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Renormalization and Mellin transforms

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We study renormalization in a kinetic scheme using the Hopf algebraic framework, first summarizing and recovering known results in this setting. Then we give a direct combinatorial description of renormalized amplitudes in terms of Mellin transform coefficients, featuring the universal property of rooted trees H_R. In particular, a special class of automorphisms of H_R emerges from the action of changing Mellin transforms on the Hochschild cohomology of perturbation series.

Furthermore, we show how the Hopf algebra of polynomials carries a refined renormalization group property, implying its coarser form on the level of correlation functions. Application to scalar quantum field theory reveals the scaling behaviour of individual Feynman graphs.

1 Introduction

As was shown in [16, 24, 6], we may decompose Feynman integrals into functions of a single *scale* parameter s only (further forking into logarithmic divergent parts multiplied by suitable powers of s) and scale-independent functions of the other kinematic variables, called *angles*. Furthermore, the Hopf algebra H_R of rooted trees suffices to encode the full structure of subdivergences in quantum field theory by [16, 8, 9].

We can therefore study such generic Feynman rules in a purely algebraic framework as pioneered in [18, 9]. Renormalizing short-distance singularities by subtraction at a reference scale μ (*kinetic scheme*) leads to amplitudes of a distinguished algebraic kind: Theorem 4.6 proves them to implement the universal property of H_R, delivering an explicit combinatorial evaluation in terms of Mellin transform coefficients.

Further investigating the role of Hochschild cohomology, in section 6 we define a class of automorphisms of H_R which transform the perturbation series in a way equivalent to changing the Feynman rules. This clarifies how exact one-cocycles describe variations.

In sections 4 and 5 we advertise to think about the renormalization group property as a Hopf algebra morphism to polynomials, determining higher logarithms in (4.8). We show how it implies the renormalization group on correlation functions and extend the *propagator-coupling-duality* of [5] which yields the functional equation (5.6).

After analysing the differences to the minimal subtraction scheme in section 7, we show explicitly how our general results manifest themselves in scalar field theory.

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2 Connected Hopf algebras

The fundamental mathematical structure behind perturbative renormalization is the Hopf algebra as discovered in [16]. We briefly summarize the results on Hopf algebras we need and recommend [21, 22] for detailed introductions with a focus on renormalization.

All vector spaces live over a field $\mathbb{K}$ of zero characteristic (in examples $\mathbb{K} = \mathbb{R}$), $\mathrm{Hom}(\cdot,\cdot)$ denotes $\mathbb{K}$-linear maps and $\operatorname{lin} M$ the linear span. Every algebra $(\mathcal{A}, m, u)$ shall be unital, associative and commutative, any bialgebras $(H, m, u, \Delta, \varepsilon)$ in addition also counital and coassociative. They split into the scalars and the *augmentation ideal* $\ker\varepsilon$ as $H = \mathbb{K}\mathbb{1} \oplus \ker\varepsilon = \operatorname{im} u \oplus \ker\varepsilon$, inducing the projection $P := \mathrm{id} - u\circ\varepsilon\colon\ H \twoheadrightarrow \ker\varepsilon$. We use Sweedler's notation $\Delta(x) = \sum_x x_1 \otimes x_2$ and $\widetilde{\Delta}(x) = \sum_x x' \otimes x''$ to abbreviate the *reduced coproduct* $\widetilde{\Delta} := \Delta - \mathbb{1} \otimes \mathrm{id} - \mathrm{id} \otimes \mathbb{1}$.

We assume a *connected grading* $H = \bigoplus_{n\geq 0} H_n$ ($H_0 = \mathbb{K}\cdot\mathbb{1}$) and write $|x| := n$ for homogeneous $0 \neq x \in H_n$, defining the *grading operator* $Y \in \mathrm{End}(H)$ by $Yx = |x| \cdot x$. Exponentiation yields a one-parameter group $\mathbb{K} \ni t \mapsto \theta_t$ of Hopf algebra automorphisms

$$\theta_t := \exp(tY) = \sum_{n\in\mathbb{N}_0} \frac{(tY)^n}{n!}, \quad \forall n \in \mathbb{N}_0 : \quad H_n \ni x \mapsto \theta_t(x) = e^{t|x|}x = e^{nt}x. \tag{2.1}$$

Given an algebra $(\mathcal{A}, m_{\mathcal{A}}, u_{\mathcal{A}})$, the associative *convolution product* on $\mathrm{Hom}(H, \mathcal{A})$ is

$$\mathrm{Hom}(H,\mathcal{A}) \ni \phi, \psi \mapsto \phi \star \psi := m_{\mathcal{A}} \circ (\phi \otimes \psi) \circ \Delta \in \mathrm{Hom}(H,\mathcal{A}),$$

with unit given by $e := u_{\mathcal{A}} \circ \varepsilon$. As outcome of the connectedness of H we stress

1. The *characters* $G^H_{\mathcal{A}} := \{\phi \in \mathrm{Hom}(H,\mathcal{A})\colon\ \phi \circ u = u_{\mathcal{A}} \text{ and } \phi \circ m = m_{\mathcal{A}} \circ (\phi \otimes \phi)\}$ (morphisms of unital algebras) form a group under $\star$.
2. Hence $\mathrm{id} \in G^H_H$ has a unique inverse $S := \mathrm{id}^{\star -1}$, called *antipode*, turning H into a Hopf algebra. For all $\phi \in G^H_{\mathcal{A}}$ we have $\phi^{\star -1} = \phi \circ S$.
3. The bijection $\exp_\star\colon\ \mathfrak{g}^H_{\mathcal{A}} \to G^H_{\mathcal{A}}$ with inverse $\log_\star\colon\ G^H_{\mathcal{A}} \to \mathfrak{g}^H_{\mathcal{A}}$ between $G^H_{\mathcal{A}}$ and the *infinitesimal characters* $\mathfrak{g}^H_{\mathcal{A}} := \{\phi \in \mathrm{Hom}(H,\mathcal{A})\colon\ \phi \circ m = \phi \otimes e + e \otimes \phi\}$ is given by the pointwise finite series
$$\exp_\star(\phi) := \sum_{n\in\mathbb{N}_0} \frac{\phi^{\star n}}{n!} \quad \text{and} \quad \log_\star(\phi) := \sum_{n\in\mathbb{N}} \frac{(-1)^{n+1}}{n} (\phi - e)^{\star n}. \tag{2.2}$$

2.1 Hochschild cohomology

The Hochschild cochain complex [8, 1, 22] we associate to H contains the functionals $H' = \mathrm{Hom}(H, \mathbb{K})$ as zero-cochains. One-cocycles $L \in \mathrm{HZ}^1_\varepsilon(H) \subset \mathrm{End}(H)$ are linear maps such that $\Delta \circ L = (\mathrm{id} \otimes L) \circ \Delta + L \otimes \mathbb{1}$ and the differential

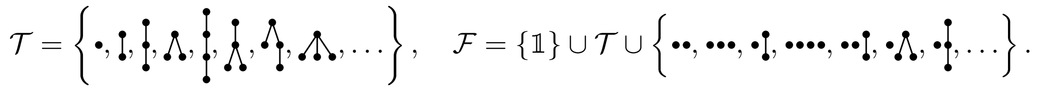
$$\delta : H' \to \mathrm{HZ}^1_\varepsilon(H), \alpha \mapsto \delta\alpha := (\mathrm{id} \otimes \alpha) \circ \Delta - u \circ \alpha \in \mathrm{HB}^1_\varepsilon(H) := \delta\left(H'\right) \tag{2.3}$$

determines the first cohomology group by $\mathrm{HH}^1_\varepsilon(H) := \mathrm{HZ}^1_\varepsilon(H)/\mathrm{HB}^1_\varepsilon(H)$.

Lemma 2.1. *Cocycles $L \in HZ^1_\varepsilon(H)$ fulfil $\operatorname{im} L \subseteq \ker\varepsilon$ and $L(\mathbb{1}) \in \mathrm{Prim}(H) := \ker\widetilde{\Delta}$ is primitive. The map $HH^1_\varepsilon(H) \to \mathrm{Prim}(H)$, $[L] \mapsto L(\mathbb{1})$ is well-defined since $\delta\alpha(\mathbb{1}) = 0$ for all $\alpha \in H'$.*

2.2 Rooted Trees

The Hopf algebra H_R of rooted trees serves as the domain of Feynman rules. As an algebra, $H_R = S(\operatorname{lin}\mathcal{T}) = \mathbb{K}[\mathcal{T}]$ is free commutative[1] generated by the *rooted trees* $\mathcal{T}$ and spanned by their disjoint unions (products) called *rooted forests* $\mathcal{F}$:



Every $w \in \mathcal{F}$ is just the monomial $w = \prod_{t\in\pi_0(w)} t$ of its multiset of tree components $\pi_0(w)$, while $\mathbb{1}$ denotes the empty forest. The number $|w| := |V(w)|$ of nodes $V(w)$ induces the grading $H_{R,n} = \operatorname{lin}\mathcal{F}_n$ where $\mathcal{F}_n := \{w \in \mathcal{F}\colon\ |w| = n\}$.

Definition 2.2. *The (linear)* grafting operator $B_+ \in \operatorname{End}(H_R)$ *attaches all trees of a forest to a new root, so for example* $B_+(\mathbb{1}) = \bullet$, $B_+(\bullet) =$ *and* $B_+(\bullet\bullet) =$.

Clearly, B_+ is homogenous of degree one with respect to the grading and restricts to a bijection $B_+\colon\ \mathcal{F} \to \mathcal{T}$. The coproduct Δ is defined to make B_+ a cocycle by requiring

$$\Delta \circ B_+ = B_+ \otimes \mathbb{1} + (\mathrm{id} \otimes B_+) \circ \Delta. \tag{2.4}$$

Lemma 2.3. *In cohomology,* $0 \neq [B_+] \in HH^1_\varepsilon(H_R)$ *is non-trivial by* $B_+(\mathbb{1}) = \bullet \neq 0$.

It characterizes H_R through the well-known (theorem 2 of [8]) *universal property* of

Theorem 2.4. *To an algebra* $\mathcal{A}$ *and* $L \in \operatorname{End}(\mathcal{A})$ *there exists a unique morphism* ${}^L\rho\colon H_R \to \mathcal{A}$ *of unital algebras such that*

$$ {}^L\rho \circ B_+ = L \circ {}^L\rho, \quad \textit{equivalently} \quad \begin{tikzcd} H_R \arrow[r, "{}^L\rho"] \arrow[d, "B_+"'] & \mathcal{A} \arrow[d, "L"] \\ H_R \arrow[r, "{}^L\rho"'] & \mathcal{A} \end{tikzcd} \quad \textit{commutes.} \tag{2.5}$$

In case of a bialgebra $\mathcal{A}$ *and a cocycle* $L \in HZ^1_\varepsilon(\mathcal{A})$, ${}^L\rho$ *is a morphism of bialgebras and even of Hopf algebras when* $\mathcal{A}$ *is Hopf.*

This morphism ${}^L\rho$ simply replaces B_+, m_{H_R} and $\mathbb{1}$ as placeholders by L, $m_{\mathcal{A}}$ and $\mathbb{1}_{\mathcal{A}}$:

$$ {}^L\rho\,(- 3\bullet) = {}^L\rho\left\{B_+\left([B_+(\mathbb{1})]^2\right) - 3B_+(\mathbb{1})\right\} = L\left([L(\mathbb{1}_{\mathcal{A}})]^2\right) - 3L(\mathbb{1}_{\mathcal{A}}).$$

Example 2.5. *The cocycle* $\int_0 \in HZ^1_\varepsilon(\mathbb{K}[x])$ *of section 4 induces the character*

$$\varphi := {}^{\int_0}\rho \in G^{H_R}_{\mathbb{K}[x]} \quad \textit{fulfilling} \quad \varphi(w) = \frac{x^{|w|}}{w!} \quad \textit{for any forest} \quad w \in \mathcal{F}, \quad \textit{using} \tag{2.6}$$

Definition 2.6. *The* tree factorial $(\cdot)! \in G^{H_R}_{\mathbb{K}}$ *is equivalently determined by requesting*

$$[B_+(w)]! = w! \cdot |B_+(w)| \quad \textit{or} \quad w! \overset{2}{=} \prod_{v\in V(w)} |w_v| \quad \textit{for all} \quad w \in \mathcal{F}. \tag{2.7}$$

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[1] We consider *unordered* trees = and forests = , sometimes called *non-planar*.

[2] By w_v we denote the subtree of w rooted at the node $v \in V(w)$.

