Deformation Theory

Michael Kemeny*

Lecture 9

1 Deformations of a scheme

Conventions. By a ring we will mean a noetherian commutative ring with unity. By a scheme we will mean a locally noetherian, seperated scheme.

Definition 1.1. Let X be a scheme over an algebraically closed field k. Let $A \in \mathrm{Art}_k$. An *infinitesemal deformation of* X *over* A is a cartesian diagram

$$\begin{tikzcd} X \arrow[r, "\iota_Y"] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

where $Y \to \operatorname{Spec} A$ is flat.

Definition 1.2. An isomorphism of two infinitesimal deformations $Y, Y' \to \operatorname{Spec} A$ of X is an isomorphism $\varphi : Y \to Y'$ over A such that $\varphi \circ \iota_Y = \iota_{Y'}$. Hence, there is a commutative diagram as follows

$$\begin{tikzcd} & X \arrow[dl, "\iota_Y"'] \arrow[dr, "\iota_{Y'}"] & \\ Y \arrow[rr, "\varphi", "\sim"'] \arrow[dr] & & Y' \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$

Definition 1.3. Let $f : A \to B \in \mathrm{Art}_k$. The *pullback* of a deformation $Y \to \operatorname{Spec} A$ of X via f is defined to be $f^*Y \to \operatorname{Spec} B$ together with the map $f^*\iota_Y : X \to f^*Y$ induced by the universal property of the pullback.

Definition 1.4. We define the *functor of infinitesimal deformations of* X as $\mathrm{Def}_X \in \mathrm{Fun}^*(\mathrm{Art}_k, (\mathrm{Sets}))$ such that

$$\mathrm{Def}_X(A) = \{\text{Infinitesimal deformations of } X \text{ over } A\} / \sim$$

where the equivalence is that of isomorphism.

*TeX by Emre Sertöz. Please email suggestions or corrections to emresertoz@gmail.com

Definition 1.5. A deformation $(Y \to \operatorname{Spec} A, \iota_Y)$ of X is called trivial if it is isomorphic to the pullback of the "deformation" $(X \to \operatorname{Spec} k, \mathrm{Id}_X)$ via the structure morphism $s_A : \operatorname{Spec} A \to \operatorname{Spec} k$. A deformation $Y \to \operatorname{Spec} A$ of X is called *locally trivial* if there exists an open cover $\{U_i\}$ of Y such that U_i is a trivial deformation of $U_i \times_{\operatorname{Spec} A} \operatorname{Spec} k$.

There is another way to phrase local triviality. Observe that as a morphism of topological spaces $\iota_Y : |X| \to |Y|$ is a homeomorphism for any infinitesimal deformation (Y, ι_Y) (see the proof of the following lemma). Therefore, if U_i is an open subscheme in X then we can restrict the structure sheaf of Y to the open set $|U_i|$ to get an open subscheme U_i' of Y. Hence, an infinitesimal deformation (Y, ι_Y) is locally trivial iff there exists an open cover $\{U_i\}$ of X such that the induced open subscheme $U_i' \subset Y$ is a trivial deformation of U_i for each i.

Notation. We define the subfunctor Def_X^* of Def_X which gives the locally trivial deformations of X.

Technically speaking, we can skirt around the following lemma to prove the final theorem of this lecture as well as its corollary. However, we provide it here for completeness. Those of you who want to take the exam should keep in mind that **the following lemma is not part of the curriculum.**

Lemma 1.6 ([Ser06] pg. 23, Lemma 1.2.3)**.** *An infinitesimal deformation of a noetherian affine scheme is noetherian affine.*

Proof. Let $Z_0 = \operatorname{Spec} R_0$ be a noetherian affine scheme over k and $A \in \mathrm{Art}_k$. Consider an infinitesimal deformation of Z_0:

$$\begin{array}{ccc} Z_0 & \xrightarrow{j} & Z \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} A \end{array}$$

Step 1

We will first show that j is a closed immersion with coherent nilpotent ideal of vanishing. This is local on the codomain so we will assume $Z = \operatorname{Spec} B$ with $A \to B$ flat. Then $Z_0 = \operatorname{Spec}(B \otimes_A k)$. Tensoring the exact sequence $0 \to \mathfrak{m}_A \to A \to k \to 0$ with the flat A-algebra B we get:

$$0 \longrightarrow \mathfrak{m}_A \otimes_A B \longrightarrow B \longrightarrow B \otimes_A k \longrightarrow 0$$

Since $\mathfrak{m}_A$ is nilpotent in A so is $\mathfrak{m}_A \otimes_A B$.

