Formal moduli

Let $X$ be a scheme over an algebraically closed field $k$.

Let $A \in \text{Aut}_k$. An infinitesimal deformation of $X$ is a cartesian diagram

$$
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec} \, k & \rightarrow & \text{Spec} \, A
\end{array}
$$

i.e. $X \cong Y \times \text{Spec} \, k$ over $\text{Spec} \, A$

where $Y$ is flat over $\text{Spec} \, A$. (i.e. $Y$ is a family of schemes over $\text{Spec} \, A$ with special fibre $X$).

If $X \rightarrow Y'$ is a second deformation of $X$,

$$
\begin{array}{ccc}
X & \rightarrow & Y' \\
\downarrow & & \downarrow \\
\text{Spec} \, k & \rightarrow & \text{Spec} \, A
\end{array}
$$

we say that the deformations $Y, Y'$ are isomorphic if $f$ is an isomorphism $f: Y \cong Y'$ over $A$ (i.e. $Y \cong Y'$) s.t. the diagram

$$
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow f \\
X & \rightarrow & Y'
\end{array}
$$

commutes, i.e. $f$ induces the identity on the closed fibre $X$.

We define the deformation functor $\text{Def}_X \in \text{Fun}$

$\text{Def}_X(CA) = \sum$ isomorphism classes of deformations of $X/k$ over $\text{Spec} \, A$.
The deformation functor is defined on morphisms via pull-back, i.e., let \( f: A \rightarrow B \in \text{Spec} \) consider the induced morphism \( f^+: \text{Spec} B \rightarrow \text{Spec} A \). If \( Y \subset \text{Def}_X(A) \) then we have an element \( \gamma \in \text{Spec} \).

\[ Y \times \text{Spec} B \subset \text{Def}_x(B) \times \text{Spec} \]

This defines the deformation functor on morphisms.

In general, the functor \( \text{Def}_X \) need not be pro-representable. However, up to a rather weak finiteness condition, it does at least have a hull, which is sufficient for many applications.

We start by studying the Zariski tangent space \( \text{Def}_X \) to the deformation functor \( Y \).

Let \( Y \in \text{Def}_X(kCE) \). We say \( Y \) is trivial if we have an isomorphism of deformations of \( X \):

\[ Y \cong X \times \text{Spec} \text{kCE} \]

We say \( Y \) is locally trivial if there exists an open cover \( \mathcal{U} \) of \( X \) s.t.

\[ Y \mid U_i \equiv U_i \times \text{Spec} \text{kCE} \text{kCE} \]
Let $\mathcal{L}_x$ be the sheaf of $\mathbb{C}$-valued differentials of a \textit{smooth} variety $X$ (i.e., integral, separated scheme of finite type over $\mathbb{C}$). Then there is a bijection

$$
\exists Y \in \operatorname{Det}_X(\mathcal{O}_X) \quad \text{if and only if} \quad 
\exists Y \in \operatorname{Det}_X(\mathcal{O}_X) \quad \text{if and only if} \quad h^1(X, \operatorname{Hom}(\mathbb{C}, \mathcal{O}_X)) \quad \text{is locally trivial.}
$$

Let $Y \in \operatorname{Det}_X(\mathcal{O}_X)$ be locally trivial. We will associate an element $[\gamma] \in h^1(X, \operatorname{Hom}(\mathbb{C}, \mathcal{O}_X))$ to $Y$. We will use Čech cohomology (Hartshorne III.4) for this.

Choose an affine open cover $\exists \mathcal{U}_{i,j} \subset X$ of $X$ such that $Y \in \operatorname{Spec} \mathcal{L}_X^\mathcal{O}_X$.

To give a Čech 1-cocycle we need to specify, for each $i < j \in I$ (we put a well-ordering on $I$) an element $\gamma_{i,j} \in \Gamma(\mathcal{U}_{i,j}, \operatorname{Hom}(\mathbb{C}, \mathcal{O}_X))$, where

$$
\mathcal{U}_{i,j} := \mathcal{U}_i \cap \mathcal{U}_j.
$$

Further, we need the collection $\gamma > \exists \mathcal{U}_{i,j} \gamma_{i,j}$ to satisfy

$$
d\gamma = 0, \quad \text{which translates to demanding} \quad \text{for each} \quad c < j < k \in I,
$$

$$
\gamma_{i,j} - \gamma_{i,k} + \gamma_{j,k} = 0 \quad \text{on} \quad \mathcal{U}_{i,j,k} := \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k.$$
For each $i \in I$ we have an isomorphism of sheaves
$$\Theta_i : U_i \times \text{Spec}(k(E)) \rightarrow Y_i \iota_i.$$  

For $i, j \in I$ define
$$\Theta_{ij} : U_{ij} \times \text{Spec}(k(E)) \rightarrow U_{ij} \times \text{Spec}(k(E)).$$

Now $X$ is assumed separated, $U_{ij}$ is affine, so let $U_{ij} \simeq \text{Spec} A_{ij}.$

$$\Gamma(\text{Spec} A_{ij}, \text{Hom}(\mathcal{E}_T, \Theta_i))$$
$$\simeq \text{Hom}_A(A_{ij}, A_{ij}, A_{ij}).$$

$$\simeq \text{Der}_k(A_{ij}, A_{ij})$$

by the universal property of the module of K"{a}hler differentials.