3 The generic model

As explained in the introduction we consider Feynman rules as characters $\phi \in G_{\mathcal{A}}^{H_R}$, mapping a rooted tree to a function of the parameter s (by proposition 3.2 it lies in the algebra $\mathcal{A} = \mathbb{K}[z^{-1}, z]][s^{-z}]$). Since B_+ mimics the insertion of a subdivergence into a fixed graph γ (restricting to a single insertion place by a result from [24]), applying ϕ yields a subintegral and therefore

Definition 3.1. *The generic Feynman rules ${}_z\phi$ are given through theorem 2.4 by*

$$ {}_z\phi_s \circ B_+ = \int_0^\infty \frac{f(\frac{\zeta}{s})\zeta^{-z}}{s} \ {}_z\phi_\zeta \ \mathrm{d}\zeta = \int_0^\infty f(\zeta)(s\zeta)^{-z} \ {}_z\phi_{s\zeta} \ \mathrm{d}\zeta. \tag{3.1} $$

The integration kernel f is specified by γ after *Wick rotation* to Euclidean space, with the asymptotic behaviour $f(\zeta) \sim \zeta^{-1}$ for $\zeta \to \infty$ generating the (logarithmic) divergences of these integrals (we do not address infrared problems and exclude any poles in f). The regulator ζ^{-z} ensures convergence when $0 < \Re(z) < 1$, with results depending analytically on z. We can perform all the integrals using this *Mellin transform*

$$ F(z) := \int_0^\infty f(\zeta)\zeta^{-z} \ \mathrm{d}\zeta = \sum_{n=-1}^{\infty} c_n z^n, \quad \text{by} \tag{3.2} $$

Proposition 3.2. *For any forest $w \in \mathcal{F}$ we have (called* BPHZ model *in [4])*

$$ {}_z\phi_s(w) = s^{-z|w|} \prod_{v \in V(w)} F\left(z\left|w_v\right|\right). \tag{3.3} $$

Proof. As both sides of (3.3) are clearly multiplicative, it is enough to prove the claim inductively for trees. Let it be valid for some forest $w \in \mathcal{F}$, then for $t = B_+(w)$ observe

$$ {}_z\phi_s \circ B_+(w) = \int_0^\infty (s\zeta)^{-z} f(\zeta) \ {}_z\phi_{s\zeta}(w) \ \mathrm{d}\zeta = \int_0^\infty (s\zeta)^{-z} f(\zeta)(s\zeta)^{-z|w|} \prod_{v \in V(w)} F\left(z\left|w_v\right|\right) \ \mathrm{d}\zeta $$

$$ = s^{-z|B_+(w)|} \left[\prod_{v \in V(w)} F\left(z\left|w_v\right|\right) \right] F\left(z\left|B_+(w)\right|\right) = s^{-z|t|} \prod_{v \in V(t)} F\left(z\left|t_v\right|\right). \qquad \square $$

Example 3.3. *Using* (3.3)*, we can directly write down the Feynman rules like*

$$ {}_z\phi_s\left(\bullet\right) = s^{-z}F(z), \quad {}_z\phi_s\left(\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}\right) = s^{-2z}F(z)F(2z) \quad \textit{and} \quad {}_z\phi_s\left(\begin{smallmatrix} & \bullet & \\ \diagup & & \diagdown \\ \bullet & & \bullet\end{smallmatrix}\right) = s^{-3z}[F(z)]^2F(3z). $$

Many examples (choices of F) are discussed in [4], the particular case of the one-loop propagator graph γ of Yukawa theory is in [5] and for scalar Yukawa theory in six dimensions one has $F(z) = \frac{1}{z(1-z)(2-z)(3-z)}$ as in [22]. Already noted in [17], the highest order pole of ${}_z\phi_s(w)$ is independent of s and just the tree factorial

$$ {}_z\phi_s(w) \in s^{-z|w|} \prod_{v \in V(w)} \left\{ \tfrac{c_{-1}}{z|w_v|} + \mathbb{K}[[z]] \right\} \underset{(2.7)}{\subseteq} \frac{1}{w!}\left(\tfrac{c_{-1}}{z}\right)^{|w|} + z^{1-|w|}\mathbb{K}[\ln s][[z]]. \tag{3.4} $$

3.1 Renormalization

Algebraically, renormalization equals a *Birkhoff decomposition* [9, 21, 22] of $\phi \in G^H_{\mathcal{A}}$ into *renormalized* rules $\phi_R := \phi_+ \in G^H_{\mathcal{A}}$ and the *counterterms* $Z := \phi_- \in G^H_{\mathcal{A}}$ such that

$$\phi = \phi_-^{\star -1} \star \phi_+ \quad \text{and} \quad \phi_\pm (\ker \varepsilon) \subseteq \mathcal{A}_\pm, \tag{3.5}$$

with respect to a splitting $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ determined by the *renormalization scheme* (the projection $R\colon \mathcal{A} \twoheadrightarrow \mathcal{A}_-$). We comment on *minimal subtraction* in 7 and now focus on

Definition 3.4. *On the target algebra $\mathcal{A}$ of regularized Feynman rules depending on a single external variable s, define the* kinetic scheme *by evaluation at $s = \mu$:*

$$\operatorname{End}(\mathcal{A}) \ni R_\mu := \operatorname{ev}_\mu = \Big(\mathcal{A} \ni f \mapsto f|_{s=\mu}\Big)\,. \tag{3.6}$$

This scheme exploits that subtraction improves the decay at infinity: Let $f(\zeta) \sim \frac{1}{\zeta}$, meaning $f(\zeta) = \frac{1}{\zeta} + \tilde{f}(\zeta)$ for some $\tilde{f}(\zeta) \in \mathcal{O}\left(\zeta^{-1-\varepsilon}\right)$ with $\varepsilon > 0$. Then ${}_z\phi_s(\bullet)$ is logarithmically divergent (would it not be for the regulator ζ^{-z}), but subtraction

$${}_z\phi_s(\bullet) - {}_z\phi_\mu(\bullet) = \int_0^\infty \left[\frac{f(\frac{\zeta}{s})}{s} - \frac{f(\frac{\zeta}{\mu})}{\mu}\right] \zeta^{-z} = \int_0^\infty \left[\frac{\tilde{f}(\frac{\zeta}{s})}{s} - \frac{\tilde{f}(\frac{\zeta}{\mu})}{\mu}\right] \zeta^{-z} \tag{3.7}$$

yields a convergent integral even for $z = 0$. As R_μ is a character of $\mathcal{A}$, the Birkhoff recursion simplifies to $Z = R_\mu \circ {}_z\phi \circ S = {}_z\phi_\mu \circ S$ and ${}_z\phi_R = {}_z\phi_\mu^{\star -1} \star {}_z\phi_s$.

Example 3.5. *We find ${}_z\phi_{R,s}(\bullet) = \left(s^{-z} - \mu^{-z}\right) F(z)$ and $S\big(\overset{\bullet}{\underset{\bullet}{\vert}}\big) = -\overset{\bullet}{\underset{\bullet}{\vert}} + \bullet\bullet$ results in*

$${}_z\phi_{R,s}\big(\overset{\bullet}{\underset{\bullet}{\vert}}\big) = \left(s^{-2z} - \mu^{-2z}\right) F(z)F(2z) - \left(s^{-z} - \mu^{-z}\right) \mu^{-z} F^2(z). \tag{3.8}$$

The goal of renormalization is to assure the *finiteness* of the *physical limit*

$${}_0\phi_R := \lim_{z \to 0} {}_z\phi_R, \tag{3.9}$$

and indeed we find the finite ${}_0\phi_{R,s}(\bullet) = -c_{-1} \ln \frac{s}{\mu}$. In the case of (3.8) check

$$\begin{aligned} {}_0\phi_{R,s}\big(\overset{\bullet}{\underset{\bullet}{\vert}}\big) = \lim_{z\to 0} \Big\{ &-\Big[-z \ln \tfrac{s}{\mu} + \tfrac{z^2}{2}\left(\ln^2 s + 2 \ln s \ln \mu - 3 \ln^2 \mu\right)\Big] \cdot \Big[\tfrac{c_{-1}^2}{z^2} + 2\tfrac{c_{-1}c_0}{z}\Big] \\ &+ \Big[-2z \ln \tfrac{s}{\mu} + 2z^2\left(\ln^2 s - \ln^2 \mu\right)\Big] \cdot \Big[\tfrac{c_{-1}^2}{2z^2} + \tfrac{3c_0 c_{-1}}{2z}\Big]\Big\} = \frac{c_{-1}^2}{2} \ln^2 \tfrac{s}{\mu} - c_{-1}c_0 \ln \tfrac{s}{\mu}, \end{aligned} \tag{3.10}$$

where all poles in z perfectly cancel. Note that ${}_0\phi_{R,s}$ maps a forest w to a polynomial in $\mathbb{K}[\ln \frac{s}{\mu}]$ of degree $\leq |w|$ without constant term (except for ${}_0\phi(\mathbb{1}) = 1$), due to the subtraction at $s = \mu$. We now prove these properties in general, extending work in [18].

3.2 Subdivergences

Inductively, the Birkhoff decomposition is constructed as $\phi_+(x) = (\mathrm{id} - R_\mu)\bar{\phi}(x)$ where

$$\bar{\phi}(x) := \phi(x) + \sum_x \phi_-(x')\phi(x'') = \phi(x) + [\phi_- \star \phi - \phi_- - \phi](x) = \phi_+(x) - \phi_-(x)$$

is the *Bogoliubov character* (*$\bar{R}$-operation*) and renormalizes the *subdivergences.* Note

Theorem 3.6. *For an endomorphism $L \in \mathrm{End}(\mathcal{A})$ consider the Feynman rules $\phi := {}^L\rho$ induced by* (2.5)*. Given a renormalization scheme $R \in \mathrm{End}(\mathcal{A})$ such that*

$$L \circ m_{\mathcal{A}} \circ (\phi_- \otimes \mathrm{id}) = m_{\mathcal{A}} \circ (\phi_- \otimes L), \tag{3.11}$$

that is to say, L is linear over the counterterms, we have

$$\bar{\phi} \circ B_+ = L \circ \phi_+. \tag{3.12}$$

Proof. This is a straightforward consequence of the cocycle property of B_+:

$$\begin{aligned}\bar{\phi} \circ B_+ &= (\phi_- \star \phi - \phi_-) \circ B_+ = m_{\mathcal{A}} \circ (\phi_- \otimes \phi) \circ [(\mathrm{id} \otimes B_+) \circ \Delta + B_+ \otimes \mathbb{1}] - \phi_- \circ B_+ \\ &= \phi_- \star (\phi \circ B_+) = \phi_- \star (L \circ \phi) \underset{(3.11)}{=} L \circ (\phi_- \star \phi) = L \circ \phi_+.\end{aligned}$$ □

As the counterterms Z of our model are independent of s, they can be moved out of the integrals in (3.1) and (3.11) is fulfilled indeed. This is a general feature of quantum field theories: The counterterms to not depend on any external variables[3].

The significance of (3.12) lies in the expression of the renormalized $\phi_{R,0}(t)$ for a tree $t = B_+(w)$ only in terms of the renormalized value ${}_z\phi_R(w)$. This allows for inductive proofs of properties of ${}_z\phi_R$ and also ${}_0\phi_R$, without having to consider the unrenormalized Feynman rules or their counterterms at all.

3.3 Finiteness

Proposition 3.7. *The physical limit ${}_0\phi_{R,s}$ exists and maps H_R into polynomials $\mathbb{K}[\ln \frac{s}{\mu}]$.*

Proof. We proceed inductively from ${}_0\phi_{R,s}(\mathbb{1}) = 1$ and as ${}_0\phi_R$ is a character only need to consider trees $t = B_+(w)$ in the induction step. Hence for this $w \in \mathcal{F}$ we already know that ${}_0\phi_{R,\zeta}(w) \in \mathcal{O}\left(\ln^N \zeta\right)$ for some $N \in \mathbb{N}_0$ such that dominated convergence yields

$$\begin{aligned}{}_0\phi_{R,s}(t) &\underset{(3.12)}{=} \lim_{z\to 0}(\mathrm{id} - R_\mu)\left[s \mapsto \int_0^\infty \frac{f(\zeta/s)}{s}\zeta^{-z}\ {}_z\phi_{R,\zeta}(w)\ \mathrm{d}\zeta\right] \\ &= \lim_{z\to 0}\int_0^\infty \left[\tfrac{f(\zeta/s)}{s} - \tfrac{f(\zeta/\mu)}{\mu}\right]\zeta^{-z}\ {}_z\phi_{R,\zeta}(w)\ \mathrm{d}\zeta = \int_0^\infty \left[\tfrac{f(\zeta/s)}{s} - \tfrac{f(\zeta/\mu)}{\mu}\right]\ {}_0\phi_{R,\zeta}(w)\ \mathrm{d}\zeta,\end{aligned}$$

[3] Even if the divergence of a Feynman graph does depend on external momenta as happens for higher degrees of divergence, the Hopf algebra is defined such that the counterterms are evaluations on certain *external structures*, given by distributions in [9]. So in any case, ϕ_- maps to scalars.

recalling the term in square brackets to be from $\mathcal{O}\left(\zeta^{-1-\varepsilon}\right)$ as in (3.7). This proves the cancellation of all z-poles in ${}_z\phi_{R,s}(t)$ and we identify ${}_0\phi_{R,s}(t)$ with the $\propto z^0$ term, which is a polynomial in $\ln s$ and $\ln\mu$ of degree $|t|$ by inspection of (3.3): Each such logarithm comes with a factor z (expanding s^{-z}) which needs to cancel with a pole $\frac{c_{-1}}{z|t_v|}$ from some $F(z\,|t_v|)$ in order to contribute to the $\propto z^0$ term. Finally the substitution $\zeta \mapsto \zeta\mu$ gives

$$ {}_0\phi_{R,s}(t) = \int_0^\infty \left[\frac{f(\zeta\frac{\mu}{s})}{\frac{s}{\mu}} - f(\zeta)\right] {}_0\phi_{R,\mu\zeta}(w)\ \mathrm{d}\zeta, \tag{3.13} $$

hence by induction ${}_0\phi_{R,\zeta\mu}$ only depends on ζ and ${}_0\phi_{R,s}$ is a function of $\frac{s}{\mu}$ only. □

Using (3.13), the physical limit of the renormalized Feynman rules can be obtained inductively by convergent integrations after performing the subtraction at $s=\mu$ on the integrand, in particular without the need of any regulator. Therefore ${}_0\phi_R$ is independent of the choice of regularization prescription, so employing a *cutoff* regulator or *dimensional regularization* yields the same renormalized result in the physical limit.