Step 2

We will show that j is a homeomorphism. This is local on the codomain so we may assume Z affine. A closed immersion of an affine scheme cut out by a nilpotent ideal is clearly a homeomorphism.

A quicker way to see this is to note $Z_0 \simeq V(N)$ as a subscheme of Z and that $|V(N)| = |Z|$ by definition of $V(N)$.

Step 3

If r is the smallest positive integer such that $N^r = 0$ then we have

$$Z = V(N^r) \supset V(N^{r-1}) \supset \cdots \supset V(N) = Z_0.$$

Therefore, our main result is true by induction on r if it is true when $r = 2$. Assuming $r = 2$ we note that N obtains the structure of an $\mathcal{O}_{Z_0}$-module. We use the homeomorphism j and the fact that cohomology of modules can instead be computed by treating them as $\mathbb{Z}$-modules to conclude that

$$\mathrm{H}^1(Z, N) = \mathrm{H}^1(Z_0, N) = 0$$

where the last equality follows because Z_0 is affine.

Consequently we have the following exact sequence, where $R := \mathrm{H}^0(Z, \mathcal{O}_Z)$:

$$0 \longrightarrow N(Z) \longrightarrow R \longrightarrow R_0 \longrightarrow 0$$

Let $Z' = \operatorname{Spec} R$, the exact sequence above gives us a map $Z_0 \to Z'$ which is a homeomorphism. Moreover the map $R \xrightarrow{=} \mathrm{H}^0(Z, \mathcal{O}_Z)$ gives us a map of schemes $\theta : Z \to Z'$ which fits into a commutative diagram as follows:

$$\begin{tikzcd}
Z \arrow[rr, "\theta"] & & Z' \\
& Z \arrow[ul, "j"] \arrow[ur] &
\end{tikzcd}$$

Since the two arrows from Z are homeomorphisms, so must θ. It remains to show that $\mathcal{O}_{Z'} \to \mathcal{O}_Z$ is an isomorphism.

Step 4

Observe that Z is quasi-compact because θ identifies the underlying topological space with that of Z', which is affine hence quasi-compact. Moreover, because Z' is seperated, intersection of any two affine subschemes of Z' are affine and in particular quasi-compact. For any $f \in R$ note that the open sets $Z_f \subset Z$ and $D(f) \subset Z'$ coincide. In particular we have shown that for any $f, g \in R$ the intersection $Z_f \cap Z_g$ is quasi-compact. Now we may apply Ex. II.2.16d of [Har77] which states that $\Gamma(Z_f, \mathcal{O}_Z) \simeq R_f$. This implies that $\mathcal{O}_{Z'} \to \mathcal{O}_Z$ is an isomorphism. $\square$

Theorem 1.7. *Any infinitesimal deformation of a smooth affine scheme is trivial.*

Proof. Using Lemma 1.6 we reduce to deformations of algebras. Let B_0 be a smooth algebra over k and let $A \in \mathrm{Art}_k$. Consider a deformation of algebras

$$\begin{tikzcd}
B_0 & B \arrow[l] \\
k \arrow[u] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

We wish to show that B is isomorphic to $B_0 \otimes_k A$ in a way that respects the map $B \otimes_A k \simeq B_0$.

