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THE ELLIPTIC DILOGARITHM FOR THE SUNSET GRAPH

by

Spencer Bloch & Pierre Vanhove

Abstract. — We study the sunset graph defined as the scalar two-point self-energy at two-loop order. We evaluate the sunset integral for all identical internal masses in two dimensions. We give two calculations for the sunset amplitude; one based on an interpretation of the amplitude as an inhomogeneous solution of a classical Picard-Fuchs differential equation, and the other using arithmetic algebraic geometry, motivic cohomology, and Eisenstein series. Both methods use the rather special fact that the amplitude in this case is a family of periods associated to the universal family of elliptic curves over the modular curve $X_1(6)$. We show that the integral is given by an elliptic dilogarithm evaluate evaluated at a sixth root of unity modulo periods. We explain as well how this elliptic dilogarithm value is related to the regulator of a class in the motivic cohomology of the universal elliptic family.

Contents

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1. Introduction

Scattering amplitudes are fundamental objects describing how particles interact. At a given loop order in the perturbative expansion in the coupling constant, there are many ways of constructing the amplitudes from first principles of quantum field theory. The result is an algebraic integral with parameters, and the physical problem of efficient evaluation of the integral is linked to the qualitative mathematical problem of classifying these multi-valued functions of the complexified kinematic invariants. The amplitudes are locally analytic, presenting branch points at the thresholds where particles can appear.

These questions can be studied order by order in perturbation. At one-loop order, around a four dimensional space-time, all the scattering amplitudes can be expanded in a basis of integral functions given by box, triangle, bubbles and tadpole integral functions, together with rational term contributions [**1, 2, 3**] (see [**4, 5**] for some reviews on this subject).

The finite part of the $\epsilon = (4-D)/2$ expansion of the box and triangle integral functions is given by dilogarithms of the proper combination of kinematic invariants. The finite part of the bubble and tadpole integral is a logarithm function of the physical parameters.

The appearance of the dilogarithm and logarithms at one-loop order is predictable from unitarity considerations since this reproduces the behaviour of the one-loop scattering amplitude under single, or double two-particle cuts in four dimensions.

The fact that one-loop amplitudes are expressed as dilogarithms and logarithms can as well be understood motivitically [**6, 7**], but the status of two-loop order scattering amplitude is far less well understood for generic amplitudes (see for instance [**8, 9, 10, 11**] for some recent progress).

The sunset integral arises as the two-loop self-energy diagram in the evaluation of higher-order correction in QED, QCD or electroweak theory precision calculations [**12**], or as a sub-topology of higher-order computation [**10**]. As a consequence, it has been the subject of numerous analysis. The integral for various configurations of vanishing masses has been analyzed using the Mellin-Barnes methods in [**13**], with two different masses and three equal masses in [**14**]. An asymptotic expansion of the sunset integral has been given in [**15**]. Various form for the integral have been considered either in geometrical terms [**16**], displaying some relations to one-loop amplitude [**17**], or a representation in terms of hypergeometric function as given in [**18, 19**] or as an integral of Bessel functions as in [**20, 21**]. Or a differential equation approach (in close relation to the method used in section 5 of the present work) was considered in [**22, 23, 24, 25, 26**]. We refer to these papers for a more complete list of references.

The question whether of the sunset integral can expressed in terms of known mathematical functions like polylogarithms has not so far been addressed.

In order to answer this question will consider the sunset graph in two space-time dimensions depicted in figure 1. The sunset integral with internal masses in two dimensions is a completely finite integral free of infra-red and ultra-violet divergences. Working with a finite integral will ease the discussion of the mathematical nature of this integral.

Although the ultimate goal is to understand the properties of two-loop amplitudes around four dimensional space-time, the restriction to two dimensions is not too bad since dimension shifting formulas, given in [**23**], relate the result in two dimensions to the finite part of the integral in four dimensions.

Another restriction of this work is to focus only on the all equal masses case with all internal masses non zero and positive. We will find in this case that the sunset integral is nicely expressed (5.28) in terms of elliptic dilogarithms obtained by q-averaging values of the dilogarithm at a sixth root of unity in (5.27) with the following q-expansion

$$E_{\ominus}(q) = \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{(-1)^{n-1}}{2\,n^2} \left(\sin\left(\frac{n\pi}{3}\right) + \sin\left(\frac{2n\pi}{3}\right)\right) \frac{1}{1-q^n}\,. \tag{1.1}$$

This expression is locally analytic in q and differs from the one for the regulator map [**27**] that would be expressed in terms of the (non-analytic) Bloch-Wigner dilogarithm $D(z)$. The reason for this difference explained in section 6 when evaluating the amplitude using motivic methods (compare equations (6.10) and (6.15)).

We give two calculations for the sunset amplitude; one based on an interpretation of the amplitude as an inhomogeneous solution of a classical Picard-Fuchs differential equation (in section 5), and the other using arithmetic algebraic geometry (in section 6), motivic cohomology (in section 7), and Eisenstein series (in section 8). Both methods use the rather special fact that the amplitude in this equal mass case is a family of periods associated to the universal family of elliptic curves over the modular curve $X_1(6)$. The elliptic fibres $\mathcal{E}_t$ are naturally embedded in $\mathbb{P}^2$ and pass through the vertices $(1,0,0), (0,1,0), (0,0,1)$ of the coordinate triangle $\Delta : XYZ = 0$. Let $P \to \mathbb{P}^2$ be the blowup of these three points, so the inverse image of Δ is a hexagon $\mathfrak{h} \subset P$. Then $\mathcal{E}_t$ lifts to P, and the amplitude period is closely related to the cohomology

$$H^2(P - \mathcal{E}_t, \mathfrak{h}^0) \tag{1.2}$$

where $\mathfrak{h}^0 := \mathfrak{h} - \mathfrak{h} \cap \mathcal{E}_t$.

The motivic picture which emerges from the basic sunset $D = 2$ equal mass case applies as well to all Feynman amplitudes. Quite generally, the Feynman amplitude will be the solution to some sort of inhomogeneous Picard-Fuchs equation $PF(x) = f(x)$ where x represent physical parameters like masses and kinematic invariants. The motive determines the function f. In the equal mass sunset case, f is constant because the amplitude motive reduces to an extension of the motive of an elliptic curve by a constant Tate motive. If the masses are distinct, the elliptic curve motive is extended by a Kummer motive which is itself an extension of constant Tate motives. The function f then involves a non-constant logarithm.

In general, our idea is to relate the physical amplitude to a regulator in the sense of arithmetic algebraic geometry applied to a class in motivic cohomology. One has a hypersurface (depending on kinematic parameters) (*cf.* equation (2.5) in the sunset case) in projective space $\mathbb{P}^n$. One has an integrand (*cf.* equation (2.7) in the sunset case) which is a top degree meromorphic form on $\mathbb{P}^n$ with a pole along X, and one has a chain of integration (2.4). Let $\Delta : x_0 x_1 \cdots x_n = 0$ be the coordinate simplex in $\mathbb{P}^n$. One first blows up faces of Δ on $\mathbb{P}^n$ in such a way that the strict transform Y of X meets all faces properly. In the sunset case, $Y = X$ and the blowup yields $P \to \mathbb{P}^2$, the blowup of the three vertices of the triangle Δ. The total transform $\mathfrak{h}$ of Δ in P is a hexagon in the sunset case. Next, one constructs a motivic cochain which is an algebraic cycle

Ξ on $P \times (\mathbb{P}^1 - \{0, \infty\})^{n-1}$ of dimension $n-1$. One then tries to interpret Ξ as a relative motivic cohomology class in $H^{n+1}_M(P, Y, \mathbb{Q}(n))$. This is possible in the sunset case, but in general one has a closed $Z \subset Y$ and Ξ represents a motivic cohomology class in $H^{n+1}_M(P-Z, Y-Z, \mathbb{Q}(n))$. When $Z = \emptyset$, the inhomogeneous term f of the differential equation will be relatively simple, involving constants and logs. The study of Z more generally should involve shrinking edges of the graph, so if the loop order is small (say two) one may hope for an inductive process describing the differential equation satisfied by the amplitude.