4 The Hopf algebra of polynomials

We summarize relevant properties of the polynomials, focusing on their Hochschild cohomology (the relevance of $\int_0$ was already mentioned in [8]). First observe

Lemma 4.1. *Requiring* $\Delta(x) = x\otimes\mathbb{1} + \mathbb{1}\otimes x$ *induces a unique Hopf algebra structure on the polynomials* $\mathbb{K}[x]$*. It is graded by degree, connected, commutative and cocommutative with* $\Delta\left(x^n\right) = \sum_{i=0}^n \binom{n}{i} x^i \otimes x^{n-i}$ *and the primitive elements are* $\operatorname{Prim}\left(\mathbb{K}[x]\right) = \mathbb{K}\cdot x$.

The *integration operator* $\int_0\colon x^n \mapsto \frac{1}{n+1}x^{n+1}$ furnishes a cocycle $\int_0 \in \mathrm{HZ}^1_\varepsilon(\mathbb{K}[x])$ as

$$ \begin{aligned} \Delta \int_0 \left(\frac{x^n}{n!}\right) &= \Delta\left(\frac{x^{n+1}}{(n+1)!}\right) = \sum_{k=0}^{n+1} \frac{x^k}{k!}\otimes\frac{x^{n+1-k}}{(n+1-k)!} \\ &= \frac{x^{n+1}}{(n+1)!}\otimes\mathbb{1} + \sum_{k=0}^{n}\frac{x^k}{k!}\otimes\int_0\left(\frac{x^{n-k}}{(n-k)!}\right) = \left[\int_0\otimes\mathbb{1} + \left(\mathrm{id}\otimes\int_0\right)\circ\Delta\right]\left(\frac{x^n}{n!}\right), \end{aligned} $$

and is not a coboundary since $\int_0 1 = x \neq 0$. In fact it generates the cohomology by

Theorem 4.2. $HH^1_\varepsilon(\mathbb{K}[x]) = \mathbb{K}\cdot[\int_0]$ *is one-dimensional as the 1-cocycles of* $\mathbb{K}[x]$ *are*

$$ HZ^1_\varepsilon(\mathbb{K}[x]) = \mathbb{K}\cdot\int_0 \oplus\, \delta\left(\mathbb{K}[x]'\right) = \mathbb{K}\cdot\int_0 \oplus\, HB^1_\varepsilon(\mathbb{K}[x]). \tag{4.1} $$

Proof. For an arbitrary cocycle $L\in\mathrm{HZ}^1_\varepsilon(\mathbb{K}[x])$, lemma 2.1 ensures $L(1) = x a_{-1}$ where $a_{-1} := \partial_0 L(1)$. Hence $\tilde{L} := L - a_{-1}\int_0 \in \mathrm{HZ}^1_\varepsilon$ fulfils $\tilde{L}(1) = 0$, so $L_0 := \tilde{L}\circ\int_0 \in \mathrm{HZ}^1_\varepsilon$ by

$$ \Delta\circ L_0 = (\mathrm{id}\otimes\tilde{L})\circ\Delta\circ\int_0 + (\tilde{L}\otimes 1)\circ\int_0 = (\mathrm{id}\otimes L_0)\circ\Delta + L_0\otimes 1 + \tilde{L}(1)\cdot\int_0. $$

Repeating the argument inductively yields $a_n := \partial_0 L_n(1) = \partial_0 \circ L \circ \int_0^{n+1}(1) \in \mathbb{K}$ and $L_{n+1} := (L_n - a_n \int_0) \circ \int_0 \in \mathrm{HZ}^1_\varepsilon$, so for any $n \in \mathbb{N}_0$ we may read off from

$$L \circ \int_0^n (1) = a_{-1} \int_0^{n+1} (1) + \ldots + a_{n-2} \int_0^2 (1) + L_{n-1}(1) = a_{-1} \int_0 \left(\int_0^n 1 \right) + \sum_{j=0}^{n-1} a_j \int_0^{n-j} (1)$$

that indeed $L = a_{-1} \int_0 + \delta\alpha$ for the functional $\alpha := \partial_0 \circ L \circ \int_0$ with $\alpha(\frac{x^n}{n!}) = a_j$. □

Lemma 4.3. *Up to subtraction $P = \delta\varepsilon = \mathrm{id} - \mathrm{ev}_0 \colon \mathbb{K}[x] \twoheadrightarrow \ker \varepsilon = x\mathbb{K}[x]$ of the constant part, direct computation exhibits $\delta\alpha$ for any $\alpha \in \mathbb{K}[x]'$ as the differential operator*

$$\delta\alpha = P \circ \sum_{n \in \mathbb{N}_0} \alpha\left(\tfrac{x^n}{n!}\right) \partial^n \in \mathrm{End}(\mathbb{K}[x]). \tag{4.2}$$

Lemma 4.4. *As any character $\phi \in G_{\mathbb{K}}^{\mathbb{K}[x]}$ of $\mathbb{K}[x]$ is fixed by $\lambda := \phi(x)$, they are the group $G_{\mathbb{K}}^{\mathbb{K}[x]} = \{\mathrm{ev}_\lambda \colon \lambda \in \mathbb{K}\}$ of evaluations (the counit $\varepsilon = \mathrm{ev}_0$ equals the neutral element)*

$$\mathbb{K}[x] \ni p(x) \mapsto \mathrm{ev}_\lambda(p) := p(\lambda) \quad \textit{with the product} \quad \mathrm{ev}_a \star \mathrm{ev}_b = \mathrm{ev}_{a+b}. \tag{4.3}$$

Proof. Note $[\mathrm{ev}_a \star \mathrm{ev}_b](x^n) = [\mathrm{ev}_a(1) \cdot \mathrm{ev}_b(x) + \mathrm{ev}_a(x) \cdot \mathrm{ev}_b(1)]^n = (b+a)^n$. □

Lemma 4.5. *The isomorphism $(\mathbb{K}, +) \ni a \mapsto \mathrm{ev}_a \in G_{\mathbb{K}}^{\mathbb{K}[x]}$ of groups is generated by the functional $\partial_0 = \mathrm{ev}_0 \circ \partial \in \mathfrak{g}_{\mathbb{K}}^{\mathbb{K}[x]}$, meaning $\log_\star \mathrm{ev}_a = a\partial_0$ and $\mathrm{ev}_a = \exp_\star(a\partial_0)$.*

Proof. Expanding the exponential series reveals $\exp_\star(a\partial_0)(x^n) = a^n$ as a direct consequence of $\partial_0^{\star k} = \varepsilon \circ \partial^{\star k} = \varepsilon \circ \partial^k = \partial_0^k$, while we appreciate $\partial^{\star k} = \partial^k$ inductively:

$$\partial \star \partial^k \left(\tfrac{x^n}{n!}\right) = \sum_{j=0}^{n} \left(\partial \tfrac{x^{n-j}}{(n-j)!}\right)\left(\partial^k \tfrac{x^j}{j!}\right) = \sum_{j=k}^{n-1} \frac{x^{n-1-k}}{(n-j-1)!(j-k)!} = \frac{x^{n-k-1}}{(n-k-1)!}. \qquad \square$$

4.1 Feynman rules induced by cocycles

Let ${}_0\phi \colon H_R \to \mathbb{K}[x]$ denote the polynomials that evaluate to the renormalized Feynman rules ${}_0\phi_{R,s} = \mathrm{ev}_\ell \circ {}_0\phi$ at $\ell = \ln \frac{s}{\mu}$. We state

Theorem 4.6. *The renormalized Feynman rules ${}_0\phi = {}^L\rho$ arise out of the universal property of theorem 2.4, where the coefficients c_n of (3.2) determine the cocycle*

$$L := -c_{-1} \int_0 + \delta\eta \in HZ^1_\varepsilon(\mathbb{K}[x]) \quad \textit{with} \quad \eta\,(x^n) := n!\,(-1)^n c_n \quad \textit{for any } n \in \mathbb{N}_0. \tag{4.4}$$

Proof. We may set $\mu = 1$ and produce logarithms of subdivergences by differentiation, exploiting analyticity of $zF(z)$ and $\frac{s^{-z}-1}{z}$ at $z = 0$ we obtain

$$\lim_{z \to 0} (\mathrm{id} - R_1) \left[s \mapsto \int_0^\infty f(\zeta)(s\zeta)^{-z} \ln^n (s\zeta)\, \mathrm{d}\zeta \right] = \left(-\frac{\partial}{\partial z}\right)^n_{z=0} (\mathrm{id} - R_1) \int_0^\infty f(\zeta)(s\zeta)^{-z}\, \mathrm{d}\zeta$$

$$= \left(-\frac{\partial}{\partial z}\right)^n_{z=0} \left[\frac{s^{-z}-1}{z} \cdot zF(z)\right] = (-1)^n \sum_{k=0}^{n} \binom{n}{k} k! \frac{(-\ln s)^{k+1}}{(k+1)!} (n-k)!\, c_{n-k-1}$$

$$= \mathrm{ev}_{\ln s}\left[-c_{-1}\frac{x^{n+1}}{n+1} + \sum_{i=1}^{n}\binom{n}{i} x^i (-1)^{n-i} c_{n-i}(n-i)!\right] = \mathrm{ev}_{\ln s} \circ L\left(x^n\right). \qquad (*)$$

By linearity we can replace $\ln^n(s\zeta)$ in the integrand by any polynomial to prove 4.6 inductively: As ${}_0\phi$ and ${}^L\rho$ are algebra morphisms, it suffices to consider a tree $t = B_+(w)$ for a forest $w \in \mathcal{F}$ already fulfilling ${}_0\phi(w) = {}^L\rho(w)$ in the induction step

$$\begin{aligned} {}_0\phi_{R,s}(t) &\underset{(3.13)}{=} \lim_{z\to 0}(\mathrm{id} - R_1)\left[s \mapsto \int_0^\infty f(\zeta)(s\zeta)^{-z}\ \mathrm{ev}_{\ln s\zeta} \circ {}_0\phi(w)\ \mathrm{d}\zeta\right] \\ &\underset{(*)}{=} \mathrm{ev}_\ell \circ L\left[{}_0\phi(w)\right] = \mathrm{ev}_\ell \circ L \circ {}^L\rho(w) \underset{2.4}{=} \mathrm{ev}_\ell \circ {}^L\rho \circ B_+(w) = \mathrm{ev}_\ell \circ {}^L\rho(t), \end{aligned}$$

where the convergence of (3.13) allows to reintroduce ζ^{-z} into the integrand. □

Corollary 4.7. *As L is a cocycle, by theorem 2.4 the physical limit ${}_0\phi\colon H_R \to \mathbb{K}[x]$ of the renormalized Feynman rules* (3.1) *is a morphism of Hopf algebras.*

This key property will imply the renormalization group in the sequel. For now observe the simple and explicit combinatorial recursion 4.6, expressing ${}_0\phi$ in terms of the Mellin transform coefficients without any need for series expansions in z, as shown in

Example 4.8. *Using* (4.4) *we rederive ${}_0\phi\left(\bullet\right) = {}^L\rho \circ B_+(\mathbb{1}) = L(1) = -c_{-1}\,x$ and also*

$$\begin{aligned} {}_0\phi\left(\overset{\bullet}{\underset{\bullet}{|}}\right) &= {}^L\rho \circ B_+\left(\bullet\right) = L \circ {}^L\rho\left(\bullet\right) = \left[-c_{-1}\int_0 + \delta\eta\right]\left(-c_{-1}x\right) = c_{-1}^2\frac{x^2}{2} - c_{-1}c_0\,x, \\ {}_0\phi\left(\overset{\bullet}{\underset{\bullet\ \bullet}{\wedge}}\right) &= {}^L\rho \circ B_+\left(\bullet\bullet\right) = L \circ {}^L\rho\left(\bullet\bullet\right) = \left[-c_{-1}\int_0 + \delta\eta\right]\left\{\left(-c_{-1}\,x\right)^2\right\} \\ &= -c_{-1}^3\frac{x^3}{3} + c_{-1}^2\left[\eta(1)\,x^2 + 2\eta(x)\,x\right] = -c_{-1}^3\frac{x^3}{3} + c_{-1}^2 c_0\,x^2 - 2c_{-1}^2 c_1\,x. \end{aligned}$$

Defining $\tilde{F}(z) := F(z) - \frac{c_{-1}}{z} = \sum_{n\in\mathbb{N}_0} c_n z^n$, (4.2) uncovers $\delta\eta = P \circ \tilde{F}(-\partial_x)$ and under the convention $\partial_x^{-1} := \int_0$ we may thus write $L = P \circ F(-\partial_x)$.