Here $\text{Der}_k(A_{ij}, A_{ij})$ is the set of $k$-derivations, i.e., additive maps $d : A_{ij} \rightarrow A_{ij}$ satisfying
$$d(ab) = ad(b) + bda,$$  

for $a, b \in A_{ij}$ and $k \in A_{ij}.$

$$\text{Der}_k(A_{ij}, A_{ij})$$

is an $A_{ij}$-module.

Now the morphism $\Theta_{ij}$ corresponds to an automorphism
$$\Theta_{ij}^* : A_{ij} \otimes_k \mathcal{E}_i \rightarrow A_{ij} \otimes_k \mathcal{E}_i$$

of rings which
$$\text{(i)} \text{ is } \mathcal{E}_i \text{-linear, i.e. } \Theta_{ij}^*(a \otimes \xi) = \Theta_{ij}^*(a) \otimes \xi,$$

$$\text{(ii)} \text{ induces the identity mod } \mathcal{E},$$

$$\text{(iii)} \text{ induces the identity mod } \mathcal{E}_i.$$
Some $\Theta_{i,j}$ is an isomorphism of deformations. From (ii) $\Theta_{i,j} - \text{id} = \Sigma_{k} \text{h}_{i,j}^{k}$, where $\text{h}_{i,j}^{k}$ is a map $A_{i,j} \otimes k \mathfrak{E}_{3} \to A_{i,j}$.

Using (i) $\Theta_{i,j} : (\lambda \otimes \xi) = (\lambda \otimes \xi) - \text{id} (\lambda \otimes \xi)$

$\Phi_{i,j} (\lambda \otimes \xi) = 0$ for $\lambda \in k \mathfrak{E}_{3}$.

We claim $\psi_{i,j} : A_{i,j} \otimes k \mathfrak{E}_{3} \to A_{i,j}$ is a $k \mathfrak{E}_{3}$ derivation.

We need to prove $\psi_{i,j}(xy) = x \psi_{i,j}(y) + \psi_{i,j}(x) y$ for $x, y \in A_{i,j} \otimes k \mathfrak{E}_{3}$.

Write $x = x_{1} \otimes 1 + \varepsilon x_{2} \otimes 1$, $x_{1} \in A_{i,j}$.

$\Theta_{i,j}(x_{1}) = \Theta_{i,j}(x_{1} \otimes 1) + \varepsilon \Theta_{i,j}(x_{2} \otimes 1)$

We may also write $\Theta_{i,j}(x) = \phi_{1}(x) + \varepsilon \phi_{2}(x)$ for $\phi_{1} : A_{i,j} \otimes k \mathfrak{E}_{3} \to A_{i,j}$ maps.

So $\Theta_{i,j}(x_{1}) = \phi_{1}(x_{1} \otimes 1) + \varepsilon (\phi_{2}(x_{1} \otimes 1) + \phi_{1}(x_{2} \otimes 1))$.

$\Theta_{i,j}(x_{2}) = \phi_{1}(z \otimes 1) \Rightarrow z \in A_{i,j}$

$\Theta_{i,j}(x) = \phi_{2}(x \otimes 1)$.

Easy exercise: Using this formula shows that $\psi_{i,j} : A_{i,j} \otimes k \mathfrak{E}_{3}$ is a derivation (you need to use $\text{h}_{i,j}^{k}$ to prove that $\Theta_{i,j}$ is a morphism of rings).

Thus $\psi_{i,j} : A_{i,j} \otimes k \mathfrak{E}_{3} \to A_{i,j}$ is a $k \mathfrak{E}_{3}$ derivation derived by $\psi_{i,j}(x) = \phi_{2}(x \otimes 1)$.
We may define a \( \sim \)-derivation

\[
\varphi_{ij} : A_{ij} \rightarrow A_{ij}
\]

by

\[
\varphi_{ij} (\xi_j) := \psi_{ij} (x_i \Theta)
\]

Thus, we have associated a collection \( \tilde{\psi}_{ij} \) in \( \operatorname{Def}(A_{ij} \times \Omega) \).