2. The sunset integral

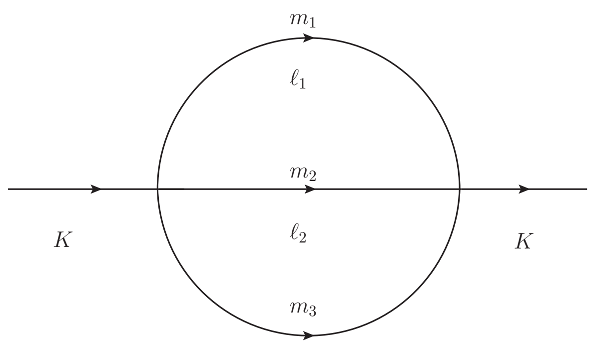

FIGURE 1. The two-loop sunset graph with three different masses. The external momentum is K, and the loop momenta are ℓ_1 and ℓ_2.

The integral associated to the sunset graph given in figure 1

$$\mathcal{I}^D_\ominus(K^2, m_1^2, m_2^2, m_3^2) = (\mu^2)^{3-D} \int_{\mathbb{R}^{2,2(D-1)}} \frac{\delta(K = \ell_1 + \ell_2 + \ell_3)}{d_1^2 d_2^2 d_3^2} \, d^D\ell_1 d^D\ell_2 \,, \tag{2.1}$$

where $d_i^2 = \ell_i^2 - m_i^2 + i\varepsilon$ with $\varepsilon > 0$ and $m_i \in \mathbb{R}^+$ and ℓ_i and K are in $\mathbb{R}^{1,D-1}$ with signature $(+-\cdots-)$. With this choice of metric, the physical region corresponds to $K^2 \geq 0$. In the following we will consider the Wick rotated integral with $\ell_i \in \mathbb{R}^D$. We have introduced the arbitrary scale μ^2 of mass-dimension two so that the integral is dimensionless.

The Feynman parametrisation is easily obtained following [**28**, sec. 6-2-3]

$$\mathcal{I}_{\ominus}^{D}=\frac{\pi^{D}\Gamma(3-D)}{(\mu^{2})^{D-3}}\int_{0}^{\infty}\int_{0}^{\infty}\frac{(x+y+xy)^{\frac{3(2-D)}{2}}\,dxdy}{((m_1^2x+m_2^2y+m_3^2)(x+y+xy)-K^2xy)^{3-D}}\,. \tag{2.2}$$

For geometric reasons, we will use an alternative symmetric representation of the integral formulated using homogeneous coordinates (see [**24, 25**])

$$\mathcal{I}_{\ominus}^{D}=\frac{\pi^{D}\Gamma(3-D)}{(\mu^{2})^{D-3}}\int_{\mathcal{D}}\frac{(xz+yz+xy)^{\frac{3(2-D)}{2}}\,(xdy\wedge dz-ydx\wedge dz+zdx\wedge dy)}{((m_1^2x+m_2^2y+m_3^2z)(xz+yz+xy)-K^2xyz)^{3-D}}\,. \tag{2.3}$$

with the domain of integration

$$\mathcal{D}=\{(x,y,z)\in\mathbb{P}^2|x,y,z\geq 0\}\,. \tag{2.4}$$

The physical parameters of the sunset integral enter the polynomial

$$A_{\ominus}(x,y,z,m_1,m_2,m_3,K^2):=(m_1^2x+m_2^2y+m_3^2z)(xz+xy+yz)-K^2xyz\,. \tag{2.5}$$

For generic values of the parameters the curve $A_{\ominus}(x,y,z)=0$ defines an elliptic curve

$$\mathcal{E}_{\ominus}(m_1,m_2,m_3,K^2):=\{A_{\ominus}(x,y,z,m_1,m_2,m_3,K^2)=0|(x,y,z)\in\mathbb{P}^2\}\,. \tag{2.6}$$

When the physical parameters varies this defines a family of elliptic curves. For special values the elliptic curve degenerates.

Before embarking a general discussion of the property of the elliptic curve, we discuss in section 3, the degeneration points $K^2=0$ and the pseudo-thresholds $K^2\in\{(-m_1+m_2+m_3)^2,(m_1-m_2+m_3)^2,(m_1+m_2-m_3)^2\}$.

In this paper we will only consider the all equal masses case $m_1^2=m_2^2=m_3^2=m^2$ and $K^2=t\,m^2$ and evaluate the integral in $D=2$ space-time dimensions

$$\mathcal{I}_{\ominus}^{2}=\frac{\pi^2\mu^2}{m^2}\int_{\mathcal{D}}\frac{xdy\wedge dz-ydx\wedge dz+zdx\wedge dy}{(x+y+z)(xz+yz+xy)-txyz}\,. \tag{2.7}$$

The restriction to two dimensions will make contact with the motivic analysis easier since the sunset integral is both ultraviolet and infrared finite for non-vanishing value of the internal mass.

If one is interested by the expression of the integral in four space-time dimensions one needs perform an $\epsilon = (4-D)/2$ expansion because the all equal masses sunset integral is not finite due to ultraviolet divergences

$$\mathcal{I}_{\ominus}^{4-2\epsilon}(K^2, m^2) = 16\pi^{4-2\epsilon}\Gamma(1+\epsilon)^2 \left(\frac{m^2}{\mu^2}\right)^{1-2\epsilon} \left(\frac{a_2}{\epsilon^2} + \frac{a_1}{\epsilon} + a_0 + O(\epsilon)\right) . \tag{2.8}$$

It was shown in [**23**] that the coefficients of this ϵ-expansion are given by derivatives of the two dimensional sunset integral, $\mathcal{J}_{\ominus}^2(t) = m^2/(\pi^2\mu^2)\,\mathcal{I}_{\ominus}^2$, with the first coefficients given by

$$\begin{aligned} a_2 &= -\frac{3}{8} \\ a_1 &= \frac{18-t}{32} \\ a_0 &= \frac{(t-1)(t-9)}{12}(1+(t+3)\frac{d}{dt})\mathcal{J}_{\ominus}^2(t) + \frac{13t-72}{128} . \end{aligned} \tag{2.9}$$

3. Special values

Before embarking into the analysis of the integral at generic values of the external momentum K^2 both the three masses case and the all equal masses case.

