# Formal moduli

Let $X$ be a scheme over an algebraically closed field $k$.

Let $A \in \mathrm{Art}_k$. An infinitesimal deformation of $X$ is a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & Y \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} A \end{tikzcd} \qquad \text{i.e. } X \xrightarrow{\sim} Y \times_{\operatorname{Spec} A} \operatorname{Spec} k$$

where $Y$ is flat over $\operatorname{Spec} A$. (i.e. $Y$ is a family of schemes $/\operatorname{Spec} A$ with special fibre $\simeq X$).

If
$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & Y' \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$
is a second deformation of $X$,

we say that the deformations $Y$, $Y'$ are isomorphic if $\exists$ an isomorphism $f: Y \xrightarrow{\sim} Y'$ over $A$ (i.e.
$$\begin{tikzcd} Y \arrow[rr, "f", "\sim"'] \arrow[dr] & & Y' \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$
) s.t. the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & Y \arrow[d, "\wr f"] \\ X \arrow[r] & Y' \end{tikzcd}$$
commutes, i.e. $f$ induces the identity on the closed fibre $X$.

We define the deformation functor $\mathrm{Def}_X \in \mathrm{Fun}$ by $\mathrm{Def}_X(A) = \{$ isomorphism classes of deformations of $X/k$ over $\operatorname{Spec} A\}$.

The deformation functor is defined on morphisms via pull-back, i.e. let $f: A \to B \in \mathcal{A}rt_k$ & consider the induced morphism $f^*: \operatorname{Spec} B \to \operatorname{Spec} A$.

If $\begin{array}{c} Y \\ \downarrow \\ \operatorname{Spec} A \end{array} \in \operatorname{Def}_X(A)$ then we have an element

$Y \times_{\operatorname{Spec} A} \operatorname{Spec} B \in \operatorname{Def}_X(B)$. This defines the deformation functor on morphisms.

$$\begin{tikzcd} Y \times_{\operatorname{Spec} A} \operatorname{Spec} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & Y \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

In general, the functor $\operatorname{Def}_X$ need not be pro-representable. However, up to a rather weak finiteness condition, it does at least have a hull, which is sufficient for many applications.

We start by studying the Zariski tangent space $t_{\operatorname{Def}_X} \simeq \operatorname{Def}_X(k[\varepsilon])$ to the deformation functor of $X$.

Let $Y \in \operatorname{Def}_X(k[\varepsilon])$. We say $Y$ is trivial if we have an isomorphism of deformations of $X$:

$$Y \simeq X \times_{\operatorname{Spec} k} \operatorname{Spec} k[\varepsilon]$$

We say $Y$ is locally trivial if there exists an open cover $\{U_i\}$ of $|X| = |Y|$ s.t.

$$Y|_{U_i} \simeq U_i \times_{\operatorname{Spec} k} \operatorname{Spec} k[\varepsilon]$$

Let $\Omega_X$ be the sheaf of Kähler differentials (Hartshorne, Ch II. 8)

<u>Thm</u> Let $X$ be a variety $/k$ (i.e. integral, separated scheme of finite type $/k$). Then there is a bijection

$$\{ Y \in \mathrm{Def}_X(k[\varepsilon]) \mid Y \text{ locally trivial} \}$$

$$\longleftrightarrow H^1(X, \mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X))$$

Pf: Let $Y \in \mathrm{Def}_X(k[\varepsilon])$ be locally trivial. We will associate an element $[Y] \in H^1(X, \mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X))$ to $Y$. We will use Čech cohomology (Hartshorne III. 4) for this.

Choose an affine open cover $\{U_i\}_{i \in I}$ of $|X|$ s.t.

$$Y|_{U_i} \simeq U_i \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k[\varepsilon]$$

To give a Čech 1-cocycle we need to specify, for each $i < j \in I$ (we put a well-ordering on $I$) an element $\alpha_{ij} \in \Gamma(U_{ij}, \mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X))$, where $U_{ij} := U_i \cap U_j$.