We need to show that we have a \( t \)-cocycle, i.e.

\[
\tilde{\psi}_{jk} - \tilde{\psi}_{ik} + \tilde{\psi}_{ij} = 0 \quad \text{on} \quad \operatorname{Spec}(A_j) \cap \operatorname{Spec}(A_i) \cap \operatorname{Spec}(A_k)
\]

We have \( \theta_{ijk} : \Theta_{ij} \circ \Theta_{jk} \circ \Theta_{ki}^{-1} = \text{id} \) on \( U_{ij} \times U_{jk} \times U_{ki} \) by definition.

So \( \Theta_{ij} \circ \Theta_{ik} \circ \theta_{ij}^{-1} = \text{id} \). We may represent

\[
\Theta_{ij} : A_{ij} \circ \Theta_{ij} \rightarrow A_{ij} \circ \Theta_{ij}
\]

as a \( 2 \times 2 \) matrix

\[
\begin{pmatrix}
\text{id} & 0 \\
\Phi_{ij} & \text{id}
\end{pmatrix}
\]

Since

\[
\begin{pmatrix}
\text{id} & 0 \\
\alpha & \text{id}
\end{pmatrix} \cdot \begin{pmatrix}
\text{id} & 0 \\
\beta & \text{id}
\end{pmatrix} = \begin{pmatrix}
\text{id} & 0 \\
\alpha + \beta & \text{id}
\end{pmatrix}
\]

(4.2) gives \( \tilde{\psi}_{jk} - \tilde{\psi}_{ik} + \tilde{\psi}_{ij} = 0 \) as required.

Thus we have associated a \( \tilde{\psi}_{ij} \)-cochain to each locally trivial deformation \( \Psi \in \operatorname{Def}(A_{ij} \times \Omega) \).

Exercise: Show that the \( t \)-cochain does not depend on the choice of the trivialising cover \( U_{ij} \times \Omega \).
The class in $H^1(\text{Hom}(\mathcal{E}_i, \mathcal{O}_X))$ does not depend on the representative for the equivalence class $[Y_i]$. Defining $\mathcal{E} \mathcal{E}$, indeed, suppose $Y'$ is a deformation of $X$ and we have an isomorphism of deformations $f: Y \to Y'$ but $\mathcal{E} \mathcal{E}$ be a cover of $|X|$ which simultaneously trivializes $Y \to Y'$, hence $x_i: \mathcal{O}_{x_i} \times \text{Spec}(k[C_2]) \to Y_{x_i} \to Y_{x_i}$, where $f_{x_i}^*: \mathcal{O}_{x_i} \times \text{Spec}(k[C_2]) \to Y_{x_i}$ are the transition functions.

We have $\Theta_{x_i} x_i = f_{x_i}^* \Theta_{x_i}$ and then $f_{x_i}^* (\Theta_{x_i}, x_i) = \Theta_{x_i} f_{x_i}^* x_i$, $f_{x_i}^* x_i = \Theta_{x_i} x_i$, $x_i = \Theta_{x_i} x_i$ on $U_i \cap U_j$.

We may write $x_i = \sum a_i$ where $a_i: A_i \to A_i$.

From the above equation becomes $\tilde{\Psi}_{x_i} + a_i - a_i = \tilde{\Psi}_{x_i}$, where $\tilde{\Psi}_{x_i}$ is the cycle associated to $\Theta_{x_i}$.

This says precisely that $\tilde{\Psi}_{x_i}$ and $\tilde{\Psi}_{x_i}$ are cohomologous.

The resulting class in $H^1(X, \text{Hom} (\mathcal{E} \mathcal{E}, \mathcal{O}_X))$ is independent of the representative for $[Y_i]$.
Conversely any 1-cocycle in $\text{H}^1(\mathbb{X}, \text{Hom}(\mathcal{E}_x, \mathcal{O}_x))$
has a representative $\tilde{\psi}_{i,j} \in \mathcal{A}_{i,j}$, where $\psi_{i,j} : A_i \to A_j$
isa $k$-derivation as above.
Define $\Theta_{i,j} : A_i \otimes_k \mathcal{E}_j \to A_j \otimes_k \mathcal{E}_j$ by the matrix
\[
\left[ \begin{array}{cc} \text{id} & 0 \\ \psi_{i,j} & \text{id} \end{array} \right] + \text{id}.
\]
\[\Theta_{i,j} : \text{Spec}(A_i) \times \text{Spec}(\mathcal{E}_j) \to \text{Spec}(A_j \otimes_k \mathcal{E}_j)
\]
be the associated isomorphism of schemes.