3.1. The case $K^2 = 0$ with three non vanishing masses. — We evaluate the integral at some special values $K^2 = 0$, assuming that the internal masses m_i are non-vanishing and positive. In $D = 2$ dimensions the integral is finite and reads

$$\mathcal{I}_{\ominus}^2(0, m_i) = \pi^2\mu^2 \int_0^{+\infty}\int_0^{+\infty} \frac{dxdy}{(x+y+xy)(m_1^2 x + m_2^2 y + m_3^2)} . \tag{3.1}$$

In this form the invariance under the permutations of the masses is seen as follows. The exchange between m_1 and m_2 corresponds to the transformation $(x, y) \to (y, x)$ and the exchange between m_1 and m_3 corresponds to the transformation $(x, y) \to (1/x, y/x)$.

Performing the integration over y we obtain

$$\mathcal{I}_{\ominus}^2(0, m_i^2) = \frac{\pi^2\mu^2}{m_1^2} J_0\left(\frac{m_1^2}{m_2^2}, \frac{m_3^2}{m_2^2}\right) , \tag{3.2}$$

with

$$\begin{aligned} J_0(a,b) &= a\int_0^\infty \frac{\log R(a,b,x)}{xR(a,b,x)-x}\,dx \\ R(a,b,x) &= (1+x^{-1})(ax+b)\,, \end{aligned} \tag{3.3}$$

We introduce z and $\bar z$ such that

$$xR(a,b,x)-x=(1+x)(ax+b)-x=a(x+z)(x+\bar z)\,, \tag{3.4}$$

with $(1-z)(1-\bar z)=1/a=m_2^2/m_1^2$ and $z\bar z=b/a=m_3^2/m_1^2$. The roots z and $\bar z$ are given by

$$z := \frac{m_1^2-m_2^2+m_3^2+\sqrt{\Delta}}{2m_1^2} \tag{3.5}$$

$$\bar z := \frac{m_1^2-m_2^2+m_3^2-\sqrt{\Delta}}{2m_1^2}\,. \tag{3.6}$$

where the discriminant

$$\Delta := M_2^2-4M_4=\sum_{i=1}^3 m_i^4-2\sum_{1\leq i<j\leq 3} m_i^2m_j^2\,. \tag{3.7}$$

The discriminant vanishes $\Delta=0$ when $m_3=m_1+m_2$ or any permutation of the masses (we assume all the masses to be positive).

3.1.1. The case $\Delta\neq 0$. — In this case in terms of these variables the integral becomes

$$\begin{aligned} J_0(z) &= \int_0^\infty j_0(z,\bar z,x)\,dx \\ j_0(z,\bar z,x) &= \frac{-\log x+\log(1+x)-\log(1-z)-\log(1-\bar z)+\log(x+z\bar z)}{(x+z)(x+\bar z)}\,. \end{aligned} \tag{3.8}$$

This integral is easily evaluated, and found to be given by

$$J_0(z)=\frac{4iD(z)}{z-\bar z}\,, \tag{3.9}$$

where $D(z)$ is the Bloch-Wigner dilogarithm function (we refer to appendix A for some details about this function). Therefore the sunset integral $\mathcal{I}^2_\ominus(0,m_i^2)$ is given by the Bloch-Wigner dilogarithm

$$\mathcal{I}^2_\ominus(0,m_i^2)=2i\pi^2\,\mu^2\,\frac{D(z)}{\sqrt{\Delta}}\,. \tag{3.10}$$

Under the permutation of the three masses (m_1, m_2, m_3) the variable z ranges over the orbit

$$\left\{z, 1-\bar z, \frac{1}{\bar z}, 1-\frac{1}{z}, \frac{1}{1-z}, -\frac{\bar z}{1-\bar z}\right\}. \tag{3.11}$$

Since Δ is invariant under the permutation of the masses and, thanks to the six functional equations for the Bloch-Wigner function in (A.4), the amplitude in (3.10) is invariant.

3.1.2. The case $\Delta = 0$. — Suppose for example $m_3 = m_1 + m_2$ then the roots in eq. (3.5) become $z = \bar z = (m_1 + m_2)/m_1$. The integral is easily evaluated and yields

$$\begin{aligned}\mathcal{I}^2_{\ominus}(0, m_1^2, m_2^2, (m_1+m_2)^2) = 4\pi^2\mu^2 \Big(&\frac{\log(m_1+m_2)}{m_1 m_2} - \frac{\log(m_1)}{m_2(m_1+m_2)} \\ &- \frac{\log(m_2)}{m_1(m_1+m_2)}\Big). \end{aligned}\tag{3.12}$$

The other cases are obtained with a permutation of the masses.

3.2. At the pseudo-thresholds with three internal masses. — We evaluate the integral at the pseudo-thresholds $K^2 = (m_1+m_2+m_3-2m_i)^2$ with $i = 1, 2, 3$. Since the sunset integral is invariant under the permutation of the internal masses we can assume that $0 < m_1 \leq m_2 \leq m_3$.

By combining direct integrations and numerical evaluations of the integrals for various values of the internal masses we find the expressions for the sunset integral at the pseudo-thresholds.

If we introduce the notation $M_i = m_1 + m_2 + m_3 - 2m_i \neq 0$ (the case $M_i = 0$ has been evaluated in section 3.1.2), we then have for $i = 1, 2, 3$

$$\mathcal{I}^2_{\ominus}(m_1^2, m_2^2, m_3^2, M_i^2) = \frac{-2i\pi^2\mu^2}{\sqrt{m_1 m_2 m_3 M_i}} \times \sum_{j=1}^{3} \frac{\partial M_i}{\partial m_j} \times \hat D\left(\sqrt{\frac{m_j^2 M_i}{m_1 m_2 m_3}}\right), \tag{3.13}$$

where we have defined $\tilde D(z)$

$$\tilde D(z) = \frac{1}{2i}\left(\mathrm{Li}_2(z) - \mathrm{Li}_2(-z) + \frac{1}{2}\log(z^2)\log\left(\frac{1-z}{1+z}\right)\right). \tag{3.14}$$

3.3. The all equal masses case. — When all the internal masses are identical $m_1 = m_2 = m_3 = m$ we set $t = K^2/m^2$. We evaluate the integral at the special values $t = 0$ and $t = 1$ for further reference in the complete evaluation done in section 5.

3.3.1. $t = 0$ case. — For $t = 0$ the integral has been evaluated in section 3.1 for generic values of the internal masses. When all the internal masses are equal $m_1 = m_2 = m_3 = m$, then the root in eq. (3.5) become the sixth-root of unity $z = \zeta_6 := (1 + i\sqrt{3})/2$. The sunset integral in (3.10) then evaluates to[1]

$$\mathcal{I}^2_\ominus(0, m^2) = i\pi^2 \frac{\mu^2}{m^2} \frac{D(\zeta_6)}{\Im\mathrm{m}(\zeta_6)} . \tag{3.15}$$

Using eq. (A.2) one can rewrite this expression for the integral as

$$\mathcal{I}^2_\ominus(0, m^2) = \frac{6i\pi^2}{\sqrt{3}} \frac{\mu^2}{m^2} \sum_{n\geq 1} \left(\sin\left(\frac{n\pi}{3}\right) - \sin\left(\frac{2n\pi}{3}\right)\right) \frac{1}{n^2} . \tag{3.16}$$

or in the expression in a form that will be useful later

$$\mathcal{I}^2_\ominus(0, m^2) = -\frac{8\pi^2}{5} \frac{\mu^2}{m^2} \left(\mathrm{Li}_2\left(\zeta_6^5\right) + \mathrm{Li}_2\left(\zeta_6^4\right) - \mathrm{Li}_2\left(\zeta_6^2\right) - \mathrm{Li}_2\left(\zeta_6\right)\right) . \tag{3.17}$$

3.3.2. $t = 1$ case. — The case $t = K^2/m^2 = 1$ corresponds to the one-particle pseudo-threshold when one internal line is cut. At this value the integral takes the form

$$\mathcal{I}^2_\ominus(K^2 = m^2, m^2) = \pi^2 \frac{\mu^2}{m^2} \int_0^\infty \int_0^\infty \frac{dxdy}{(x+1)(y+1)(x+y)} . \tag{3.18}$$

This integral is readily evaluated to give

$$\mathcal{I}^2_\ominus(m^2, m^2) = 2\pi^2 \frac{\mu^2}{m^2} \frac{\pi^2}{8} . \tag{3.19}$$

One checks that this expression is special case of section 3.2 since $\tilde{D}(1) = -i\pi^2/8$.