Further, we need the collection $\alpha = \{\alpha_{ij}\}_{i<j}$ to satisfy $d\alpha = 0$, which translates to demanding, for each $i < j < k \in I$,

$$\alpha_{j,k} - \alpha_{i,k} + \alpha_{i,j} = 0 \text{ on } U_{i,j,k} := U_i \cap U_j \cap U_k$$

For each $i \in I$ we have an isomorphism of deformations

$$\Theta_i : U_i \times \operatorname{Spec}(k[\varepsilon]) \to Y|_{U_i}$$

For $i, j \in I$ define

$$\Theta_{ij} := \Theta_i^{-1}\Theta_j : U_{ij} \times \operatorname{Spec}(k[\varepsilon]) \to U_{ij} \times \operatorname{Spec}(k[\varepsilon]).$$

Now $X$ is assumed separated $\Rightarrow U_{ij}$ is affine, so let $U_{ij} = \operatorname{Spec} A_{ij}$.

$$\Gamma(\operatorname{Spec} A_{ij}, \mathcal{H}om(\Omega^1_X, \mathcal{O}_X))$$

$$\simeq \operatorname{Hom}_{A_{ij}}(\Omega_{A_{ij}}, A_{ij})$$

$$\simeq \operatorname{Der}_k(A_{ij}, A_{ij})$$ by the universal property of the module of Kähler differentials.

Here $\operatorname{Der}_k(A_{ij}, A_{ij})$ is the set of $k$-derivations, i.e. additive maps $d : A_{ij} \to A_{ij}$ satisfying

(i) $d(ab) = a\,db + b\,da$, (ii) $d\lambda = 0$ for $\lambda \in k$

$$\left( \begin{array}{l} k \hookrightarrow A_{ij} \\ \lambda \mapsto \lambda \cdot 1 \end{array} \right)$$

$\operatorname{Der}_k(A_{ij}, A_{ij})$ is an $A_{ij}$-module

Now the morphism $\Theta_{ij}$ corresponds to an automorphism

$$\Theta_{ij}^{\#} : A_{ij} \otimes_k k[\varepsilon] \xrightarrow{\sim} A_{ij} \otimes_k k[\varepsilon]$$ of

rings which

(i) is $k[\varepsilon]$ linear, i.e. $\Theta_{ij}^{\#}(a \otimes \lambda) = \Theta_{ij}^{\#}(a \otimes 1) \otimes 1 \otimes \lambda$

(ii) induces the identity mod $\varepsilon$,

since $\Theta_{i,j}$ is an isomorphism of deformations.

From (ii), $\Theta^t_{i,j} - id = \varepsilon \psi_{i,j}$, where $\psi_{i,j}$ is a map $A_{i,j} \otimes_k k[\varepsilon] \to A_{i,j}$

Using (i), $\Theta^t_{i,j}(1 \otimes \lambda) = 1 \otimes \lambda = id(1 \otimes \lambda)$

$\Rightarrow \psi_{i,j}(1 \otimes \lambda) = 0$ for $\lambda \in k[\varepsilon]$

We claim $\psi_{i,j} : A_{i,j} \otimes k[\varepsilon] \to A_{i,j}$ is a $k[\varepsilon]$ derivation.

We need to prove $\psi_{i,j}(xy) = \bar{x}\psi_{i,j}(y) + \bar{y}\psi_{i,j}(x)$ for

$x, y \in A_{i,j} \otimes k[\varepsilon]$, $\bar{x}, \bar{y} \in A_{i,j}$

Write $x = x_1 \otimes 1 + \varepsilon x_2 \otimes 1$, $x_i \in A_{i,j}$

$\Theta^t_{i,j}(x) = \Theta^t_{i,j}(x_1 \otimes 1) + \varepsilon \Theta^t_{i,j}(x_2 \otimes 1)$

We may also write $\Theta^t(\ ) = \phi_1(\ ) + \varepsilon \phi_2(\ )$ for

$\phi_i : A_{i,j} \otimes k[\varepsilon] \to A_{i,j}$ maps.