Corollary 4.9. *As in η only $-c_{-1}\int_0$ increases the degree in x, the highest order (called* leading log*) of ${}_0\phi$ is the tree factorial (note the analogy to* (3.4)*): For any forest $w \in \mathcal{F}$,*

$$ {}_0\phi(w) \in {}^{\left[-c_{-1}\int_0\right]}\rho(w) + \mathcal{O}\left(x^{|w|-1}\right) \underset{2.5}{=} \frac{\left(-c_{-1}x\right)^{|w|}}{w!} + \mathbb{K}[x]_{<|w|}. \qquad (4.5)$$

4.2 Feynman rules as Hopf algebra morphisms

As ${}_0\phi\colon H_R \to \mathbb{K}[x]$ is a morphism of Hopf algebras, the induced map $G_{\mathbb{K}}^{\mathbb{K}[x]} \to G_{\mathbb{K}}^{H_R}$ given by $\mathrm{ev}_a \mapsto {}_0\phi_a := \mathrm{ev}_a \circ {}_0\phi$ becomes a morphism of groups. In particular note

Corollary 4.10. *Using* (4.3) *we obtain the* renormalization group equation *(as in [17])*

$$ {}_0\phi_a \star {}_0\phi_b = {}_0\phi_{a+b}, \quad \textit{for any} \quad a, b \in \mathbb{K}. \tag{4.6}$$

Before we obtain the generator of this one-parameter group in corollary 4.13, note how this result gives non-trivial relations between individual trees (graphs) like

$$\begin{aligned} {}_0\phi_a \star {}_0\phi_b \left(\overset{\bullet}{\bullet}\right) &= {}_0\phi_a \left(\overset{\bullet}{\bullet}\right) + {}_0\phi_a (\bullet)\, {}_0\phi_b (\bullet) + {}_0\phi_b \left(\overset{\bullet}{\bullet}\right) \\ &\underset{(3.10)}{=} c_{-1}^2 \frac{a^2+b^2}{2} - c_{-1}c_0\,(a+b) + c_{-1}^2 ab \underset{(3.10)}{=} {}_0\phi_{a+b}\left(\overset{\bullet}{\bullet}\right). \end{aligned}$$

Proposition 4.11. *Let H be any connected bialgebra and $\phi\colon H \to \mathbb{K}[x]$ a morphism of bialgebras.*[4] *Then $\log_\star \phi$ is given by the linear term in x through*

$$\log_\star \phi = x \cdot \partial_0 \circ \phi. \tag{4.7}$$

Proof. Letting $\phi\colon C \to H$ and $\psi\colon H \to \mathcal{A}$ denote morphisms of coalgebras and algebras, exploiting $(\psi \circ \phi - u_{\mathcal{A}} \circ \varepsilon_C)^{\star n} = \psi \circ (\phi - u_H \circ \varepsilon_H)^{\star n} = (\psi - u_{\mathcal{A}} \circ \varepsilon_H)^{\star n} \circ \phi$ in (2.2) proves $(\log_\star \psi) \circ \phi = \log_\star(\psi \circ \phi) = \psi \circ \log_\star \phi$. Now set $\psi = \mathrm{ev}_a$ and use lemma 4.5. $\square$

Example 4.12. *In the leading-*log *case* (2.6) *we read off $\partial_0 \circ \varphi = Z_\bullet \in \mathfrak{g}_{\mathbb{K}}^{H_R}$ where $Z_\bullet(w) := \delta_{w,\bullet}$. Comparing $\varphi = \exp_\star(xZ_\bullet)$ with* (2.6) *shows $|w|! = w! \cdot Z_\bullet^{\star|w|}(w)$, hence*[5]

$$\frac{|w|}{w!} = \frac{1}{(|w|-1)!}\sum_w Z_\bullet(w_1) Z_\bullet^{\star|w|-1}(w_2) = \sum_{w\colon\ w_1 = \bullet} \frac{1}{|w_2|!} Z_\bullet^{\star|w_2|}(w_2) = \sum_{w\colon\ w_1=\bullet} \frac{1}{w_2!}.$$

Corollary 4.13. *The character ${}_0\phi$ is fully determined by the* anomalous dimension

$$H_R' \supset \mathfrak{g}_{\mathbb{K}}^{H_R} \ni \gamma := -\partial_0 \circ {}_0\phi \quad \textit{such that} \quad {}_0\phi = \exp_\star(-x\cdot\gamma) = \sum_{n\in\mathbb{N}_0} \frac{\gamma^{\star n}}{n!}(-x)^n. \tag{4.8}$$

An analogous phenomenon happens with the counterterms in the minimal subtraction scheme: The first order poles $\propto z^{-1}$ alone already determine the full counterterm via the *scattering formula* proved in [10]. However, (4.8) is much simpler as illustrated in

Example 4.14. *Reading off $\gamma(\bullet) = c_{-1}$, $\gamma\left(\overset{\bullet}{\bullet}\right) = c_{-1}c_0$ and $\gamma\left(\overset{\bullet}{\bullet\bullet}\right) = 2c_{-1}^2 c_1$ from the example 4.8 above, corollary 4.13 determines the higher powers of x through*

$$\begin{aligned} {}_0\phi\left(\overset{\bullet}{\bullet}\right) &\underset{(2.2)}{=} \left[e - x\gamma + x^2\frac{\gamma\star\gamma}{2}\right]\left(\overset{\bullet}{\bullet}\right) = 0 - x\gamma\left(\overset{\bullet}{\bullet}\right) + x^2\frac{\gamma^2(\bullet)}{2} = -c_{-1}c_0\,x + c_{-1}^2\frac{x^2}{2}, \\ {}_0\phi\left(\overset{\bullet}{\bullet\bullet}\right) &= 0 - x\gamma\left(\overset{\bullet}{\bullet\bullet}\right) + x^2\frac{\gamma\otimes\gamma}{2}\left(2\bullet\otimes\overset{\bullet}{\bullet} + \bullet\bullet\otimes\bullet\right) - x^3\frac{\gamma\otimes\gamma\otimes\gamma}{6}\left(2\bullet\otimes\bullet\otimes\bullet\right) \\ &= -\gamma^3(\bullet)\frac{x^3}{3} + x^2\gamma(\bullet)\gamma\left(\overset{\bullet}{\bullet}\right) - 2c_{-1}^2c_1\,x = -c_{-1}^3\frac{x^3}{3} + c_{-1}^2c_0\,x^2 - 2c_{-1}^2c_1\,x. \end{aligned}$$

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[4] This already implies ϕ to be a morphism of Hopf algebras.

[5] This combinatorial relation among tree factorials, noted in [17], thus drops out of $\Delta\varphi = (\varphi\otimes\varphi)\circ\Delta$.

Note how the fragment $\bullet\bullet \otimes \bullet$ of $\Delta\left(\;\right)$ does not contribute to the quadratic terms $\frac{x^2}{2}\gamma \star \gamma$, as γ vanishes on products. We will exploit this in (5.5) of section 5.1 and close with a method of calculating γ emerging from

Lemma 4.15. *From* $\gamma \circ B_+ = -\partial_0 \circ L \circ {}_0\phi = \mathrm{ev}_0 \circ [zF(z)]_{-\partial_x} \circ \exp_\star(-x\gamma)$ *we obtain the inductive formula* $\gamma \circ B_+ = \sum_{n\in\mathbb{N}_0} c_{n-1}\gamma^{\star n}$.

Example 4.16. *We can recursively calculate* $\gamma(\bullet) = c_{-1}\varepsilon(\mathbb{1}) = c_{-1}$, *similarly also*

$$\gamma\left(\;\right) = c_{-1}\varepsilon(\bullet) + c_0\gamma(\bullet) = c_{-1}c_0,$$

$$\gamma\left(\;\right) = c_{-1}\varepsilon\left(\;\right) + c_0\gamma\left(\;\right) + c_1\gamma\star\gamma\left(\;\right) = c_{-1}c_0^2 + c_1[\gamma(\bullet)]^2 = c_{-1}c_0^2 + c_{-1}^2c_1,$$

$$\gamma\left(\;\right) = c_{-1}\varepsilon(\bullet\bullet) + c_0\gamma(\bullet\bullet) + c_1\gamma\star\gamma(\bullet\bullet) = 2c_1[\gamma(\bullet)]^2 = 2c_{-1}^2c_1 \quad \textit{and so on.}$$

5 Dyson-Schwinger equations and correlation functions

We now study the implications for the *correlation functions* (5.3) as formal power series in the *coupling constant* g. For simplicity we restrict to a single equation and refer to [24] for systems. With detailed treatments in [1, 11], for our purposes suffices

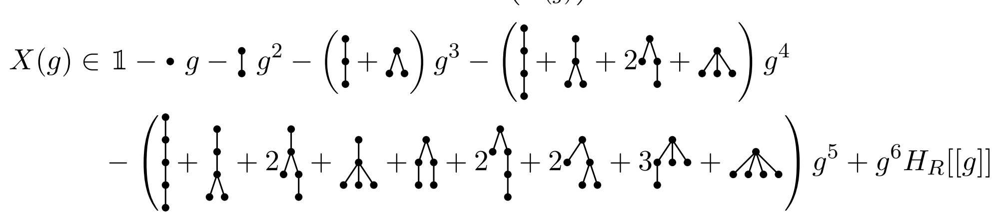
Definition 5.1. *To a parameter* $\kappa \in \mathbb{K}$ *and a family of cocycles* $B_\cdot\colon \mathbb{N} \to HZ^1_\varepsilon(H_R)$ *we associate the* combinatorial Dyson-Schwinger equation[6]

$$X(g) = \mathbb{1} + \sum_{n\in\mathbb{N}} g^n B_n\left(X^{1+n\kappa}(g)\right). \tag{5.1}$$

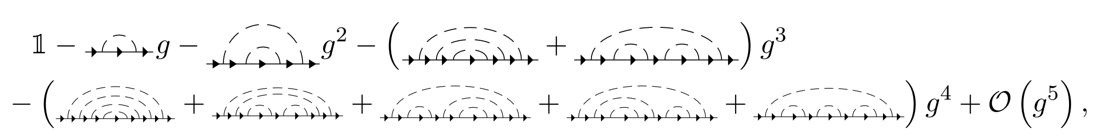
Lemma 5.2. *As* perturbation series $X(g) = \sum_{n\in\mathbb{N}_0} x_n g^n \in H_R[[g]]$, *equation* (5.1) *has a unique solution. It begins with* $x_0 = \mathbb{1}$ *while* x_{n+1} *is determined recursively from* $x_0, \ldots, x_n$. *These coefficients generate a Hopf sub algebra, explicitly we find*[7]

$$\Delta X(g) = \sum_{n\in\mathbb{N}_0} [X(g)]^{1+n\kappa} \otimes g^n x_n \in (H_R \otimes H_R)[[g]]. \tag{5.2}$$

Example 5.3. *In [5, 22],* $X(g) = \mathbb{1} - gB_+\left(\frac{1}{X(g)}\right)$ *features* $\kappa = -2$, *summing all trees*



with a symmetry factor. Physically these correspond to (Yukawa) propagators



arising from insertions of the one-loop graph into itself.

[6] As $x_0 = \mathbb{1}$, for arbitrary p the series $[X(g)]^p := \sum_{n\in\mathbb{N}_0}\binom{p}{n}[X(g)-\mathbb{1}]^n \in H_R[[g]]$ is well defined.

[7] A proof of (5.2) may be found in [11] and [12, 13] study systems of Dyson-Schwinger equations.

Definition 5.4. *The* correlation function $G(g)$ *evaluates the renormalized Feynman rules* ${}_0\phi\colon H_R \to \mathbb{K}[\ell]$ *on the perturbation series* $X(g)$*, yielding the formal power series*

$$G(g) := {}_0\phi \circ X(g) = \sum_{n\in\mathbb{N}_0} {}_0\phi(x_n) g^n \in (\mathbb{K}[\ell])\,[[g]]. \tag{5.3}$$

We call $\widetilde{\gamma}(g) := \gamma \circ X(g) = -\partial_\ell|_0 G(g) \in \mathbb{K}[[g]]$ *the* physical anomalous dimension.