[1] Since $D(\zeta_6) = Cl_2(\pi/3)$, the second Clausen sum, this evaluation is in agreement with the result of [**23**].

4. Families of elliptic $K3$-surfaces

We now turn to the discussion of the nature of the sunset integral for generic values of $t = K^2/m^2$. We define the integral

$$\mathcal{J}^2_{\ominus}(t) := \frac{m^2}{\mu^2\pi^2}\,\mathcal{I}^2_{\ominus}(K^2, m^2) = \int_0^\infty\int_0^\infty \frac{dxdy}{A_{\ominus}(x,y,t)}\,, \tag{4.1}$$

and

$$A_{\ominus}(x,y,t) = (1+x+y)(x+y+xy) - xyt\,. \tag{4.2}$$

We have a family of elliptic curves

$$\mathcal{E}_t := \{A_{\ominus}(x,y,t) = 0, x, y \in \mathbb{P}^2\} \tag{4.3}$$

for $t \in \mathbb{P}^1$. The following change of variables

$$\begin{aligned}(x,y,z) &= \left(\eta\,\frac{2\zeta+\sigma(t-1)}{2\sigma(\eta-\sigma)}, \eta\,\frac{-2\zeta+\sigma(t-1)}{2\sigma(\eta-\sigma)}, 1\right) \\ (\sigma,\zeta,\eta) &= \left(\frac{x+y+z(1-t)}{x+y}, \frac{(t-1)(x-y)(x+y+z(1-t))}{2(x+y)^2}, 1\right)\,,\end{aligned} \tag{4.4}$$

brings elliptic curve into its standard Weierstraß form

$$\zeta^2\eta = \sigma(t\eta^2 + \frac{(t-3)^2-12}{4}\,\sigma\eta + \sigma^2)\,. \tag{4.5}$$

The discriminant is given by

$$\Delta = 256\,(t-9)(t-1)^3t^2\,, \tag{4.6}$$

and the j-invariant is given by

$$J(t) = \frac{(t-3)^3(t^3-9t^2+3t-3)^3}{(t-9)(t-1)^3t^2}\,. \tag{4.7}$$

This family of elliptic curves

$$\mathcal{S}_{\ominus} = \{(x,y,z,t) \in \mathbb{P}^2\times\mathbb{P}^1, A_{\ominus}(x,y,z,t) = 0\}\,, \tag{4.8}$$

defines a pencil of elliptic curves in $\mathbb{P}^2$ corresponding to a modular family of elliptic curves $f : \mathcal{E} \to X_1(6) = \{\tau \in \mathbb{C}|\Im\mathrm{m}(\tau) > 0\}/\Gamma_1(6)$. This will play an important role in section 8 when expressing the motive for the sunset graph in term of the Eisenstein series in $E_{\ominus}(q)$.

From the discriminant in eq. (4.6) we deduce there are four singular fibers at $t = 0, 1, 9, \infty$ of respective Kodeira type I_6, I_1, I_3, I_2. This is one of the $K3$ family of elliptic curves given by Beauville in [**29**].

For a given $t \in \mathbb{P}^1$, the elliptic curve $\mathcal{E}_t$ intersects the domain of integration $\mathcal{D}$ in eq. (2.4) at the following six points

$$(x,y,z) = \Big\{(0,1,0),(1,0,0),(0,0,1),(1,-1,0),(0,-1,1),(-1,0,1)\Big\} \tag{4.9}$$

which are mapped to using (4.4)

$$(\sigma,\zeta,\eta) = \Big\{(0,1,0),(0,0,1),(1,\frac{t-1}{2},1),(1,\frac{1-t}{2},1), \\ (t,\frac{t(t-1)}{2},1),(t,\frac{t(1-t)}{2},1)\Big\}. \tag{4.10}$$

Lemma 4.11. — *The points in the list in* (4.10) *are always of torsion for any values of* t *of the elliptic curve in* (4.5)*. The torsion group is* $\mathbb{Z}/6\mathbb{Z}$ *generated by* $(t, t(t-1)/2, 1)$ *or* $(t, -t(t-1)/2, 1)$*.*

Proof. — As is standard for the Weierstraß model, we set the point $O = (0,1,0)$ to be the identity. Then $(0,0,1)$ is of order 2 (any points of the form $(x,0,1)$ are of order 2). A point $P = (x,y,1)$ is of order 3, if and only if $3P = O$ or equivalently $2P = -P$, which implies for the elliptic curve in eq. (4.5) that

$$x = \frac{(t-x^2)^2}{x\left(((t-6)t-3)x+4t+4x^2\right)} \tag{4.12}$$

This has a solution $x = 1$ and $y = \pm(1-t)/2$, so the points $P_3 = (1, \frac{t-1}{2}, 1)$ and $P_4 = (1, \frac{1-t}{2}, 1)$ are of order 3. For the remaining two points $P_5 = (t, \frac{t(t-1)}{2}, 1)$ and $P_6 = (t, \frac{t(1-t)}{2}, 1)$, one checks that $2P_5 = P_3$ and $2P_6 = P_4$. Therefore the points P_5 and P_6 are of order 6. We have as well that $5P_5 = 2P_3 + P_5 = P_5 - P_3 = P_6$ and $5P_6 = P_5$. We conclude that the torsion group is $\mathbb{Z}/6\mathbb{Z}$ generated by P_5 or P_6. □

The Picard-Fuchs equation associated with this family of elliptic curve given in [**30**] reads for $t \in \mathbb{P}^1 \backslash \{0,1,9\}$

$$L_t F(t) := \frac{d}{dt}\left(t(t-1)(t-9)\frac{dF(t)}{dt}\right) + (t-3)F(t) = 0\,. \tag{4.13}$$

Two independent solutions of the Picard-Fuchs equation providing the real period ϖ_r (chosen to be positive), and the imaginary period ϖ_c (with

$\Im\mathrm{m}(\varpi_c) > 0$) are given by [**30, 31**]

$$\varpi_c(t) = \frac{2\pi}{(t-3)^{\frac{1}{4}}(t^3-9t^2+3t-3)^{\frac{1}{4}}}\, {}_2F_1\left(\begin{matrix}\frac{1}{12}, \frac{5}{12}\\ 1\end{matrix}; \frac{1728}{J(t)}\right), \tag{4.14}$$

and

$$\varpi_r(t) = \frac{12\sqrt{3}}{t-9}\,\varpi_c\left(9\frac{t-1}{t-9}\right). \tag{4.15}$$

This real period can as well be computed by the following integral

$$\varpi_r(t) = 6\int_{-1}^{0} \frac{dx}{\frac{\partial A_\ominus(x,y)}{\partial y}|_{\mathcal{E}}} = 6\int_{-1}^{0} \frac{dx}{\sqrt{(1+(3-t)x+x^2)^2 - 4x(1+x)^2}} \tag{4.16}$$

Using the result of Maier in [**31**], we can derive the expression for the $\Gamma_1(6)$ invariant Hauptmodul, t as modular form function of $q := \exp(2i\pi\tau)$ with τ the period ratio $\tau = \varpi_c/\varpi_r$

$$t = 9 + 72\frac{\eta(q^2)}{\eta(q^3)}\left(\frac{\eta(q^6)}{\eta(q)}\right)^5. \tag{4.17}$$

The four cusps of $\Gamma_1(6)$ located $\tau = 0, 1/2, 1/3, +i\infty$ are mapped to the values $t = +\infty, 1, 0, 9$ respectively.