So $\Theta^t_{i,j}(x) = \phi_1(x_1 \otimes 1) + \varepsilon(\phi_2(x_1 \otimes 1) + \phi_1(x_2 \otimes 1))$

(ii) $\Rightarrow \phi_1(z \otimes 1) = z \quad \forall z \in A_{i,j} \Rightarrow \Theta^t_{i,j}(x) = x + \varepsilon\phi_2(x_1 \otimes 1)$

so $\psi_{i,j}(x) = \phi_2(x_1 \otimes 1)$

Easy exercise Using this formula, show that $\psi_{i,j} : A_{i,j} \otimes k[\varepsilon]$ is a derivation (you need to use that $\Theta^t_{i,j}$ is a morphism of rings)

Thus $\psi_{i,j} : A_{i,j} \otimes k[\varepsilon] \to A_{i,j}$ is a $k[\varepsilon]$ derivation.

defined by $\psi_{i,j}(x) = \phi_2(x_1 \otimes 1)$

We may define a $k$-derivation

$$\tilde{\psi}_{ij} : A_{ij} \to A_{ij} \quad \text{by}$$

$$\tilde{\psi}_{ij}(x_1) := \psi_{ij}(x_1 \otimes 1)$$

Thus, we have associated a collection $\{\tilde{\psi}_{ij}\}_{i<j}$ with $\tilde{\psi}_{ij} \in \Gamma(\operatorname{Spec} A_{ij}, \mathcal{H}om(\Omega^1_{X}, \mathcal{O}_X)) \simeq \operatorname{Der}_k(A_{ij}, A_{ij})$

We need to show that we have a 1-cocycle, i.e.

$$\tilde{\psi}_{j,k} - \tilde{\psi}_{i,k} + \tilde{\psi}_{i,j} = 0 \quad \text{on } \operatorname{Spec}(A_i) \cap \operatorname{Spec}(A_j) \cap \operatorname{Spec}(A_k)$$

We have $\theta_{ij} \circ \theta_{jk} \circ \theta_{ik}^{-1} = id$ on $U_{ijk}$

by definition

so $\theta_{ij}^{t} \circ \theta_{jk}^{t} \circ (\theta_{ik}^{t})^{-1} = id$ (A). We may represent

$\theta_{ij}^{t} : A_{ij} \otimes k[\varepsilon] \to A_{ij} \otimes k[\varepsilon]$ as a 2-2 matrix $\begin{pmatrix} id & 0 \\ \tilde{\psi}_{ij} & id \end{pmatrix}$ with entries maps $A_{ij} \to A_{ij}$

since

$$\begin{pmatrix} id & 0 \\ \alpha & id \end{pmatrix} \circ \begin{pmatrix} id & 0 \\ \beta & id \end{pmatrix} = \begin{pmatrix} id & 0 \\ \alpha+\beta & id \end{pmatrix}$$

(A) give $\tilde{\psi}_{j,k} - \tilde{\psi}_{i,k} + \tilde{\psi}_{i,j} = 0$ as reqd

Thus we have associated a Čech 1-cocycle to each locally trivial deformation $Y \in \operatorname{Def}_X(k[\varepsilon])$

Exercise: Show the 1-cocycle does not depend on the choice of the trivialising cover $\{U_i\}_{i \in I}$.