Example 5.5. *The Feynman rules* φ *from* (2.6) *result in the convergent series* $G(g) = \sqrt{1-2g\ell}$ *and* $\widetilde{\gamma}(g) \underset{4.12}{=} -Z_{\bullet} \circ X(g) = g$ *for the propagator of example 5.3. Perturbatively,*

$$G(g) = 1 - \frac{(g\ell)}{\bullet!} - \frac{(g\ell)^2}{\substack{\bullet\\ \bullet}!} - \frac{(g\ell)^3}{\substack{\bullet\\ \bullet\\ \bullet}!} - \frac{(g\ell)^3}{\substack{\bullet\\ \bullet\,\bullet}!} - \ldots = 1 - g\ell - \frac{1}{2}(g\ell)^2 - \frac{1}{2}(g\ell)^3 + \mathcal{O}\left((g\ell)^4\right).$$

5.1 Propagator coupling duality

The Hopf subalgebra of the perturbation series allows to calculate convolutions in

Lemma 5.6. *Let* $\psi \in \mathfrak{g}_{\mathcal{A}}^{H_R}$ *denote an infinitesimal character,* $\Psi \in G_{\mathcal{A}}^{H_R}$ *a character and* $\lambda \in \mathrm{Hom}(H_R, \mathcal{A})$ *a linear map. Then (in suggestive notation)*

$$\begin{aligned}(\Psi \star \lambda) \circ X(g) &= [\Psi \circ X(g)] \cdot \lambda \circ X\left(g\,[\Psi \circ X(g)]^{\kappa}\right) && (5.4)\\ &:= [\Psi \circ X(g)] \cdot \sum_{n\in\mathbb{N}_0} \lambda(x_n) \cdot \left(g\,[\Psi \circ X(g)]^{\kappa}\right)^n \in \mathcal{A}[[g]] \\ (\psi \star \lambda) \circ X(g) &= [\psi \circ X(g)] \cdot (\mathrm{id} + \kappa g \partial_g)\,[\lambda \circ X(g)] \in \mathcal{A}[[g]]. && (5.5)\end{aligned}$$

Proof. These are immediate consequences of lemma 5.2, for (5.5) consider

$$\psi\left([X(g)]^{1+n\kappa}\right) \cdot g^n = \sum_{i\in\mathbb{N}_0} \binom{1+n\kappa}{i} \psi\left([X(g)-\mathbb{1}]^i\right) g^n = \psi\left(X(g)-\mathbb{1}\right) \cdot (1+n\kappa) g^n. \quad \square$$

Example 5.7. *Continuing 5.3 we deduce* $Z_{\bullet}^{\star 2}(X(g)) = -g(1-2g\partial_g)(-g) = -g^2$ *and*

$$Z_{\bullet}^{\star n+1}(X(g)) \underset{(5.5)}{=} -g^{n+1}(2n-1)(2n-3)\cdots(1) = -g^{n+1}\frac{(2n)!}{2^n n!},$$

proving $\varphi(x_{n+1}) = -2^{-n} C_n \ell^{n+1}$ *with the* Catalan numbers C_n *already noted in [20]. From 5.5 we find their generating function* $2g\sum_{n\in\mathbb{N}_0} g^n C_n = 1 - \sqrt{1-4g}$.

Corollary 5.8. *As* ${}_0\phi$ *is a morphism of Hopf algebras, for any* $a, b \in \mathbb{K}$ *we can factor*

$$G_{a+b}(g) = ({}_0\phi_a \star {}_0\phi_b) \circ X(g) \underset{(5.4)}{=} G_a(g) \cdot G_b\left[g G_a^{\kappa}(g)\right] = G_b(g) \cdot G_a\left[g G_b^{\kappa}(g)\right]. \tag{5.6}$$

These functional equations of formal power series make sense for the non-perturbative correlation functions as well. Relating the scale- with the coupling-dependence, this integrated form of the renormalization group equation becomes infinitesimally

Corollary 5.9. *From* $-\frac{\mathrm{d}}{\mathrm{d}x}\, {}_0\phi = \gamma \star {}_0\phi = {}_0\phi \star \gamma$ *or differentiating* (5.6) *by* b *at zero note*

$$G_\ell(g) \cdot \widetilde{\gamma}\left[g G_\ell^\kappa(g)\right] \underset{(5.4)}{=} -\partial_\ell G_\ell(g) \underset{(5.5)}{=} \widetilde{\gamma}(g) \cdot (1 + \kappa g \partial_g)\, G_\ell(g). \tag{5.7}$$

The first of these equations generalizes the *propagator coupling duality* in [5, 20]. For any fixed coupling g, it expresses the correlation function as the solution of the ode

$$-\frac{\mathrm{d}}{\mathrm{d}\ell} \ln G_\ell(g) = \widetilde{\gamma}\left[g e^{\kappa \ln G_\ell(g)}\right] \quad \text{with} \quad \ln G_0(g) = 0, \tag{5.8}$$

determining $G_\ell(g)$ completely from $\widetilde{\gamma}(g)$ in a non-perturbative manner as in (5.11).

Example 5.10. *The* leading-log *expansion takes only the highest power of* ℓ *in each* g*-order. Equally,* $\widetilde{\gamma}(g) = c g^n$ *for constants* $c \in \mathbb{K}$, $n \in \mathbb{N}$ *and* (5.8) *integrates to*

$$G_{\textit{leading-}\mathrm{log}}(g) = \left[1 + c n \kappa \ell g^n\right]^{-\frac{1}{n\kappa}}. \tag{5.9}$$

As a special case we recover example 5.5 for $n = c = 1$ *and* $\kappa = -2$.

Example 5.11. *In the linear case* $\kappa = 0$, (5.6) *states* $G_{a+b}(g) = G_a(g) \cdot G_b(g)$ *in accordance with the* scaling solution $G_\ell(g) = e^{-\ell \widetilde{\gamma}(g)}$ *of* (5.8)*, well-known from [19].*

Example 5.12. *For vertex insertions as in [2] we have* $\kappa = 1$*, so* $G_{a+b}(g) = G_b(g) \cdot G_a\big[\widetilde{G}_b(g)\big]$ *expresses the running of the coupling constant* $\widetilde{G} := g \cdot G$*: A change in scale by* b *is (up to a multiplicative constant) equivalent to replacing the coupling* g *by* $\widetilde{G}_b(g)$.

5.2 The physicist's renormalization group

To cast (5.6) and (5.7) into the common forms of (7.3.15) and (7.3.21) in [7], we introduce the β*-function* $\beta(g) := -\kappa g \widetilde{\gamma}(g)$ and the *running coupling* $g(\mu)$ as the solution of

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g(\mu) = \beta(g(\mu)), \quad \text{hence} \quad \mu \frac{\mathrm{d}}{\mathrm{d}\mu} G\left(g(\mu), \ln \tfrac{s}{\mu}\right) \underset{(5.7)}{=} \widetilde{\gamma}(g(\mu)) G\left(g(\mu), \ln \tfrac{s}{\mu}\right). \tag{5.10}$$

Integration relates the correlation functions for different renormalization points μ in

$$G\left(g(\mu_2), \ln \tfrac{s}{\mu_2}\right) = G\left(g(\mu_1), \ln \tfrac{s}{\mu_1}\right) \cdot \exp\left[\int_{\mu_1}^{\mu_2} \widetilde{\gamma}(g(\mu)) \tfrac{\mathrm{d}\mu}{\mu}\right] \underset{(5.10)}{=} G\left(g(\mu_1), \ln \tfrac{s}{\mu_1}\right) \cdot \left[\frac{g(\mu_2)}{g(\mu_1)}\right]^{-\frac{1}{\kappa}}.$$

Setting $\mu_1 = s$ we may thus write $G_\ell(g)$ explicitly in terms of $\widetilde{\gamma}(g)$ as

$$G_\ell(g) = \left[\frac{g}{g(s)}\right]^{-\frac{1}{\kappa}}, \quad \text{with } g(s) \text{ subject to} \quad \ell = \ln \frac{s}{\mu} = \int_g^{g(s)} \frac{\mathrm{d}g'}{\beta(g')}. \tag{5.11}$$

5.3 Relation to Mellin transforms

We finally exploit the analytic input from theorem 4.6 to the perturbation series in

$$G_\ell(g) \underset{(5.1)}{=} 1 + \sum_{n\in\mathbb{N}} g^n {}_0\phi \circ B_n \left(X(g)^{1+n\kappa}\right) \underset{4.6}{=} 1 + \sum_{n\in\mathbb{N}} g^n \left[-c_{-1}^{(n)} \int_0 + P \circ \widetilde{F}_n(\partial_{-\ell})\right] G_\ell(g)^{1+n\kappa},$$

with Mellin transforms $F_n(z) = \frac{1}{z} c_{-1}^{(n)} + \widetilde{F}_n(z)$ corresponding to the insertions[8] B_n.

Corollary 5.13. *By $G_\ell(0) = 1$, the power series $G_\ell(g) \in \mathbb{K}[\ell][[g]]$ is fully determined by*

$$\partial_{-\ell} G_\ell\,(g) \underset{(3.2)}{=} \sum_{n\in\mathbb{N}} g^n \left[zF_n(z)\right]_{z=-\partial_\ell} \left(G_\ell(g)^{1+n\kappa}\right). \tag{5.12}$$

Restricting to a single cocycle $F_k(z) = F(z)\delta_{k,n}$, choosing $F(z) = \frac{c_{-1}}{z}$ reproduces (5.9) from $\partial_{-\ell} G_\ell(g) = g^n c_{-1} G_\ell(g)^{1+n\kappa}$. More generally, for any rational $F(z) = \frac{p(z)}{q(z)} \in \mathbb{K}(z)$ with $q(0) = 0$, (5.12) collapses to a finite order ode $q(-\partial_\ell)G_\ell(g) = g^n p(-\partial_\ell) G_\ell(g)^{1+n\kappa}$ that makes perfect sense non-perturbatively (extending the algebraic $\partial_\ell \in \mathrm{End}(\mathbb{K}[\ell])$ to the analytic differential operator).

Example 5.14. *For $F(z) = \frac{1}{z(1-z)}$, the propagator ($\kappa = -2$ as in example 5.3) fulfils*

$$\frac{g}{G_\ell(g)} = \partial_{-\ell}\,(1 - \partial_{-\ell})\, G_\ell(g) \underset{(5.7)}{=} \widetilde{\gamma}(g)\,(1 - 2g\partial_g)\,[1 - \widetilde{\gamma}(g)\,(1 - 2g\partial_g)]\, G_\ell(g).$$

At $\ell = 0$ this evaluates to $\widetilde{\gamma}(g) - \widetilde{\gamma}(g)(1 - 2g\partial_g)\widetilde{\gamma}(g) = g$, which is studied in [24, 5].

6 Automorphisms of H_R

Applying the universal property to H_R itself, adding coboundaries to B_+ leads to

Definition 6.1. *For any $\alpha \in H_R'$, theorem 2.4 defines the Hopf algebra morphism*

$$^{\alpha}\chi := {}^{B_+ + \delta\alpha}\rho\colon\ H_R \to H_R \quad \textit{such that} \quad {}^{\alpha}\chi \circ B_+ = [B_+ + \delta\alpha] \circ {}^{\alpha}\chi. \tag{6.1}$$

Example 6.2. *The action on the simplest trees yields*

$$\begin{aligned}
{}^{\alpha}\chi\,(\bullet) &= {}^{\alpha}\chi \circ B_+(\mathbb{1}) = B_+(\mathbb{1}) + (\delta\alpha)(\mathbb{1}) = B_+(\mathbb{1}) = \bullet, \\
{}^{\alpha}\chi\,\Big(\overset{\bullet}{\underset{\bullet}{|}}\Big) &= {}^{\alpha}\chi \circ B_+(\bullet) = (B_+ + \delta\alpha)\,{}^{\alpha}\chi\,(\bullet) = \overset{\bullet}{\underset{\bullet}{|}} + \delta\alpha\,(\bullet) = \overset{\bullet}{\underset{\bullet}{|}} + \alpha(\mathbb{1})\bullet, \\
{}^{\alpha}\chi \left(\begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix}\right) &= \begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix} + 2\alpha(\mathbb{1})\overset{\bullet}{\underset{\bullet}{|}} + \left\{[\alpha(\mathbb{1})]^2 + \alpha\,(\bullet)\right\}\bullet \quad \textit{and} \quad {}^{\alpha}\chi\,\Big(\underset{\bullet\ \bullet}{\overset{\bullet}{\wedge}}\Big) = \underset{\bullet\ \bullet}{\overset{\bullet}{\wedge}} + 2\alpha\,(\bullet)\,\bullet + \alpha(\mathbb{1})\bullet\bullet.
\end{aligned}$$

These morphisms capture the change of ${}^{L}\rho$ under a variation of L by a coboundary in

[8] For this generality we need decorated rooted trees as commented on in section 6.1