Plugging the expression for t in the one for the periods in (4.14) and (4.15), and performing the q-expansion with `Sage` [**32**], one easily identifies their expression as a modular form using the `Online encyclopedia of integer sequences` [**33**]

$$\varpi_r = \frac{\pi}{\sqrt{3}}\frac{\eta(q)^6\eta(q^6)}{\eta(q^2)^3\eta(q^3)^2}. \tag{4.18}$$

5. The elliptic dilogarithm for the sunset integral

The sunset integral is not annihilated by the Picard-Fuchs for our elliptic curve given in (4.13), but satisfies an inhomogenous Picard-Fuchs equation

$$\frac{d}{dt}\left(t(t-1)(t-9)\frac{d}{dt}\mathcal{J}^2_\ominus(t)\right) + (t-3)\mathcal{J}^2_\ominus(t) = -6. \tag{5.1}$$

This differential equation has been derived in [**23**] using the properties of the Feynman integrals or in [**24**] using cohomological methods.

We provide here another derivation of this differential equation. We rewrite the sunset integral in eq. (4.1) as

$$\mathcal{J}^2_\ominus(t) = \int_0^\infty \int_0^\infty \frac{1}{(1+x+y)(1+\frac{1}{x}+\frac{1}{y}) - t} \frac{dxdy}{xy} \tag{5.2}$$

Since $x + x^{-1} \geq 2$ for $x > 0$,

$$(1+x+y)(1+\frac{1}{x}+\frac{1}{y}) \geq 9, \quad \forall x \geq 0, y \geq 0 \tag{5.3}$$

This implies that the integral has a branch cut for $t > 9$ and the integral is analytic for $t \in \mathbb{C}\backslash[9, +\infty[$. The value $t = 9$ corresponds to the three-particle threshold when all the internal lines are cut and the integral has a logarithmic singularity $\mathcal{J}^2_\ominus(t) \propto \log(9-t)$ for $t \sim 9$.

For $t < 9$ we can perform the series expansion

$$\begin{aligned}
\mathcal{J}^2_\ominus(t) &= \sum_{n\geq 0} I_n\, t^n \\
I_n &:= \int_0^\infty \int_0^\infty ((1+x+y)(1+x^{-1}+y^{-1}))^{-n-1} \frac{dxdy}{xy} \\
&= \frac{1}{n!^2} \int_{[0,\infty]^4} u^{n+1}\, v^{n+1}\, e^{-u(1+x+y)-v(1+x^{-1}+y^{-1})} \frac{dxdydudv}{xyuv} \\
&= \frac{1}{4^{n-1}\, n!^2} \int_0^\infty x^{2n+1} K_0(x)^3\, dx\,.
\end{aligned} \tag{5.4}$$

Resumming the series we have the following integral representation in terms of Bessel functions

$$\mathcal{J}^2_\ominus(t) = 4 \int_0^\infty x I_0(\sqrt{t}x)\, K_0(x)^3\, dx \qquad |t| < 9\,. \tag{5.5}$$

For the generic case of three different masses it is immediate to derive using the same method, the following representation for the sunset integrals for $K^2 < (m_1+m_2+m_3)^2$

$$\mathcal{I}^2_\ominus(K^2, m_1, m_2, m_3) = 4\pi^2\mu^2 \int_0^\infty x I_0(\sqrt{K^2}x) \prod_{i=1}^3 K_0(m_i x) dx\,. \tag{5.6}$$

These integrals are particular cases of the Bessel moment appearing Feynman integral discussed in [**20, 21**]. Using the t expansion in eq. (5.4), we obtain the following action of the Picard-Fuchs operator

$$\frac{d}{dt}\left(t(t-1)(t-9)\frac{d\mathcal{J}^2_\ominus(t)}{dt}\right)+(t-3)\mathcal{J}^2_\ominus(t)=-3I_0+9I_1 \\ +\sum_{n\geq 1}\left(n^2I_{n-1}-(3+10n+10n^2)\,I_n+9(n+1)^2I_{n+1}\right)t^n\,. \quad (5.7)$$

It is easy to show that $I_1=(\mathcal{J}^2_\ominus(0)-2)/3$ therefore $-3I_0+9I_1=-6$. Introducing

$$\hat I_n := \frac{1}{4^{n-1}n!^2}\int_0^\infty x^{2n+1}\,K_0(x)K_1(x)^2\,dx\,, \quad (5.8)$$

such that $\hat I_1=1/3$ and $\hat I_n$ is convergent for $n>0$. Setting $v_n:=\begin{pmatrix}I_n\\ \hat I_n\end{pmatrix}$ a repeated use of integration by part identities gives the matrix relation for $n>1$

$$\begin{aligned} v_{n+1} &= \frac{1}{9^n\,(n+1)!^2}\,M(n)\cdot M(n-1)\cdots M(1)\,v_1\\ M(n) &:= \begin{pmatrix}(3+7n)(n+1) & -6n^2\\ -2n(n+1) & 3n^2\end{pmatrix}\,. \end{aligned} \quad (5.9)$$

Using this expression one finds that for $n\geq 1$

$$n^2v_{n-1}-(3+10n(1+n))\,v_n+9(n+1)^2v_{n+1}=\begin{pmatrix}0 & 0\\ -\frac{2n(n+1)}{9^{n-1}n!^2} & \frac{2n^2-1}{9^{n-2}n!^2}\end{pmatrix}v_{n-1}\,. \quad (5.10)$$

This implies that for $n\geq 1$

$$n^2I_{n-1}-(3+10n+10n^2)\,I_n+9(n+1)^2I_{n+1}=0 \quad (5.11)$$

which is one of Apéry-like recursion considered by Zagier in [**34**].

Therefore for $|t|<9$ the sunset integral $\mathcal{J}^2_\ominus(t)$ satisfies the differential equation in eq. (5.1).

5.1. Solving the picard-fuchs equation. — We turn to the resolution of the differential equation (5.1) for the sunset integral for $t\in\mathbb{C}\backslash[9,+\infty[$.