The class in $H^1(\mathcal{H}om(\Omega_X, \mathcal{O}_X))$ does not depend on the representative for the equivalence class $[Y] \in \mathrm{Def}_X(k[\varepsilon])$.
Indeed, suppose $Y' \to \mathrm{Spec}\, k[\varepsilon]$ is a deformation of $X$ and we have an isomorphism of deformations $f: Y \to Y'$.
Let $\{U_i\}$ be a cover of $|X|$ which simultaneously trivialises $Y$ & $Y'$. Have

$$\alpha_i : U_i \times \mathrm{Spec}(k[\varepsilon]) \xrightarrow{\theta_i} Y|_{U_i} \xrightarrow[\sim]{f|_{U_i}} Y'|_{U_i} \xrightarrow{\theta_i'^{-1}} U_i \times \mathrm{Spec}\, k[\varepsilon],$$

where $\theta_i' : U_i \times \mathrm{Spec}\, k[\varepsilon] \xrightarrow{\sim} Y'|_{U_i}$ are the trivialisations.
We have $\theta_i' \alpha_i = f|_{U_i} \theta_i$ & then

$$(\theta_i' \alpha_i)^{-1} (\theta_j' \alpha_j) = \theta_i^{-1} f_{U_i}^{-1} \circ f_{U_j} \theta_j = \theta_i^{-1} \theta_j \quad \text{on } U_i \cap U_j$$

$$\Rightarrow \quad \alpha_i^{-1} \theta'_{ij} \alpha_j = \theta_{ij} \quad \text{on } U_i \cap U_j$$

We may write $\alpha_i = \begin{bmatrix} id & 0 \\ a_i & id \end{bmatrix}$, $a_i : A_i \to A_i$ as above.
Then the above equation becomes

$$\tilde{\psi}'_{ij} + a_j - a_i = \tilde{\psi}_{ij}, \text{ where } \{\tilde{\psi}'_{ij}\} \text{ is}$$

the 1-cycle associated to $\{\theta_i'\}$.
This says precisely that

$$\{\tilde{\psi}_{ij}\} \text{ and } \{\tilde{\psi}'_{ij}\} \text{ are cohomologous.}$$

$\Rightarrow$ the resulting class in $H^1(X, \mathcal{H}om(\Omega^1_X, \mathcal{O}_X))$ is independent of the representative for $[Y] \in \mathrm{Def}_X(k[\varepsilon])$.

Conversely any 1-cocycle in $H^1(X, \mathcal{H}om(\Omega^1_X, \mathcal{O}_X))$ has a representative $\{\tilde{\varphi}_{ij}\}$, where $\varphi_{ij} : A_{ij} \to A_{ij}$ is a $k$-derivation as above.

Define $\theta_{ij}^{+} : A_{ij} \otimes k[\varepsilon] \xrightarrow{\sim} A_{ij} \otimes k[\varepsilon]$ by the matrix $\begin{bmatrix} id & 0 \\ \tilde{\varphi}_{ij} & id \end{bmatrix}$, & let

$$\theta_{ij} : \operatorname{Spec}(A_{ij}) \times \operatorname{Spec} k[\varepsilon] \xrightarrow{\sim} \operatorname{Spec} A_{ij} \times \operatorname{Spec} k[\varepsilon]$$

be the associated isomorphism of schemes.

Since $\{\tilde{\varphi}_{ij}\}$ is a 1-cocycle, we get the relation

$$\theta_{ij}\, \theta_{jk}\, \theta_{ik}^{-1} = id \quad \text{on } U_i \cap U_j \cap U_k \times \operatorname{Spec} k[\varepsilon]$$

$(U_i = \operatorname{Spec} A_i$ etc$)$

Thus we may glue the affine opens $U_i \times \operatorname{Spec} k[\varepsilon]$ to a scheme $Y$. The projections glue to give a flat morphism $Y \to \operatorname{Spec} k[\varepsilon]$ which is obviously locally trivial.

These two operations are clearly inverse, so we have a bijection

$$\{Y \in \operatorname{Def}_X(k[\varepsilon]) \mid Y \text{ locally trivial}\} \longleftrightarrow H^1(X, \mathcal{H}om(\Omega^1_X, \mathcal{O}_X)).$$

$\square$

The theorem above is particularly useful when $X$ is smooth. In this case $\mathcal{H}om(\Omega^1_X, \mathcal{O}_X) \simeq T_X$ is just the tangent bundle.