Theorem 6.3. *Let H denote a bialgebra, $L \in HZ^1_\varepsilon(H)$ a 1-cocycle and further $\alpha \in H'$ a functional. Then for ${}^L\rho, {}^{L+\delta\alpha}\rho\colon\ H_R \to H$ given through theorem 2.4 and ${}^{\alpha\circ{}^L\rho}\chi\colon\ H_R \to H_R$ from definition 6.1, we have*

$$
{}^{L+\delta\alpha}\rho = {}^L\rho \circ {}^{\left[\alpha \circ {}^L\rho\right]}\chi, \quad \textit{equivalently} \quad
\begin{tikzcd}
H_R \arrow[r, "{}^{L+\delta\alpha}\rho"] \arrow[d, "{}^{\alpha\circ{}^L\rho}\chi"'] & H \\
H_R \arrow[ur, "{}^L\rho"'] &
\end{tikzcd}
\quad \textit{commutes.} \tag{6.2}
$$

Proof. As both sides of (6.2) are algebra morphisms, it suffices to prove it inductively for trees: Let it be true for a forest $w \in \mathcal{F}$, then it holds as well for the tree $B_+(w)$ by

$$
\begin{aligned}
&{}^L\rho \circ {}^{\left[\alpha\circ{}^L\rho\right]}\chi \circ B_+(w) \underset{(2.5)}{=} {}^L\rho \circ \left[B_+ + \delta\left(\alpha \circ {}^L\rho\right)\right] \circ {}^{\left[\alpha\circ{}^L\rho\right]}\chi(w) \\
&= \left\{L \circ {}^L\rho + (\delta\alpha) \circ {}^L\rho\right\} \circ {}^{\left[\alpha\circ{}^L\rho\right]}\chi(w) = \{L + \delta\alpha\} \circ \underbrace{{}^L\rho \circ {}^{\left[\alpha\circ{}^L\rho\right]}\chi(w)}_{{}^{L+\delta\alpha}\rho(w)} \underset{(2.5)}{=} {}^{L+\delta\alpha}\rho \circ B_+(w).
\end{aligned}
$$

We used $(\delta\alpha) \circ {}^L\rho = {}^L\rho \circ \delta\left(\alpha \circ {}^L\rho\right)$, following from ${}^L\rho$ being a morphism of bialgebras. ☐

Hence the action of a coboundary $\delta\alpha$ on the universal morphisms induced by L is given by ${}^{\alpha\circ{}^L\rho}\chi$. This turns out to be an automorphism of H_R as shown in

Theorem 6.4. *The map ${}^{\cdot}\chi\colon\ H'_R \to \mathrm{End}_{\textit{Hopf}}(H_R)$, taking values in the space of Hopf algebra endomorphisms of H_R, fulfils the following properties:*

1. *For any $w \in \mathcal{F}$ and $\alpha \in H'_R$, ${}^\alpha\chi(w)$ differs from w only by lower order forests:*

$$
{}^\alpha\chi(w) \in w + H_R^{|w|-1} = w + \bigoplus_{n=0}^{|w|-1} H_{R,n}. \tag{6.3}
$$

2. *${}^{\cdot}\chi$ maps H'_R into the Hopf algebra automorphisms $\mathrm{Aut}_{\textit{Hopf}}(H_R)$. Its image is closed under composition, as for any $\alpha, \beta \in H'_R$ we have ${}^\alpha\chi \circ {}^\beta\chi = {}^\gamma\chi$ taking*

$$
\gamma = \alpha + \beta \circ {}^\alpha\chi^{-1}. \tag{6.4}
$$

3. *The maps $\delta\colon H'_R \to HZ^1_\varepsilon(H_R)$ and ${}^{\cdot}\chi\colon H'_R \to \mathrm{Aut}_{\textit{Hopf}}(H_R)$ are injective, thus the subgroup $\mathrm{im}\,{}^{\cdot}\chi = \{{}^\alpha\chi\colon\ \alpha \in H'_R\} \subset \mathrm{Aut}_{\textit{Hopf}}(H_R)$ induces a group structure on H'_R with neutral element 0 and group law $\triangleright$ given by*

$$
\alpha \triangleright \beta := {}^{\cdot}\chi^{-1}\left({}^\alpha\chi \circ {}^\beta\chi\right) \underset{(6.4)}{=} \alpha + \beta \circ {}^\alpha\chi^{-1} \quad \textit{and} \quad \alpha^{\triangleright -1} = -\alpha \circ {}^\alpha\chi. \tag{6.5}
$$

Proof. Statement (6.3) is an immediate consequence of $\delta\alpha(H_R^n) \subseteq H_R^n$: Starting from ${}^\alpha\chi(\bullet) = \bullet$, suppose inductively (6.3) to hold for forests $w, w' \in \mathcal{F}$. Then it obviously also holds for $w \cdot w'$ as well and even so for $B_+(w)$ through

$$
{}^\alpha\chi \circ B_+(w) = [B_+ + \delta\alpha] \circ {}^\alpha\chi(w) \subseteq [B_+ + \delta\alpha]\left(w + H_R^{|w|-1}\right) \subseteq B_+(w) + H_R^{|w|}.
$$

This already implies bijectivity of ${}^{\alpha}\chi$, but applying (6.2) to $L = B_+ + \delta\alpha$ and ${}^{\tilde{\alpha}}\chi$ for $\tilde{\alpha} := -\alpha \circ {}^{\alpha}\chi$ shows $\mathrm{id} = {}^{\alpha}\chi \circ {}^{\tilde{\alpha}}\chi$ directly. We deduce bijectivity of all ${}^{\alpha}\chi$ and thus ${}^{\alpha}\chi \in \mathrm{Aut}_{\mathrm{Hopf}}(H_R)$ with the inverse ${}^{\alpha}\chi^{-1} = {}^{\tilde{\alpha}}\chi$. Now (6.4) follows from

$$^{\left[\alpha+\beta\,\circ\,{}^{\alpha}\chi^{-1}\right]}\chi = {}^{[B_+ + \delta\alpha]+\delta\left(\beta\,\circ\,{}^{\alpha}\chi^{-1}\right)}\rho \underset{(6.2)}{=} {}^{[B_++\delta\alpha]}\rho \circ {}^{\left[\beta\,\circ\,{}^{\alpha}\chi^{-1}\,\circ\,{}^{(B_++\delta\alpha)}\rho\right]}\chi = {}^{\alpha}\chi \circ {}^{\beta}\chi.$$

Finally consider $\alpha, \beta \in H_R'$ with ${}^{\alpha}\chi = {}^{\beta}\chi$, then $0 = ({}^{\alpha}\chi - {}^{\beta}\chi) \circ B_+ = \delta \circ (\alpha - \beta) \circ {}^{\alpha}\chi$ reduces the injectivity of ${}^{\cdot}\chi$ to that of δ. But if $\delta\alpha = 0$, for all $n \in \mathbb{N}_0$

$$0 = \delta\alpha\left(\bullet^{n+1}\right) = \sum_{i=0}^{n} \binom{n+1}{i} \alpha\left(\bullet^{i}\right) \bullet^{n+1-i} \quad \text{implies} \quad \alpha\left(\bullet^{n}\right) = 0.$$

Given an arbitrary forest $w \in \mathcal{F}$ and $n \in \mathbb{N}$, the expression

$$0 = \delta\alpha\left(\bullet^{n} w\right) = w \underbrace{\alpha\left(\bullet^{n}\right)}_{0} + \sum_{w} \sum_{i=0}^{n} \binom{n}{i} \bullet^{i} w' \alpha\left(\bullet^{n-i} w''\right) + \sum_{i=1}^{n} \binom{n}{i} \left[\bullet^{i} w \underbrace{\alpha\left(\bullet^{n-i}\right)}_{0} + \bullet^{i} \alpha\left(w \bullet^{n-i}\right)\right]$$

simplifies upon projection onto $\mathbb{K}\bullet$ to $\alpha\left(w\bullet^{n-1}\right) = -\frac{1}{n}\sum_{w:\ w'=\bullet} \alpha\left(\bullet^{n} w''\right)$. Iterating this formula exhibits $\alpha(w)$ as a scalar multiple of $\alpha(\bullet^{|w|}) = 0$ and proves $\alpha = 0$. $\square$

6.1 Decorated rooted trees

Our observations generalize straight forwardly to the Hopf algebra $H_R(\mathcal{D})$ of rooted trees with decorations drawn from a set $\mathcal{D}$. In this case, the universal property assigns to each $\mathcal{D}$-indexed family $L_{\cdot}\colon\ \mathcal{D} \to \mathrm{End}(\mathcal{A})$ the unique algebra morphism

$${}^{L_{\cdot}}\rho\colon H_R(\mathcal{D}) \to \mathcal{A} \quad \text{such that} \quad {}^{L_{\cdot}}\rho \circ B_+^{d} = L_d \circ {}^{L_{\cdot}}\rho \quad \text{for any } d \in \mathcal{D}.$$

For cocycles $\mathrm{im}\, L_{\cdot} \subseteq HZ^1_{\varepsilon}(\mathcal{A})$ this is a morphism of bialgebras and even of Hopf algebras (should $\mathcal{A}$ be Hopf). For a family $\alpha_{\cdot}\colon\ \mathcal{D} \to H_R'(\mathcal{D})$ of functionals, setting $L_d^{\alpha_{\cdot}} := B_+^d + \delta\alpha_d$ yields an automorphism ${}^{\alpha_{\cdot}}\chi := {}^{L_{\cdot}^{\alpha_{\cdot}}}\rho$ of the Hopf algebra $H_R(\mathcal{D})$. Theorems 6.3 and 6.4 generalize in the obvious way.

In view of the Feynman rules, decorations d denote different graphs into which B_+^d inserts a subdivergence. Hence we gain a family of Mellin transforms $F_{\cdot}$ and theorem 4.6 generalizes straightforwardly as ${}_0\phi \circ B_+^d = P \circ F_d(-\partial_\ell) \circ {}_0\phi$.

6.2 Subleading corrections under variations of Mellin transforms

As an application of (6.2) consider a change of the Mellin transform F to a different F' that keeps c_{-1} fixed but alters the other coefficients c_n. With $\alpha := \eta' - \eta$,

$${}_0\phi' = {}^{L'}\rho = {}^{L+\delta\alpha}\rho = {}^{L}\rho \circ {}^{\left[\alpha\,\circ\,{}^{L}\rho\right]}\chi = {}_0\phi \circ {}^{\left[\alpha\,\circ\,{}_0\phi\right]}\chi$$

translates the new renormalized Feynman rules ${}_0\phi'$ into the original ${}_0\phi$. For $c_{-1} = -1$, this relates ${}_0\phi$ to $\varphi = \int_0 \rho$ using example 6.2 together with $\eta \circ \varphi(w) = (-1)^{|w|} \frac{|w|!}{w!} c_{|w|}$ as

$$
{}_0\phi\left(\bullet\right) = x = \varphi\left(\bullet\right) = \varphi \circ {}^{\eta\circ\varphi}\chi\left(\bullet\right), \qquad {}_0\phi\left(\substack{\bullet\\\bullet}\right) = \frac{x^2}{2} + c_0 x = \varphi\left\{\substack{\bullet\\\bullet} + \eta(1)\bullet\right\} = \varphi \circ {}^{\eta\circ\varphi}\chi\left(\substack{\bullet\\\bullet}\right),
$$

$$
{}_0\phi\left(\substack{\bullet\\\bullet\\\bullet}\right) = \frac{x^3}{6} + x^2 c_0 + x(c_0^2 - c_1) = \varphi\left\{\substack{\bullet\\\bullet\\\bullet} + 2c_0 \substack{\bullet\\\bullet} + \left[c_0^2 - c_1\right]\bullet\right\} = \varphi \circ {}^{\eta\circ\varphi}\chi\left(\substack{\bullet\\\bullet\\\bullet}\right) \quad \text{and}
$$

$$
{}_0\phi\left(\overset{\bullet}{\bullet\bullet}\right) = \frac{x^3}{3} + c_0 \cdot x^2 - 2c_1 \cdot x = \varphi\left\{\overset{\bullet}{\bullet\bullet} + c_0 \bullet\bullet - 2c_1 \bullet\right\} = \varphi \circ {}^{\eta\circ\varphi}\chi\left(\overset{\bullet}{\bullet\bullet}\right).
$$

Corollary 6.5. *The new correlation function ${}_0\phi \circ X = \varphi \circ \widetilde{X}$ equals the original φ applied to a modified perturbation series $\widetilde{X}(g)$, fulfilling a Dyson-Schwinger equation differing by coboundaries. By* (6.3) *the leading logs coincide and explicitly*

$$
\widetilde{X}(g) := {}^{\eta\circ\varphi}\chi \circ X(g) = \mathbb{1} + \sum_{n\in\mathbb{N}} g^n \left(B_n + \delta\eta_n\right)\left(\widetilde{X}(g)^{1+n\kappa}\right).
$$

7 Locality, finiteness and minimal subtraction

Consider the regulated but unrenormalized Feynman rules ${}_z\phi$. Now setting $\mathcal{A} := \mathbb{C}[z^{-1}, z]]$ and $\phi := {}_z\phi_1 \in G_{\mathcal{A}}^{H_R}$, (3.3) fixes the scale dependence ${}_z\phi_s = \phi \circ \theta_{-sz}$.