The periods ϖ_r and ϖ_c are solutions of the homogeneous equation, and a solution to the inhomogeneous equation is given by

$$-\frac{\mathcal{J}^2_\ominus(t)}{6}=\alpha\,\varpi_c(t)+\beta\,\varpi_r(t)+\frac{1}{12\pi}\int_0^t(\varpi_c(t)\varpi_r(x)-\varpi_c(x)\varpi_r(t))\,dx \quad (5.12)$$

where α and β are constant that will be determined later. The action of the Picard-Fuchs operator this expression gives

$$L_t\left(-\frac{\mathcal{J}_\ominus^2(t)}{6}\right)=\frac{t(t-1)(t-9)}{12\pi}\left(\varpi_c'(t)\varpi_r(t)-\varpi_c(t)\varpi_r'(t)\right). \tag{5.13}$$

The quantity $W(t):=\varpi_c'(t)\varpi_r(t)-\varpi_c(t)\varpi_r'(t)$ is the Wronskian for the second order differential equation in (4.13), therefore $t(t-1)(t-9)W(t)$ is a constant. Using the expression for the periods given in the previous section we find that this constant is equal to 12π. This shows that $\mathcal{J}_\ominus^2(t)$ satisfies the inhomogeneous equation (5.1).

Changing variable from x to $q:=\exp(2i\pi\tau(x))$ and using the relation given by Maier in [**31**]

$$\frac{1}{2i\pi}\frac{dt}{d\tau}=\frac{dt}{d\log q}=72\,\frac{\eta(q^2)^8\eta(q^3)^6}{\eta(q)^{10}}\,. \tag{5.14}$$

and finally using the expression in terms of η-function in (4.18), one can rewrite the integral as

$$-\frac{\mathcal{J}_\ominus^2(t)}{6}=\alpha\,\varpi_c(t)+\beta\,\varpi_r(t)-2\sqrt{3}\,\varpi_c(t)\int_{-1}^{q}\frac{\eta(\hat q^6)\eta(\hat q^2)^5\eta(\hat q^3)^4}{\eta(\hat q)^4}\,d\log\hat q$$
$$+\frac{\sqrt{3}}{\pi}\,\varpi_r(t)\int_{-1}^{q}\frac{\eta(\hat q^6)\eta(\hat q^2)^5\eta(\hat q^3)^4}{\eta(\hat q)^4}\,\log\hat q\,d\log\hat q\,. \tag{5.15}$$

Integrating by part the second line leads to

$$-\frac{\mathcal{J}_\ominus^2(t)}{6}=\alpha\,\varpi_c(t)+\beta\,\varpi_r(t)+\varpi_r(t)\int_{q_1}^{q_t}L(\hat q)\,d\log\hat q\,, \tag{5.16}$$

where we have introduced

$$L(q):=\frac{\sqrt{3}}{\pi}\int_{-1}^{q}\frac{\eta(\hat q^6)\eta(\hat q^2)^5\eta(\hat q^3)^4}{\eta(\hat q)^4}\,d\log\hat q\,. \tag{5.17}$$

For evaluating this expression we start by performing the q-expansion of the integrand

$$\frac{\eta(q^6)\eta(q^2)^5\eta(q^3)^4}{\eta(q)^4}=\sum_{k\geq 0}k^2\left(\frac{q^k}{1+q^k+q^{2k}}+\frac{q^{2k}}{1+q^{2k}+q^{4k}}\right), \tag{5.18}$$

which we integrate term by term using the result of the integral for $|X_0|, |X| \leq 1$

$$\int_{X_0}^{X} \left(\frac{x}{1+x+x^2} + \frac{x^2}{1+x^2+x^4} \right) d\log x = \frac{i}{\sqrt{3}} \left(f(X) - f(X_0) \right) \quad (5.19)$$

with

$$f(x) := \tanh^{-1}\left(\frac{x}{\zeta_6}\right) - \tanh^{-1}\left(\frac{x}{\bar{\zeta}_6}\right) , \quad (5.20)$$

where $\zeta_6 := \exp(i\pi/3) = 1/2 + i\sqrt{3}/2$ is a sixth root of unity. Since $f(1) = -f(-1) = \pi/(2\sqrt{3})$, using the ζ-regularization, ie $\zeta(0) = -1/2$ and $\zeta(-1) = -1/12$, we have

$$\frac{i}{\pi} \sum_{k\geq 0} k\, f((-1)^k) = \frac{i}{4\sqrt{3}} . \quad (5.21)$$

Therefore $L(q)$ in eq. (5.17) has the q-expansion

$$L(q) = \frac{i}{4\sqrt{3}} + \frac{i}{\pi} \sum_{k\geq 0} k \left(\tanh^{-1}\left(\frac{q^k}{\zeta_6}\right) - \tanh^{-1}\left(\frac{q^k}{\bar{\zeta}_6}\right) \right) . \quad (5.22)$$

For evaluating the Integral in (5.16), we integrate this series term by term, using now that, for $|X_0|, |X| \leq 1$,

$$\int_{X_0}^{X} \left(\tanh^{-1}\left(\frac{x}{\zeta_6}\right) - \tanh^{-1}\left(\frac{x}{\bar{\zeta}_6}\right) \right) d\log x = h(X) - h(X_0) \quad (5.23)$$

where

$$h(x) := \frac{i}{2} \left(\mathrm{Li}_2\left(x\zeta_6^5\right) + \mathrm{Li}_2\left(x\zeta_6^4\right) - \mathrm{Li}_2\left(x\zeta_6^2\right) - \mathrm{Li}_2\left(x\zeta_6\right) \right) . \quad (5.24)$$

Since $h(1) = -h(-1) = \frac{5}{4\sqrt{3}} \mathcal{J}_{\ominus}^2(0)$ obtained using (3.17), we therefore find that

$$\int_{-1}^{q} L(\hat{q})\, d\log \hat{q} = \frac{i}{4\sqrt{3}} \log(-q) + \frac{i}{\pi} \sum_{k\geq 0} h(q^k) . \quad (5.25)$$

We find that the solution to the inhomogeneous Picard-Fuchs equation in eq. (5.12) is given by

$$-\frac{\mathcal{J}_{\ominus}^2(t)}{6} = \alpha\, \varpi_c(t) + \beta \varpi_r(t) - \frac{\varpi_r(t)}{\pi} E_{\ominus}(q) . \quad (5.26)$$

where we introduced $E_\ominus(q)$ defined by

$$\begin{aligned}
E_\ominus(q) &:= \sum_{n\geq 0} h(q^n) - \frac{h(1)}{2} \\
&= -\frac{1}{2i}\sum_{n\geq 0}\left(\mathrm{Li}_2\left(q^n\zeta_6^5\right)+\mathrm{Li}_2\left(q^n\zeta_6^4\right)-\mathrm{Li}_2\left(q^n\zeta_6^2\right)-\mathrm{Li}_2\left(q^n\zeta_6\right)\right) \\
&+ \frac{1}{4i}\left(\mathrm{Li}_2\left(\zeta_6^5\right)+\mathrm{Li}_2\left(\zeta_6^4\right)-\mathrm{Li}_2\left(\zeta_6^2\right)-\mathrm{Li}_2\left(\zeta_6\right)\right) . \qquad (5.27)
\end{aligned}$$

Matching the values of the sunset integral at $t=0$ and $t=1$ we find that $\alpha = i\pi/3$ and $\beta = -i\pi/6$

$$-\frac{\mathcal{J}^2_\ominus(t)}{6} = -i\frac{\pi}{6}\left(-2\varpi_c(t)+\varpi_r(t)\right) + \frac{\varpi_r(t)}{\pi}\, E_\ominus(q) \,. \qquad (5.28)$$

This shows that the one-mass sunset integral in two dimensions is expressed in term of the elliptic dilogarithm $E_\ominus(q)$ with the real period ϖ_r and complex periods ϖ_c respectively defined in (4.15) and (4.14).