<u>Prop</u> Let $X$ be smooth. Then any deformation $Y \in \mathrm{Def}_X(k[\varepsilon])$ is locally trivial.

In particular $\mathrm{Def}_X\, k[\varepsilon]$ is in 1-1 correspondence with $H^1(X, T_X)$.

Pf. Let $\operatorname{Spec} A$ be any affine open in $Y$.

We will show that $\operatorname{Spec} A \to \operatorname{Spec} k[\varepsilon]$ is trivial.

Let $B := A \otimes_{k[\varepsilon]} k$. $\operatorname{Spec} B$ is an open affine of the fibre $X$, hence $B$ is regular (k.

We have a commutative diagram

$$\begin{tikzcd} A \arrow[r] & B \\ k[\varepsilon] \arrow[u, "f"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & k \arrow[u] \end{tikzcd}$$

Now $f$ is flat & the closed fibre is regular $\Rightarrow f$ is a smooth morphism. (Hartshorne, II ch10).

We now quote the following property of smooth morphisms (Serre's, Appendix C)

Let $R$ be a ring and $R \to S$ a smooth morphism, with $S$ of finite type over $R$.

Then, for any surjection $A \twoheadrightarrow A'$ of $R$-algebras s.t. the kernel $I \subseteq A$ satisfies $I^2 = 0$, the map

$$\mathrm{Hom}_{R\text{-alg}}(S, A) \to \mathrm{Hom}_{R\text{-alg}}(S, A')$$

is surjective.

Exercise: look up the definition of smooth morphisms of functors & compare it to the above definition.

Consider $B \otimes_k k[\varepsilon]$. We have a commutative diagram

$$\begin{tikzcd} A \arrow[r] & B \\ k[\varepsilon] \arrow[u, "f"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & B \otimes_k k[\varepsilon] \arrow[u] \end{tikzcd}$$

, where the morphism

$B \otimes_k k[\varepsilon] \to B$ is given by tensoring $k[\varepsilon] \twoheadrightarrow k$ by $\otimes_k B$.

Thus $B \otimes_k k[\varepsilon] \overset{g}{\twoheadrightarrow} B$ is a surjection of $k[\varepsilon]$ algebras, with $\ker(g)^2 = 0$.

As $f$ is smooth we thus get a morphism $h: A \to B \otimes_k k[\varepsilon]$ of $k[\varepsilon]$ algebras, fitting into the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "h"] & B \\ k[\varepsilon] \arrow[u, "f"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & B \otimes_k k[\varepsilon] \arrow[u] \end{tikzcd}$$

Thus $h$ induces a morphism of deformations $\operatorname{Spec} B \times \operatorname{Spec} k[\varepsilon] \to \operatorname{Spec} A$. We claim $h$ is an iso.

Now let $m \subseteq k[\varepsilon]$ be the ideal generated by $\varepsilon$.

Let $C = \operatorname{Coker} h$. We have

$$A \to B \otimes_k k[\varepsilon] \to C \to 0 \quad \&$$

tensoring by $k[\varepsilon]/m \simeq k$ gives

$$A \otimes_{k[\varepsilon]} k \xrightarrow[\text{isomorphism}]{\sim} B \to C/mC \to 0.$$

Thus $C = mC = m^2 C = 0 \Rightarrow h$ surjective.

Now let $D = \operatorname{Ker} h$

$$0 \to D \to A \to B \otimes_k k[\varepsilon] \to 0$$

Now $B \otimes_k k[\varepsilon]$ is flat as a $k[\varepsilon]$ module, so we have

$$0 \to D/mD \to A \otimes_{k[\varepsilon]} k \xrightarrow{\sim} B \to 0$$

$$\Rightarrow D = mD = m^2 D = 0$$

$\Rightarrow h$ is an iso $\square$