We will do this by induction on $\dim A$. The base case $\dim A = 1$ implies $A = k$ and there is nothing to prove. If $\dim A \geq 2$, choose $0 \neq t \in \mathfrak{m}_A$ and consider the following principal small extension

$$0 \longrightarrow (t) \longrightarrow A \longrightarrow A' \longrightarrow 0$$

We have the following diagram

$$\begin{tikzcd} B \arrow[r] & B \otimes_A A' \arrow[r] & B_0 \\ A \arrow[u] \arrow[r] & A' \arrow[u] \arrow[r] & k \arrow[u] \end{tikzcd}$$

By induction hypothesis $B \otimes_A A' \simeq B_0 \otimes_k A'$. There is a map $B_0 \otimes_k A \to B_0 \otimes_k A'$ and we wish to lift the map $B \to B_0 \otimes_k A'$ to $B_0 \otimes_k A$.

Since $A \to B$ is flat and the only geometric fiber of the map is smooth we conclude that B is smooth over A (see [Har77] Thm III.10.2). Moreover we have the following exact sequence

$$0 \longrightarrow B_0 \otimes_k (t) \longrightarrow B_0 \otimes_k A \longrightarrow B_0 \otimes_k A' \longrightarrow 0$$

where the kernel is a square zero ideal because $(t) \subset A$ is. Therefore we may use the smoothness criterion ([Ser06] Theorem C.9) to conclude that

$$\hom_A(B, B_0 \otimes_k A) \to \hom_A(B, B_0 \otimes_k A')$$

is surjective. In particular, the lift we are after exists.

Exercise. Compare the definition of smoothness for functors with this lifting property of smooth algebras.

It remains to show that the map $B \to B_0 \otimes_k A$ is an isomorphism. Clearly $B_0 \otimes_k A$ is free and thus flat over A, the map is an isomorphism on the special fiber from which it follows that the map itself is an isomorphism by Lemma 3.1 of Lecture 5 (or [Ser06] Lemma A.4). $\square$

We will now study locally trivial deformations. Something that is locally trivial is formed by gluing trivial objects by isomorphisms. Therefore the following lemma will be crucial for our understanding of the functor Def_X^*. Note however that this is a simplified version of [Ser06] Lemma 1.2.6. We will eventually need the stronger version but this will do for now.

Lemma 1.8. *Let A be a k-algebra and $A[\varepsilon] = A \times_k k[\varepsilon]$. Denote by $\mathrm{Aut}^*_{k[\varepsilon]}(A[\varepsilon])$ the group of $k[\varepsilon]$-algebra automorphisms of $A[\varepsilon]$ that induce the identity on $A[\varepsilon]/(\varepsilon) \simeq A$. Denote by $\mathrm{Der}_k(A)$ the k-derivations of A into itself.*

There is a natural isomorphism of groups

$$\mathrm{Aut}^*_{k[\varepsilon]}(A[\varepsilon]) \xrightarrow{\sim} \mathrm{Der}_k(A).$$

Proof. Since there is a natural splitting $A[\varepsilon] = A \oplus \varepsilon A$ as a k-vector space, we may define two projections from $A[\varepsilon]$ to A which we will denote by π_1 and π_2, such that the first projection is a morhpism of algebras whereas the second projection is only k-linear. Letting $\psi \in \mathrm{Aut}^*_{k[\varepsilon]}(A[\varepsilon])$, we may write $\psi = \psi_1 + \varepsilon \cdot \psi_2$ where $\psi_i = \pi_i \circ \psi$.

By $k[\varepsilon]$-linearity we have $\psi(a + \varepsilon b) = \psi(a) + \varepsilon\psi(b)$. Expanding this out gives the following identities:

$$\begin{array}{rcl} \psi_1(a + \varepsilon b) & = & \psi_1(a) \\ \psi_2(a + \varepsilon b) & = & \psi_2(a) + \psi_1(b). \end{array}$$

Finally we combine this with our assumption that ψ descends to the identity on A to conclude $\psi_1 = \pi_1$. Therefore, given our restrictions on ψ only $\mathrm{d} := \psi_2|_A$ is not completely determined.

We now prove that d is a k-derivation of A. Indeed, for $c \in k$ we have $\psi(c) = c\psi(1)$ and thus $\mathrm{d}(c) = 0$. If $a, b \in A$ then expanding out $\psi(ab) = \psi(a)\psi(b)$ gives $\mathrm{d}(ab) = a\,\mathrm{d}(b) + b\,\mathrm{d}(a)$.