Proposition 7.1. *For any character $\phi \in G_{\mathcal{A}}^{H_R}$, the following are equivalent:*

1. *$\phi^{\star -1} \star (\phi \circ Y) = \phi \circ (S \star Y)$ maps into $\frac{1}{z}\mathbb{C}[[z]]$, so $\lim_{z\to 0} \phi^{\star -1} \star (z\phi \circ Y)$ exists.*
2. *For every $n \in \mathbb{N}_0$, $\phi^{\star -1} \star (\phi \circ Y^n) = \phi \circ (S \star Y^n)$ maps into $z^{-n}\mathbb{C}[[z]]$.*
3. *For any $s \in \mathbb{K}$, $\phi^{\star -1} \star (\phi \circ \theta_{sz}) = \phi \circ (S \star \theta_{sz})$ maps into $\mathbb{C}[[z]]$.*

Proof. We refer to the accounts in [21, 5, 10], however only 1. ⇒ 2. is non-trivial and

$$
\phi \circ \left(S \star Y^{n+1}\right) = \phi \circ (S \star Y^n) \circ Y + [\phi \circ (S \star Y)] \star [\phi \circ (S \star Y^n)]
$$

yields an inductive proof. It exploits $(S \circ Y) \star \mathrm{id} = -S \star Y$ in the formula (α arbitrary)

$$
S \star (\alpha \circ Y) - (S \star \alpha) \circ Y = -(S \circ Y) \star \alpha = -\left[(S \circ Y) \star \mathrm{id}\right] \star S \star \alpha = S \star Y \star S \star \alpha. \quad \square
$$

Note that condition 3. is equivalent to the finiteness 3.7 of the physical limit ${}_0\phi$ as

$$
{}_z\phi_{R,s} = {}_z\phi_\mu^{\star -1} \star {}_z\phi_s = \phi \circ \left[(S \circ \theta_{-z\ln\mu}) \star \theta_{-zs}\right] = \phi \circ (S \star \theta_{-z\ln\frac{s}{\mu}}) \circ \theta_{-z\ln\mu}.
$$

Corollary 7.2. *The anomalous dimension can be obtained from the $\frac{1}{z}$-pole coefficients*

$$
\gamma = -\partial_0 \circ {}_0\phi = -\partial_0 \circ \lim_{z\to 0} \phi \circ (S \star \theta_{-zx}) = \mathrm{Res}\left[\phi \circ (S \star Y)\right]. \tag{7.1}
$$

The minimal subtraction scheme R_{MS} projects onto the pole parts such that $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ where $\mathcal{A}_- := z^{-1}[z^{-1}]$ and $\mathcal{A}_+ := [[z]]$. Though it renders finiteness trivial, its counterterms might depend on the scale s and violate locality. So from [10] we need

Definition 7.3. *A Feynman rule $\phi \in G_{\mathcal{A}}^{H_R}$ is called* local *iff in the minimal subtraction scheme, the counterterm $(\phi \circ \theta_{sz})_-$ is independent of $s \in \mathbb{K}$.*

Proposition 7.4. *Locality of $\phi \in G_{\mathcal{A}}^{H_R}$ is equivalent to the conditions of proposition 7.1.*

Proof. In case of 7.1, $\phi \circ \theta_{sz} = (\phi_-)^{\star -1} \star \{\phi_+ \star [\phi^{\star -1} \star (\phi \circ \theta_{sz})]\}$ is a Birkhoff decomposition by condition 3. such that $(\phi \circ \theta_{sz})_- = \phi_-$ from uniqueness. Conversely, for local ϕ,

$$0 = R_{\text{MS}} \circ (\phi \circ \theta_{sz})_+ = R_{\text{MS}} \circ \Big[(\phi \circ \theta_{sz})_- \star (\phi \circ \theta_{sz})\Big] = R_{\text{MS}} \circ [\phi_- \star (\phi \circ \theta_{sz})]$$

implies $\mathbb{C}[[z]] = \ker R_{\text{MS}} \supseteq \operatorname{im} \phi_- \star (\phi \circ \theta_{sz})$ and convolution with $\phi_+^{\star -1} = \phi^{\star -1} \star \phi_-^{\star -1} : H_R \to \mathbb{C}[[z]]$ yields condition 3. of 7.1. □

So we showed algebraically that the problems of finiteness in the kinetic scheme and locality in minimal subtraction are precisely the same. These schemes are related by

Lemma 7.5. *If ${}_z\phi_{MS,s}$ denotes the R_{MS}-renormalized Feynman rule, then its scale dependence is given by ${}_0\phi$ through ${}_z\phi_{MS} = (R_\mu \circ {}_z\phi_{MS}) \star {}_z\phi_R$ (as already exploited in [4]).*

Proof. Locality of the minimal subtraction counterterms ϕ_- implies $R_\mu \circ \phi_- = \phi_-$, hence

$$(R_\mu \circ {}_z\phi_{\text{MS}}) \star {}_z\phi_R = [R_\mu \circ (\phi_- \star {}_z\phi)] \star (R_\mu \circ {}_z\phi)^{\star -1} \star {}_z\phi = (R_\mu \circ \phi_-) \star {}_z\phi = {}_z\phi_{\text{MS}}. \quad □$$

The physical limit $\text{ev}_{\ln s} \circ {}_0\phi_{\text{MS}} = \lim_{z \to 0} {}_z\phi_{\text{MS},s}$ yields polynomials ${}_0\phi_{\text{MS}}$ and 7.5 becomes

Corollary 7.6. *The characters ${}_0\phi_{MS}, {}_0\phi : H_R \to \mathbb{K}[x]$ fulfil the relations*

$${}_0\phi_{MS} = (\varepsilon \circ {}_0\phi_{MS}) \star {}_0\phi, \quad \textit{equivalently} \quad \Delta \circ {}_0\phi_{MS} = ({}_0\phi_{MS} \otimes {}_0\phi) \circ \Delta. \tag{7.2}$$

In particular, the constant parts $\varepsilon \circ {}_0\phi_{\text{MS}} = \text{ev}_0 \circ {}_0\phi_{\text{MS}} \in G_{\mathbb{K}}^{H_R}$ determine ${}_0\phi_{\text{MS}}$ completely as the scale dependence is governed by ${}_0\phi$. Using ${}_0\phi = \exp_\star(-x\gamma)$, the β-*functional* ${}_0\phi_{\text{MS}} = \exp_\star(x\beta) \star (\varepsilon \circ {}_0\phi_{\text{MS}})$ from [10] relates to γ by conjugation:

$$\beta \star (\varepsilon \circ {}_0\phi_{\text{MS}}) = -(\varepsilon \circ {}_0\phi_{\text{MS}}) \star \gamma.$$

Corollary 7.7. *Applying* (5.4) *to* (7.2) *expresses the correlation function of the R_{MS}-scheme to the kinetic scheme by a redefinition of the coupling constant:*

$$G_{\text{MS},\ell}(g) = G_{\text{MS},0}(g) \cdot G_\ell\Big(g \cdot [G_{\text{MS},0}(g)]^\kappa\Big).$$

8 Feynman graphs and logarithmic divergences

In a typical renormalizable scalar quantum field theory, the vertex function is logarithmically divergent and may be renormalized by a simple subtraction as studied above. Referring to [6] for quadratic divergences, we now restrict to logarithmically divergent graphs with only logarithmic subdivergences, in D dimensions of space-time.

Following the notation established in [3], the renormalized amplitude of a graph Γ in the Hopf algebra H of Feynman graphs is given by the *forest formula*[9]

$$\Phi_+(\Gamma) = \int \Omega_\Gamma \sum_{F \in \mathcal{F}(\Gamma)} \frac{(-1)^{|F|}}{\psi_F^{D/2}} \ln \frac{\frac{\varphi}{\psi}_{\Gamma/F} + \sum_{\Gamma \neq \gamma \in F} \frac{\tilde{\varphi}}{\psi}_{\gamma/F}}{\frac{\tilde{\varphi}}{\psi}_{\Gamma/F} + \sum_{\Gamma \neq \gamma \in F} \frac{\tilde{\varphi}}{\psi}_{\gamma/F}}. \tag{8.1}$$

The forests $\mathcal{F}(\Gamma)$ account for subdivergences, the first and second *Symanzik polynomials* $\psi_\Gamma, \varphi_\Gamma$ depend on the edge variables α_e and we integrate over $\mathbb{RP}_{>0}^{|E(\Gamma)|-1}$ in projective space with canonical volume form Ω_Γ. Apart from a scale s, φ_Γ depends on dimensionless *angle variables* $\Theta = \left\{ \frac{m^2}{s} \right\} \cup \left\{ \frac{p_i \cdot p_j}{s} \right\}$ built from the mass m and external momenta p_i. We abbreviate $\frac{\varphi}{\psi}_\Gamma := \frac{\varphi_\Gamma}{\psi_\Gamma}$ and denote evaluation at the renormalization point $(\tilde{s}, \tilde{\Theta})$ of the kinetic scheme by a tilde or $\cdot|_R := \cdot|_{(s,\Theta)\mapsto(\tilde{s},\tilde{\Theta})}$.

Definition 8.1. *Holding the angles Θ fixed, the* period functional $\mathcal{P} \in H'$ *is given by*

$$\mathcal{P}(\Gamma) := -\frac{\partial}{\partial \ln s} \Phi_+(\Gamma)\Big|_R \quad \textit{for any} \quad \Gamma \in H. \tag{8.2}$$

Corollary 8.2. *For any graph $\Gamma \in H$, the value $\mathcal{P}(\Gamma)$ is a period in the sense of [15] (provided that $\tilde{s}$ and all $\theta \in \tilde{\Theta}$ are rational) by the formula*

$$\mathcal{P}(\Gamma) \underset{(8.1)}{=} \int \Omega_\Gamma \sum_{F \in \mathcal{F}(\Gamma)} \frac{(-1)^{1+|F|}}{\psi_F^{D/2}} \frac{\frac{\tilde{\varphi}}{\psi}_{\Gamma/F}}{\frac{\tilde{\varphi}}{\psi}_{\Gamma/F} + \sum_{\Gamma \neq \gamma \in F} \frac{\tilde{\varphi}}{\psi}_{\gamma/F}}. \tag{8.3}$$

For primitive (subdivergence free) graphs, [23] gives equivalent definitions of this period in momentum and position space. The product rule, (8.2) and $\Phi_+|_R = \varepsilon$ show

Corollary 8.3. *The period is an infinitesimal character $\mathcal{P} \in \mathfrak{g}_{\mathbb{K}}^H$ (it vanishes on any graph that is not connected).*

8.1 Renormalization group

Proposition 8.4. *Holding the angles Θ fixed, differentiation by the scale results in*[10]

$$-\frac{\partial}{\partial \ln s} \Phi_+ = \mathcal{P} \star \Phi_+. \tag{8.4}$$

[9] We prefer to work in the *parametric* representation as introduced in [14, section 6-2-3].

[10] This simple form circumvents the decomposition into one-scale graphs utilized in [6] and therefore holds in the original renormalization Hopf algebra H.

Proof. Adding $0 = \mathcal{P}(\Gamma) - \mathcal{P}(\Gamma)$ and collecting the contributions of $\frac{\tilde{\varphi}}{\psi}_{\gamma/F}$ in $(*)$ we find

$$-\frac{\partial}{\partial \ln s}\Phi_+(\Gamma) \underset{(8.1)}{=} \int \Omega_\Gamma \left\{ \frac{1}{\psi_\Gamma^{D/2}} + \sum_{\{\Gamma\} \neq F \in \mathcal{F}(\Gamma)} \frac{(-1)^{1+|F|}}{\psi_F^{D/2}} \frac{\frac{\varphi}{\psi}_{\Gamma/F}}{\frac{\varphi}{\psi}_{\Gamma/F} + \sum_{\Gamma \neq \delta \in F} \frac{\tilde{\varphi}}{\psi}_{\delta/F}} \right\}$$

$$\underset{(8.3)}{=} \mathcal{P}(\Gamma) + \int \Omega_\Gamma \sum_{\{\Gamma\} \neq F \in \mathcal{F}(\Gamma)} \frac{(-1)^{1+|F|}}{\psi_F^{D/2}} \frac{\left(\frac{\varphi}{\psi}_{\Gamma/F} - \frac{\tilde{\varphi}}{\psi}_{\Gamma/F}\right) \sum_{\Gamma \neq \gamma \in F} \frac{\tilde{\varphi}}{\psi}_{\gamma/F}}{\left[\frac{\varphi}{\psi}_{\Gamma/F} + \sum_{\Gamma \neq \delta \in F} \frac{\tilde{\varphi}}{\psi}_{\delta/F}\right] \cdot \left[\frac{\tilde{\varphi}}{\psi}_{\Gamma/F} + \sum_{\Gamma \neq \delta \in F} \frac{\tilde{\varphi}}{\psi}_{\delta/F}\right]}$$

$$\underset{(*)}{=} \mathcal{P}(\Gamma) + \int \Omega_\Gamma \sum_{\substack{\gamma \prec \Gamma \\ |\pi_0(\gamma)|=1}} \sum_{\gamma \in F \in \mathcal{F}(\Gamma)} \frac{(-1)^{1+|F|}}{\psi_F^{D/2}} \frac{\left(\frac{\varphi}{\psi}_{\Gamma/F} - \frac{\tilde{\varphi}}{\psi}_{\Gamma/F}\right) \frac{\tilde{\varphi}}{\psi}_{\gamma/F}}{\left[\frac{\varphi}{\psi}_{\Gamma/F} + \sum_{\Gamma \neq \delta \in F} \frac{\tilde{\varphi}}{\psi}_{\delta/F}\right] \cdot \sum_{\delta \in F} \frac{\tilde{\varphi}}{\psi}_{\delta/F}}.$$