For obtaining the q-expansion of this elliptic dilogarithm we first rewrite $h(x)$ in eq. (5.24) as

$$h(x) = \sum_{n\geq 1}\psi(n)\,\frac{x^n}{n^2} \qquad (5.29)$$

where we have set

$$\psi(n) := \frac{(-1)^{n-1}}{\sqrt{3}}\left(\sin(\frac{n\pi}{3}) + \sin(\frac{2n\pi}{3})\right) \qquad (5.30)$$

so that $\psi(n)$ is a character such that $\psi(n)=1$ for $n=1 \mod 6$, and $\psi(n)=-1$ for $n=5 \mod 6$, and zero otherwise. Then,

$$E_\ominus(q) = \sqrt{3}\sum_{\substack{n\geq 0\\ k\geq 1}}\frac{\psi(k)}{k^2}\,q^{nk} - \frac{\sqrt{3}}{2}\sum_{k\geq 1}\frac{\psi(k)}{k^2}\,, \qquad (5.31)$$

summing over n and using that $\psi(-k)=-\psi(k)$ we have

$$E_\ominus(q) = \frac{\sqrt{3}}{2}\sum_{k\in\mathbb{Z}\backslash\{0\}}\frac{\psi(k)}{k^2}\,\frac{1}{1-q^k}\,. \qquad (5.32)$$

We remark that the expression for this elliptic dilogarithm is invariant under the transformation $q \to 1/q$.

In section 8 we will show how this object can be derived from the motif associated with the elliptic curve. See in particular eq. (8.33) and the argument preceeding that equation.

Finally we notice that the invariance of the Hauptmodul t in (4.17) under $\Gamma_1(6)$ implies the following transformations of the elliptic dilogarithm sum

$$\begin{aligned}\tau' = \tau + 1 \quad &: \quad E_\ominus(q') = E_\ominus(q) - \frac{i\pi^2}{3}\\ \tau' = \frac{5\tau-1}{6\tau-1} \quad &: \quad E_\ominus(q') = \frac{E_\ominus(q)}{-6\tau+1} - \frac{i\pi^2}{3}\,\frac{3\tau-1}{6\tau-1}\\ \tau' = \frac{7\tau-3}{12\tau-5} \quad &: \quad E_\ominus(q') = \frac{E_\ominus(q)}{12\tau-5}\,.\end{aligned} \tag{5.33}$$

In particular, the elliptic dilogarithm $E_\ominus(q)$ is not modular invariant.

6. The motive for the sunset graph: Hodge structures

The next three sections are devoted to the motivic calculation of the amplitude.

Let $H = H_\mathbb{Q}$ be a finite dimensional $\mathbb{Q}$-vector space. A *pure Hodge structure* of weight n on H is a decreasing filtration (Hodge filtration) $F^*H_\mathbb{C}$ on $H_\mathbb{C} = H_\mathbb{Q} \otimes \mathbb{C}$ such that writing $\overline{F}^p$ for the complex conjugate filtration and $H^{p,q} := F^p \cap \overline{F}^q$, we have

$$H_\mathbb{C} = \bigoplus_p H^{p,n-p}. \tag{6.1}$$

A *mixed Hodge Structure* on H is a pair of filtrations

(i) $W_*H_\mathbb{Q}$ an increasing filtration (finite, separated, exhaustive) called the weight filtration.

(ii) $F^*H_\mathbb{C}$ a decreasing filtration (finite, separated, exhaustive), the Hodge filtration, defined on $H_\mathbb{C} = H_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{C}$. Note that the graded pieces for the weight filtration $gr_n^W H := W_nH/W_{n-1}H$ inherit a Hodge filtration

$$\begin{aligned}F^p(gr_n^W H)_\mathbb{C} = (F^pH_\mathbb{C} \cap W_{n,\mathbb{C}})/(F^pH_\mathbb{C} \cap W_{n-1,\mathbb{C}}) \cong\\ (F^pH_\mathbb{C} \cap W_{n,\mathbb{C}} + W_{n-1,\mathbb{C}})/W_{n-1,\mathbb{C}}.\end{aligned} \tag{6.2}$$

For a mixed Hodge structure we require that with this filtration $gr_n^W H$ should be a pure Hodge structure of weight n for all n.

Fundamental Example. The notion of mixed Hodge structure has extremely wide application. All Betti groups of algebraic varieties carry canonical and functorial Hodge structures [**35, 36, 37**]. Here "Betti group" means Betti cohomology, relative cohomology, cohomology with compact supports, cohomology of diagrams, homology, etc. "Functorial" means functorial for algebraic maps.

Examples 6.3. — *(i)* $H^1(C,\mathbb{Q})$ *is a pure Hodge structure of weight* 1*, where* C *is a compact Riemann surface. This is just the statement that cohomology classes with* $\mathbb{C}$*-coefficients can be decomposed into types* $(1,0)$ *and* $(0,1)$*. More generally, the Hodge decomposition for cohomology in degree* p *for any smooth projective variety* X *and any degree* p*, says that* $H^p(X,\mathbb{Q})$ *is a pure Hodge structure of weight* p*.*
(ii) $\mathbb{Q}(n)$ *is the pure Hodge structure of weight* $-2n$ *with underlying vector space a* 1*-dimensional* $\mathbb{Q}$*-vector space and Hodge structure* $\mathbb{Q}(n)_{\mathbb{C}} = \mathbb{Q}(n)_{\mathbb{C}}^{-n,-n}$*.*

The categories of pure and of mixed Hodge structures are abelian. In the mixed case this is a real surprise, because categories of filtered vector spaces do not tend to be abelian. Let V and W be filtered vector spaces, and let $\phi : V \to W$ be a linear map compatible with the filtrations. The image of ϕ, $\Im\mathrm{m}(\phi)$, has in general two possible filtrations; one can take $fil_n\Im\mathrm{m}(\phi)$ to be either the image of fil_nV or else the intersection $\Im\mathrm{m}(\phi) \cap fil_nW$. Technically, the image and coimage differ in the category of filtered vector spaces, and this prevents the category from being abelian. In the case of Hodge structures, the two filtrations can be seen to coincide. Of particular importnace for us is that an exact sequence of mixed Hodge structures induces exact sequences on the level of W_n and on the level of F^p for all possible n,p.

Categories of Hodge structures carry natural tensor product structures induced by ordinary tensor product on the underlying vector spaces with the usual notion of tensor products of filtrations. For example if H is a Hodge structure, then $H(1) := H \otimes \mathbb{Q}(1)$ is the Hodge structure with underlying vector space H with weight and Hodge structures given by $W_n(H(1)) = W_{n+2}H$ and $F^pH(1)_{\mathbb{C}} = F^{p+1}H$.