Conversely, begin with a k-derivation $\mathrm{d} : A \to A$ and define

$$\psi := \pi_1 + \varepsilon(\mathrm{d} + \pi_2) : A[\varepsilon] \to A[\varepsilon].$$

It is left as an exercise to show that ψ is a $k[\varepsilon]$-algebra automorphism.

We have thus established a bijection between automorphisms of $A[\varepsilon]$ and k-derivations of A. It is clear that this bijection respects the group structure and is thus a group isomorphism. $\square$

Let X be a scheme, then we denote $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ by $\mathcal{T}_X$ and call it *the tangent bundle of* X.

Theorem 1.9. *Let X be a variety over k (ie. integral, seperated scheme of finite type over k). Then the tangent space of the functor of locally trivial deformations of X is canonically isomorphic to $\mathrm{H}^1(X, \mathcal{T}_X)$.*

Proof.

Step 1: From locally trivial families to cohomology classes

Let $\mathcal{X} \to \operatorname{Spec} k[\varepsilon]$ be a locally trivial infinitesimal deformation of X. Choose an affine open cover $\{U_i \to X\}$ such that $\mathcal{X}|_{U_i}$ is a trivial deformation of U_i. Choose isomorphisms $\theta_i : U_i \times \operatorname{Spec} k[\varepsilon] \to \mathcal{X}|_{U_i}$ and by pulling back these isomorphisms we define automorphisms $\theta_{ij} := \theta_i^{-1}\theta_j : U_{ij} \times \operatorname{Spec} k[\varepsilon] \to U_{ij} \times \operatorname{Spec} k[\varepsilon]$ where $U_{ij} := U_i \cap U_j = U_i \times_X U_j$.

Each U_{ij} is affine because X is seperated and thus Lemma 1.8 corresponds to each automorphisms θ_{ij} a derivations of $\Gamma(U_{ij}, \mathcal{O}_X)$, equivalently an element $d_{ij} \in \Gamma(U_{ij}, \mathcal{T}_X)$. Since $\theta_{ij}\theta_{jk}\theta_{ik}^{-1} = \mathrm{Id}$ we conclude $d_{ij} + d_{jk} - d_{ik} = 0$. Thus $\{d_{ij}\}$ is a Check 1-cocycle and therefore gives a class in $\mathrm{H}^1(X, \mathcal{T}_X)$.

Step 2: Showing invariance on the isomorphism class

We made three choices in constructing the cohomology class from the isomorphism class of a locally trivial family. First we chose a representing family, second we chose a trivializing cover $\{U_i\}$ and then we chose trivializing isomorphisms θ_i. With one stroke we can show the invariance of the resulting cohomology class on the first and third choices whilst the second one is an easy exercise:

Exercise. Show the cohomology class does not depend on the trivializing cover $\{U_i\}$.

Let $\mathcal{X}' \to \operatorname{Spec} k[\varepsilon]$ be another deformation and let $\Psi : \mathcal{X}' \to \mathcal{X}$ be an isomorphism of deformations. The cover $\{U_i \to X\}$ also trivializes $\mathcal{X}'$ and therefore we choose trivializing isomorphisms $\theta_i' : U_i \times \operatorname{Spec} k[\varepsilon] \to \mathcal{X}'|_{U_i}$. Let $d_{ij}' \in \Gamma(U_{ij}, \mathcal{T}_X)$ be constructed as above, this time using θ_i''s. Let $\mu_i : U_i \times \operatorname{Spec} k[\varepsilon] \to U_i \times \operatorname{Spec} k[\varepsilon]$ be the isomorphism given by $\theta_i^{-1}\Psi\theta_i'$. Observe the following:

$$\begin{aligned} \theta_i'^{-1}\theta_j' &= \theta_i'^{-1}\Psi\Psi^{-1}\theta_j' \\ &= (\theta_i\mu_i)^{-1}(\theta_j\mu_j) \\ &= \mu_i^{-1}\theta_i^{-1}\theta_j\mu_j \end{aligned}$$

Therefore, letting $u_i \in \Gamma(U_i, \mathcal{T}_X)$ be the derivation associated to μ_i we get $d_{ij}' = d_{ij} + u_j - u_i$. In other words, the cohomology classes associated to $\{d_{ij}'\}$ and $\{d_{ij}\}$ are equal.