With $\gamma \prec \Gamma$ denoting a subdivergence $\gamma \neq \Gamma$, the forests $F \in \mathcal{F}(\Gamma)$ containing γ correspond bijectively to the forests of γ and Γ/γ by

$$\mathcal{F}_\gamma(\Gamma) := \{F \in \mathcal{F}(\Gamma)\colon\ \gamma \in F\} \ni F \mapsto \left(F|_\gamma,\ F/\gamma\right) \in \mathcal{F}(\gamma) \times \mathcal{F}(\Gamma/\gamma), \quad \text{using}$$
$$F|_\gamma := \{\delta \in F\colon\ \delta \preceq \gamma\} \quad \text{and} \quad F/\gamma := \{\delta/\gamma\colon\ \delta \in F \quad \text{and} \quad \delta \npreceq \gamma\}.$$

This is an immediate consequence of the definition of a forest, as for $F \in \mathcal{F}_\gamma(\Gamma)$, each $\delta \in F$ is either disjoint to γ or strictly containing γ (in both cases it is mapped to $\delta/\gamma \in F/\gamma$) or itself a subdivergence of γ. Thus integrating $\int_0^\infty \frac{A-\tilde{A}}{(A+tB)(\tilde{A}+tB)} \mathrm{d}t = B^{-1} \ln \frac{A}{\tilde{A}}$ in

$$= \mathcal{P}(\Gamma) + \int \sum_{\substack{\gamma \prec \Gamma \\ |\pi_0(\gamma)|=1}} \Omega_\gamma \wedge \Omega_{\Gamma/\gamma} \sum_{\substack{F_\gamma \in \mathcal{F}(\gamma) \\ F \in \mathcal{F}(\Gamma/\gamma)}} \frac{(-1)^{1+|F_\gamma|+|F|}}{\psi_{F_\gamma}^{D/2} \cdot \psi_F^{D/2}}$$

$$\times \int_0^\infty \frac{\mathrm{d}t_\gamma}{t_\gamma} \frac{\left(\frac{\varphi}{\psi}_{\Gamma/F} - \frac{\tilde{\varphi}}{\psi}_{\Gamma/F}\right) \cdot t_\gamma \cdot \frac{\tilde{\varphi}}{\psi}_{\gamma/F_\gamma}}{\left[\frac{\varphi}{\psi}_{\Gamma/F} + \sum_{\Gamma \neq \delta \in F} \frac{\tilde{\varphi}}{\psi}_{\delta/F} + t_\gamma \cdot \sum_{\delta \in F_\gamma} \frac{\tilde{\varphi}}{\psi}_{\delta/F_\gamma}\right] \cdot \left[\sum_{\delta \in F} \frac{\tilde{\varphi}}{\psi}_{\delta/F} + t_\gamma \cdot \sum_{\delta \in F_\gamma} \frac{\tilde{\varphi}}{\psi}_{\delta/F_\gamma}\right]}$$

$$= \mathcal{P}(\Gamma) + \int \sum_{\substack{\gamma \prec \Gamma \\ |\pi_0(\gamma)|=1}} \Omega_\gamma \wedge \Omega_{\Gamma/\gamma} \sum_{\substack{F_\gamma \in \mathcal{F}(\gamma) \\ F \in \mathcal{F}(\Gamma/\gamma)}} \frac{(-1)^{1+|F_\gamma|+|F|}}{\psi_{F_\gamma}^{D/2} \cdot \psi_F^{D/2}} \cdot \frac{\frac{\tilde{\varphi}}{\psi}_{\gamma/F_\gamma}}{\sum_{\delta \in F_\gamma} \frac{\tilde{\varphi}}{\psi}_{\delta/F_\gamma}} \cdot \ln \frac{\frac{\varphi}{\psi}_{(\Gamma/\gamma)/F} + \sum_{\delta \in F \setminus \{\Gamma/\gamma\}} \frac{\tilde{\varphi}}{\psi}_{\delta/F}}{\sum_{\delta \in F} \frac{\tilde{\varphi}}{\psi}_{\delta/F}}$$

reduces to the projective $\int \Omega_\gamma$ in the edge variables of the subgraph γ, making use of

$$|F| = \left|F|_\gamma\right| + |F/\gamma|, \quad \frac{\varphi}{\psi}_{\delta/F} = \begin{cases} \frac{\varphi}{\psi}_{(\delta/\gamma)/(F/\gamma)}, & \text{if } \gamma \npreceq \delta \in F \\ \frac{\varphi}{\psi}_{\delta/\,F|_\gamma}, & \text{if } \gamma \succeq \delta \in F \end{cases} \quad \text{and} \quad \psi_F = \psi_{F|_\gamma} \cdot \psi_{F/\gamma}.$$

The apparent factorization into $\mathcal{P}(\gamma)$ and $\Phi_+(\Gamma/\gamma)$ shows that we obtain convergent integrals for each $\gamma \prec \Gamma$ individually and may therefore separate into

$$= \mathcal{P}(\Gamma) + \sum_{\substack{\gamma \prec \Gamma \\ |\pi_0(\gamma)|=1}} \int \Omega_\gamma \sum_{F_\gamma \in \mathcal{F}(\gamma)} \frac{(-1)^{1+|F_\gamma|}}{\psi_{F_\gamma}^{D/2}} \cdot \frac{\frac{\tilde{\varphi}}{\psi}_{\gamma/F_\gamma}}{\sum_{\delta \in F_\gamma} \frac{\tilde{\varphi}}{\psi}_{\delta/F_\gamma}}$$

$$\times \int \Omega_{\Gamma/\gamma} \sum_{F \in \mathcal{F}(\Gamma/\gamma)} \frac{(-1)^{|F|}}{\psi_F^{D/2}} \cdot \ln \frac{\frac{\varphi}{\psi}_{(\Gamma/\gamma)/F} + \sum_{\delta \in F \setminus \{\Gamma/\gamma\}} \frac{\tilde{\varphi}}{\psi}_{\delta/F}}{\sum_{\delta \in F} \frac{\tilde{\varphi}}{\psi}_{\delta/F}} = \mathcal{P} \star \Phi_+(\Gamma).$$

Note that the terms $\gamma \otimes \Gamma/\gamma$ of $\Delta(\Gamma)$ with $|\pi_0(\gamma)| > 1$ do not contribute here by 8.3. $\square$

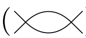
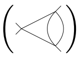
Together with 8.3 and the connected graduation of H, this shows

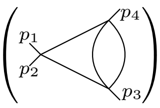
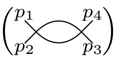
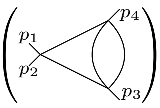
$$\Phi_+ = \sum_{n \in \mathbb{N}_0} \frac{(-\ell)^n}{n!} \left[\left(-\frac{\partial}{\partial \ln s} \right)^n \Phi_+ \right]_{s=\tilde{s}} \underset{(8.4)}{=} \sum_{n \in \mathbb{N}_0} \frac{(-\ell \cdot \mathcal{P})^{\star n}}{n!} \star \Phi_+|_{s=\tilde{s}},$$

where we set $\ell := \ln \frac{s}{\tilde{s}}$ and the series is pointwise finite. Hence note

Corollary 8.5. *The renormalized Feynman rules $\Phi_+ = \Phi_+|_{\Theta=\tilde{\Theta}} \star \Phi_+|_{s=\tilde{s}}$ factorize ([6] gives a different decomposition) into the angle-dependent part $\Phi_+|_{s=\tilde{s}}$ and the scale-dependence $\Phi_+|_{\Theta=\tilde{\Theta}}$ given as the Hopf algebra morphism*

$$\Phi_+|_{\Theta=\tilde{\Theta}} = \exp_\star(-\ell\mathcal{P}) : \quad H \to \mathbb{K}[\ell]. \tag{8.5}$$

Example 8.6. *For primitive $\Gamma \in \mathrm{Prim}(H)$, $\Phi_+(\Gamma) = -\ell \cdot \mathcal{P}(\Gamma) + \Phi_+|_{s=\tilde{s}}(\Gamma)$ disentangles the scale- and angle-dependence. Subdivergences evoke higher powers of ℓ with angle-dependent factors.* Dunce's cap *of ϕ^4-theory gives* $\mathcal{P}$ () $= \mathcal{P}$ () $= 1$ *such that*

Φ_+ () $= \frac{\ell^2}{2} - \ell - \ell\, \Phi_+|_{s=\tilde{s}}$ () $+ \Phi_+|_{s=\tilde{s}}$ ().

8.2 Dimensional regularization

The *dimensional regularization* of [7] assigns a Laurent series ${}_z\Phi(\Gamma)$ in z to each Feynman graph $\Gamma \in H$, which for large $\Re z$ is given by the convergent parametric integral

$${}_z\Phi(\Gamma) = \left[\prod_{e \in E(\Gamma)} \int_0^\infty \alpha_e \right] \frac{e^{-\frac{\varphi}{\psi}_\Gamma}}{\psi_\Gamma^{D/2-z}}. \tag{8.6}$$

As $\frac{\varphi}{\psi}_\Gamma$ is linear in the scale s and homogeneous of degree one in the edge variables, simultaneously rescaling of all α_e yields (for logarithmically divergent graphs)

Corollary 8.7. *The scale dependence ${}_z\Phi = {}_z\Phi|_{s=\tilde{s}} \circ \theta_{-z\ell}$ of (8.6) is induced from the grading Y of H given by the loop number.*

Thus the finiteness of the physical limit $\Phi_+|_{\Theta=\tilde{\Theta}} = \lim_{z\to 0} {}_z\Phi|_R \circ (S \star \theta_{-z\ell})$ results by 7.1 in the local character ${}_z\Phi|_R \in G^H_{\mathcal{A}}$, evaluated at the renormalization point $(\tilde{s}, \tilde{\Theta})$.

Corollary 8.8. *In dimensional regularization, the period* (8.3) *is the* $\frac{1}{z}$*-pole coefficient*

$$\mathcal{P} \underset{(7.1)}{=} \operatorname{Res} \circ \ {}_z\Phi|_R \circ (S \star Y). \tag{8.7}$$

8.3 Dilatations and conformal symmetry

For $\lambda > 0$, consider the *dilatation operator* Λ_λ scaling masses $m \mapsto \lambda \cdot m$ and momenta $p_i \mapsto \lambda \cdot p_i$. It fixes all angles Θ, multiplies the scale s with λ^2 and therefore acts as

$$\Phi_+ \circ \Lambda_\lambda = \exp_\star \left(-\mathcal{P} \ln \frac{s}{\tilde{s}} \right) \star \Phi_+|_{s=\tilde{s}} \circ (s \mapsto s \cdot \lambda^2) = \exp_\star (-2\mathcal{P} \ln \lambda) \star \Phi_+.$$

In other words, the dilatations $\mathbb{R}_{>0} \ni \lambda \mapsto \Lambda_\lambda \mapsto \exp_\star (-2\mathcal{P} \ln \lambda) \star \cdot$ are represented on the group $G^H_{\mathcal{A}}$ of characters by a left convolution. As the unrenormalized logarithmically divergent graphs are dimensionless and naively invariant under Λ_λ, $\mathcal{P}$ precisely measures how renormalization breaks this symmetry, giving rise to *anomalous dimensions*.

9 Conclusion

We stress that the physical limit of the renormalized Feynman rules results in a morphism ${}_0\phi\colon H_R \to \mathbb{K}[x]$ of Hopf algebras in case of the kinetic scheme. This compatibility with the coproduct allows to obtain ${}_0\phi$ from the linear terms γ only. As we just exemplified, these relations are statements about individual Feynman graphs unraveling scale- and angle-dependence in a simple way. Again we recommend [6] for further reading.

Secondly we revealed how Hochschild cohomology governs not only the perturbation series through Dyson-Schwinger equations, but also determines the Feynman rules. Addition of exact one-cocycles captures variations of Feynman rules and the anomalous dimension γ can efficiently be calculated in terms of Mellin transform coefficients.

Note how this feature is lost upon substitution of the kinetic scheme by minimal subtraction: We do not obtain a Hopf algebra morphism anymore due to the constant terms, which are also more difficult to obtain in terms of the Mellin transforms F.

Finally we want to emphasize the remarks in section 5 towards a non-perturbative framework. Though this relation between $F(z)$ and the anomalous dimension $\widetilde{\gamma}(g)$ is still under investigation and so far only fully understood in special cases, these already give interesting results [5, 24].

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