$ to sets with $R(1) = \mathbb{S}$. We say that $F$ admits a tangent-obstruction theory if there exist finite dimensional $R$-vector spaces $T_1$ and $T_2$ such that the following holds:

(1) For any small extension $0 \to M \to R \to A \to 0$ the existence of an exact sequence of sets

\[ T_1 \otimes M \to F(R) \to F(A) \to T_2 \otimes M \]

This is defined to mean the following:

(a) $g \circ f$ are maps \( f(x) \in \text{FCA} \), then \( g(y) \in \text{FCA} \) with \( g(y) = x \)

\[ \Leftrightarrow \, \text{for } (x) = 0 \text{ ("exactness of } \text{FCA} \text{")} \]

(b) we are given an action of $T_1 \otimes M$ on the fibres of $F(R)$ and furthermore, this action is transitive. ("exactness of $\text{FCA}$")

(2) In case $A = k$, the action of $T_1 \otimes M$ on fibres is free (i.e. $g \circ x = x \implies g = 0 \in T_1 \otimes M$)

(3) The exact sequence in (1) is functorial. This means the following:

Let $0 \to M \otimes B \to A \to 0$

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ 0 \to M' \otimes B' \to A' \to 0 \]

be a commutative diagram, with rows small
Extensions. Then the following diagram is commutative
\[ T_e M \rightarrow R(A) \rightarrow F(A) \rightarrow T_2 e M \]
\[ \downarrow \text{id}_M \downarrow R(\epsilon) \downarrow R(A) \downarrow \text{id}_M \]
\[ T_{(e M)} \rightarrow \mathbb{P}B \rightarrow R(A) \rightarrow T_2 e M \]

Commutativity on the leftmost square is desired to mean that the action of \( T e M \) resp. \( T_{(e M)} \) are compatible (i.e., \( R(\epsilon)(g \cdot x) = \text{id} \circ \text{id}_{M}(g) \cdot x \)). \( R(\epsilon)(x) \)

we will prove in a few weeks the following:

Let \( F \) be a pre-representable functor with \( F = \text{Hom}_R \)
\( R = A^e \). Then
(i) \( F \) admits a tangent obstruction theory with
\[ T \cong F(L_{E_1}) \cong \text{Hom}_R (R, L_{E_1}). \]
(ii) Let \( d = \dim T \). Then we have an
\[ R \cong \mathbb{P} \mathbb{E}_X, \ldots, Xd(1)/5 \]
\[ S \circ M \cong (x_0, \ldots, x_d) \cdot v \]
We may choose \( T_2 = C^{S/M} \)
The reason for the terminology "we may choose" is that in general, the obstruction space is not unique.

Note that the theorem above shows that finding a target obstruction theory for a pro-representable functor gives a lot of information about the complete local ring $R$ representing $F$.

Exercise Suppose $F \in \text{Fun}$ admits a target obstruction theory with target space $T_1$ and obstruction space $T_2$. Let $T_2'$ be any 1-vector space containing $T_2$. Then $F$ admits a $T \to 0$ theory at target space $T_1$ and obstruction space $T_2'$.

In contrast to the obstruction space, the target space of a functor admitting a $T \to 0$ theory is unique.

Prop. Let $F \in \text{Fun}$ be a functor admitting a target obstruction theory $(T_1, T_2)$. Then there is a canonical isomorphism $T_1 \cong T$.

Consider the small extension
$$0 \to C \to D \to \mathbb{Z}/2 \to 0$$

by definition of a $T \to 0$ theory, this square defines a free and transitive action of
$T_i$ on $P(b) \times 1$

Now there is a canonical element $0 \in T_i$ defined as the image of the unique element in $P(b)$ under the morphism $1 \to b \times 1$

We have a canonical bijection

$v: T_i \to T_i$

$v \mapsto v \circ 0$

We will show that we can define a $k$-vector space structure on $T_i$ s.t. $e_i$ is $k$-linear (Note: we do not know if $T_i$ has a good definition yet.)

For $1 \leq k \leq n$ we have a morphism

$\psi_k: b \times E_i \to b \times E_i$

$e_i \mapsto e_i^k$

This gives $P(\psi_k): T_i \to T_i$ and define multiplication by $\psi$. We also define addition:

we have small extensions

$P_i^c: b \times E_i \to b \times E_i$ for $i=1,2$

where $b \times E_i \times E_j := b \times E_i \times (E_j)^2$

and $P_i^c(e_i^c) = e_i^c \in E_i$

And $(P_i^c, P_i^c)$ define a map $\phi: P(\psi_1 \times E_i \times E_j) \to T_i \times T_j \times T_i$
We claim that this is a bijection. Its inverse is defined by the action of \( T_i \) on \( \mathbb{R}^2 \) at \( \mathbf{p} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \).

Namely, \( \pi_1 \circ \pi_2 \) define two different actions, say \( i \) and \( \pi_2 \) of \( T_i \) on \( \mathbb{R}^2 \).

Let \( \phi \in \mathbb{R}^2 \) be the image of \( \phi \in \mathbb{R} \) under \( \phi \rightarrow \mathbb{R}^2 \).

We define \( T_i \times T_i \rightarrow \mathbb{R}^2 \cdot \mathbb{R}^2 \):

\[
(\mathbf{v}_1, \mathbf{v}_2) \mapsto \mathbf{v}_2 \cdot (\mathbf{v}_1 \cdot \phi)
\]

This defines an inverse of \( \phi \) (use functoriality).

Exercise:

We have a morphism \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \):

\[
\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3
\]

so \( T_i \times T_i \rightarrow \mathbb{R}^2 \cdot \mathbb{R}^2 \rightarrow \mathbb{R} \).

This defines the addition.

Exercise: Show the bijection \( \phi : T_i \rightarrow T_i \) is linear.

In practice, the main reason to want a tangent-destabilization theory for a prerepresentable functor is that it gives dimension estimate for the complete...
ring $R$ representing the functor. These dimension estimates are extremely useful in practice.

Indeed, we will show next week that if $F$ is a nonrepresentable functor and $d = \dim (\mathcal{T}_i)$, $r = \dim (\mathcal{T}_2)$ for a t.o. theory $\mathcal{T}_1, \mathcal{T}_2$, and $F = h_p$, then

\[ d \geq \dim R \geq d - r \]