If we take $\mathcal{X}' = \mathcal{X}$ and $\Psi = \mathrm{Id}$ then this also proves the independence of the class on the choice of trivializing isomorphisms θ_i.

Step 3: From cohomology classes to deformations

Now we would like to reverse the operation and get a locally trivial deformation of X from a cohomology class $\xi \in \mathrm{H}^1(X, \mathcal{T}_X)$. Choose an affine open cover $\{U_i \to X\}$. By [Har77] Theorem III.4.5 we may calculate the sheaf cohomology using Čheck cohomology with the cover $\{U_i\}$. Thus $\xi = \{d_{ij}\}$ with $d_{ij} \in \Gamma(U_{ij}, \mathcal{T}_X)$ and $U_{ij} = U_i \cap U_j$ as before.

By Lemma 1.8 each d_{ij} corresponds to an isomorphism $\theta_{ij} : U_{ij} \times \operatorname{Spec} k[\varepsilon] \to U_{ij} \times \operatorname{Spec} k[\varepsilon]$. We glue the schemes $U_i \times \operatorname{Spec} k[\varepsilon]$ along the open sets $U_{ij} \times \operatorname{Spec} k[\varepsilon]$ via these isomorphisms θ_{ij} to obtain a scheme $\mathcal{X}$ (see [Har77] Ex. II.2.12).

Since the automorphisms θ_{ij} don't interfere with the projection onto $\operatorname{Spec} k[\varepsilon]$ we can glue these projections to get a map $\mathcal{X} \to \operatorname{Spec} k[\varepsilon]$. Similarly, the maps $U_i \to \mathcal{X}$ glue to a map $X \to \mathcal{X}$ because each automorphism θ_{ij} is an automorphism of deformations, therefore it fixes the special fiber $U_{ij} \to U_{ij} \times \operatorname{Spec} k[\varepsilon]$. This gives us the following diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[\varepsilon] \end{array}$$

The map $\mathcal{X} \to \operatorname{Spec} k[\varepsilon]$ is flat because flatness is local on the domain and locally this map is just the projection from a trivial family. Moreover, the map $\mathcal{X} \to \operatorname{Spec} k[\varepsilon]$ is proper because $X \to \operatorname{Spec} k$ is proper (since any morphism from $k[\varepsilon]$ to a DVR kills ε the implication is immediate).

Step 4: Invariance on the representative of the cohomology class

We would like to show now that the isomorphism class of the deformation constructed from $\xi \in \mathrm{H}^1(X, \mathcal{T}_X)$ is invariant with respect to the choice of a representative in ξ. This requires first that we show invariance under refinements of the cover we use for Čheck cohomology, which is easy, and then invariance for two different representatives $\{d_{ij}\}, \{d'_{ij}\} \in \xi$. Let $\mathcal{X}$ correspond to $\{d_{ij}\}$ and $\mathcal{X}'$ correspond to $\{d'_{ij}\}$. We will show that these two deformations are isomorphic. To do this, use the fact that $\{d_{ij} - d'_{ij}\} = \{a_j - a_i\}$ where $a_i \in \Gamma(U_i, \mathcal{T}_X)$. The a_i's correspond to isomorphisms $\alpha_i : U_i \times \operatorname{Spec} k[\varepsilon] \to U_i \times \operatorname{Spec} k[\varepsilon]$ and these α_i's glue to an isomorphism $\mathcal{X} \to \mathcal{X}'$. □

Using Theorem 1.7 together with Theorem 1.9 we have the following.

Corollary 1.10. *If X is a smooth variety over k then its first order deformations are in bijection with* $\mathrm{H}^1(X, \mathcal{T}_X)$.

References

[Har77] R. Hartshorne, *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496, Graduate Texts in Mathematics, No. 52.

[Ser06] E. Sernesi, *Deformations of algebraic schemes*, ser. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006, vol. 334, pp. xii+339.