Since $\tilde{\psi}_{i,j}$ is a 1-cocycle, we get the relation
\[\Theta_{i,j} \Theta_{j,k} \Theta_{k,i}^{-1} = \text{id}, \quad \text{on } \text{Spec}(A_i \otimes_k \mathcal{E}_j) \times \text{Spec}(\mathcal{E}_k).
\]

Thus we may glue the affine opens $\text{Spec}(A_i \otimes_k \mathcal{E}_i)$ to a scheme $Y$. The projections glue to give a fortiori

Thus, these two operations are clearly inverse, so we have a bijection
\[\Sigma \to \text{Def}_X(\mathcal{E}_X) | Y \text{ locally trivial} \leftrightarrow \text{H}^1(\mathbb{X}, \text{Hom}(\mathcal{E}_x, \mathcal{O}_x)).
\]

The theorem above is particularly useful when $X$ is smooth. In this case $\text{H}^1(\mathbb{X}, \mathcal{O}_X) = \mathcal{O}_X$
is just the tangent bundle.
Prop let $X$ be smooth. Then any deformation $Y \in \text{Def}_X (kCEJ)$ is locally trivial.
In particular $\text{Def}_X kCEJ$ is in 1-1 correspondence with $H^1(X, T_X)$.

Let $\text{Spec} A$ be any affine open in $Y$.
We will show that $\text{Spec} A \to \text{Spec} kCEJ$ is trivial.
Let $B := A \otimes_k k$. $\text{Spec} B$ is an open affine set in the fibre $X$, hence $B$ is regular.

We have a commutative diagram

\[
\begin{array}{ccc}
A & \to & B \\
\uparrow f & & \uparrow c \\
\text{Spec} B & \to & k
\end{array}
\]

Now $f$ is flat if the closed fibre is regular.
$f$ is a smooth morphism. (Hartshorne, III, Ch.10).

We now quote the following property of smooth morphisms (Serre, Appendix C).

Let $f$ be a ring and $R \to S$ a smooth morphism with $S$ of finite type over $R$.
Then, for any surjective $A \to A'$ of $f$-algebras, the kernel $\text{Ker}(A \to A')$ is surjective.

For subjective.
Exercise: hook up the definition of smooth morphisms of functors to compare it to the above definition.

Consider \( \mathbb{B} \cdot \mathbb{C} \mathbb{E} \). We have a commutative diagram

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\uparrow f & & \uparrow g \\
\mathbb{C} \mathbb{E} \mathbb{J} & \rightarrow & \mathbb{B} \cdot \mathbb{C} \mathbb{E} \mathbb{J} \\
\end{array}
\]

\( \mathbb{B} \cdot \mathbb{C} \mathbb{E} \mathbb{J} \rightarrow B \) is given by functors

\[
\mathbb{C} \mathbb{E} \mathbb{J} \rightarrow k \text{ by } \mathbb{B} \cdot \mathbb{P}.
\]

They \( \mathbb{B} \cdot \mathbb{C} \mathbb{E} \mathbb{J} \rightarrow B \) is a subfunctor of \( \mathbb{C} \mathbb{E} \mathbb{J} \) algebra with \( \ker(g)^2 = 0 \).

As \( f \) is smooth we thus get a morphism \( h : A \rightarrow \mathbb{B} \cdot \mathbb{C} \mathbb{E} \mathbb{J} \) of \( \mathbb{C} \mathbb{E} \mathbb{J} \) algebra fitting into the commutative diagram

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\uparrow h & & \uparrow g \\
\mathbb{C} \mathbb{E} \mathbb{J} & \rightarrow & \mathbb{B} \cdot \mathbb{C} \mathbb{E} \mathbb{J} \\
\end{array}
\]

They \( h \) induce a morphism of deformation

\[
\text{Spec } B \times \text{Spec } \mathbb{C} \mathbb{E} \mathbb{J} \rightarrow \text{Spec } A.
\]

We claim \( h \) is an iso.

Now let \( MS \mathbb{C} \mathbb{E} \mathbb{J} \) be the ideal generated by \( S \).
Let $C = \text{coker } h$. We see

$$A \rightarrow B \otimes_k \mathcal{C} \rightarrow C \rightarrow 0 \rightarrow 0$$

lifting by $\mathcal{C} \otimes_k A \rightarrow \mathcal{C}$ give

$$A \otimes_k C \rightarrow \mathcal{C} \otimes_k C \rightarrow 0.$$ 

This is an isomorphism.

Thus $C = mC = m^2 C = 0 \rightarrow h$ surjective.

Now let $D = \ker h$

$$0 \rightarrow D \rightarrow A \rightarrow B \otimes_k \mathcal{C} \rightarrow 0.$$ 

Now $B \otimes_k \mathcal{C}$ is flat as a $\mathcal{C}$ module, so we have

$$0 \rightarrow D/\mathcal{C} \rightarrow A \otimes_k \mathcal{C} \rightarrow B \otimes_k \mathcal{C} \rightarrow 0$$

Thus $D = mD = m^2 D = 0$.

$h$ is an iso.