Example 6.4. — *We shall need certain extensions of Hodge structures; in particular we shall need to understand* $Ext^1_{HS}(\mathbb{Q}(0),H)$ *for a given Hodge structure* H*. Let*

$$0 \to H \to \mathbb{E} \to \mathbb{Q}(0) \to 0 \tag{6.5}$$

be such an extension. The interesting case is when $H = W_{<0}H$*, in other words when the weights of* H *are* < 0*, so we assume that. (Exercise: show in general that the inclusion* $W_{<0}H \subset H$ *induces an isomorphism* $Ext^1_{HS}(\mathbb{Q}(0), H) \cong Ext^1_{HS}(\mathbb{Q}(0), W_{<0}H)$*.) This then determines the weight filtration on* $\mathbb{E}$*, namely* $W_i\mathbb{E} = W_iH$ *for* $i < 0$ *and* $W_0\mathbb{E} = \mathbb{E}$*. It remains to understand the Hodge filtration. Let* $1 \in \mathbb{Q}(0)$ *be a basis, and let* $s : \mathbb{Q}(0) \to \mathbb{E}_{\mathbb{Q}}$ *be a vector space splitting, so we can identify* $\mathbb{E}_{\mathbb{Q}} = H_{\mathbb{Q}} \oplus \mathbb{Q} \cdot s(1)$ *as a vector space. For the Hodge filtration we have*

$$0 \to F^pH_{\mathbb{C}} \to F^p\mathbb{E}_{\mathbb{C}} \to F^p\mathbb{Q}(0)_{\mathbb{C}} \to 0 \tag{6.6}$$

from which one sees that $F^p\mathbb{E}_{\mathbb{C}} = F^pH_{\mathbb{C}}$ *for* $p \geq 1$ *and* $F^p\mathbb{E}_{\mathbb{C}} = F^pH_{\mathbb{C}} + F^0\mathbb{E}_{\mathbb{C}}$ *for* $p \leq 0$*. Thus, the only variable is* $F^0\mathbb{E}_{\mathbb{C}}$*. Since* $F^0\mathbb{E}_{\mathbb{C}} \twoheadrightarrow \mathbb{Q}(0)_{\mathbb{C}}$ *there exists* $h \in H_{\mathbb{C}}$ *with* $h - s(1) \in F^0\mathbb{E}_{\mathbb{C}}$*. Clearly* h *is well-defined upto the choice of splitting* s *and an element in* $F^0H_{\mathbb{C}}$*. It follows that*

$$Ext^1_{HS}(\mathbb{Q}(0), H) \cong H_{\mathbb{C}}/(H_{\mathbb{Q}} + F^0H_{\mathbb{C}}). \tag{6.7}$$

To understand how Hodge structures are related to amplitudes, consider the dual $M^\vee$ of a Hodge structure M. We have

$$M^\vee := \mathrm{Hom}(M, \mathbb{Q}); \quad W_nM^\vee = (W_{-n-1}M)^\perp; \quad F^pM^\vee_{\mathbb{C}} = (F^{-p+1}M_{\mathbb{C}})^\perp. \tag{6.8}$$

The dual of (6.5) reads

$$0 \to \mathbb{Q}(0) \to \mathbb{E}^\vee \to H^\vee \to 0 \tag{6.9}$$

For convenience we write $M = \mathbb{E}^\vee$ so we have an inclusion of Hodge structures $\mathbb{Q}(0) \hookrightarrow M$. Let $0 = F^{p+1}M_{\mathbb{C}} \subset F^pM_{\mathbb{C}} \neq (0)$ be the smallest level of the Hodge filtration, and suppose we are given $\omega \in F^pM_{\mathbb{C}}$. Dualizing we get $M^\vee \twoheadrightarrow \mathbb{Q}(0)$. We choose a splitting $s : \mathbb{Q}(0) \hookrightarrow M^\vee$. The period is then

$$\langle \omega, s(1) \rangle \in \mathbb{C}. \tag{6.10}$$

Because the Hodge filtration is not defined over $\mathbb{Q}$, the period is not rational.

The period in (6.10) is the Feynman integral or amplitude for instance in eq. (4.1) for the one mass sunset graph. In the precise relation to the Feynman integral or amplitude, one needs to pay attention to the important fact that the period in (6.10) depends on the choice of splitting s. This may seem odd to the physicist, who doesn't think of amplitude calculations as involving any choices. Imagine, however, that the amplitude depends on external momenta. The resulting function of momenta is generally multiple-valued. Concretely, as momenta vary, the chain of integration, which is typically the positive quadrant in some $\mathbb{R}^{4n}$, has to deform to avoid the polar locus of the integrand. When the momenta return to their original values, the chain of integration may differ from the original chain. Thus the choice of section s is a reflection of the multiple-valued nature of the amplitude. This is a crucial point in identifying periods with Feynman integrals.

Example 6.11. — *Let E be an elliptic curve and suppose M arises as an extension*

$$0 \to \mathbb{Q}(0) \to M \to H^1(E, \mathbb{Q}(-1)) \to 0. \tag{6.12}$$

(We will construct a family of such extensions in a minute.) The dual sequence[2] *is*

$$0 \to H^1(E, \mathbb{Q}(2)) \to M^\vee \to \mathbb{Q}(0) \to 0. \tag{6.13}$$

We have $F^2 M_{\mathbb{C}} \cong F^2 H^1(E, \mathbb{C})(-1) \cong F^1 H^1(E, \mathbb{C}) = \Gamma(E, \Omega^1)$ *so the choice of a holomorphic* 1*-form* ω *determines an element in the smallest Hodge filtration piece* $F^2 M_{\mathbb{C}}$*. In this case, we can understand the relation between the amplitude* (6.10) *and the extension class* (6.15) *as follows. We have* $F^0 H^1(E, \mathbb{C})(2) = F^2 H^1(E, \mathbb{C}) = (0)$ *so* $F^0 M^\vee_{\mathbb{C}} \cong F^0 \mathbb{C}(0)$ *and there is a canonical lift* s_F *of* $1 \in \mathbb{Q}(0)$ *to* $F^0 M^\vee_{\mathbb{C}}$*. The class of* $M^\vee \in Ext^1_{HS}(\mathbb{Q}(0), H^1(E, \mathbb{Q}(2))) \cong H^1(E, \mathbb{C}(2))/H^1(E, \mathbb{Q}(2))$ *is given by* $-s_F + s(1)$*. The pairing* $\langle\rangle : M \times M^\vee \to \mathbb{Q} = \mathbb{Q}(0)$ *can be viewed as a pairing of Hodge structures, so* $\langle \omega, s_F \rangle \in F^2\mathbb{C}(0) = (0)$*. With care, the* $\mathbb{Q}$ *can be replaced by* $\mathbb{Z}$*, and the amplitude becomes a map*

$$\langle \omega, ? \rangle : Ext^1_{HS}(\mathbb{Z}(0), H^1(E, \mathbb{Z}(2))) \cong \\ H^1(E, \mathbb{C})/H^1(E, \mathbb{Z}(2)) \to \mathbb{C}/\langle \omega, H^1(E, \mathbb{Z}(2)) \rangle. \tag{6.14}$$

We have seen in section 5 for the sunset all equal masses case with mass m and external momentum K, the amplitude is a solution of an inhomogeneous Picard-Fuchs equation in eq. (5.1) *for a family of elliptic curves, where the parameter of the family is* $t := K^2/m^2$*. The quotient* $\langle \omega, H^1(E, \mathbb{Z}(2)) \rangle$ *reflects the fact that the inhomogeneous solution is multiple-valued with variation given by a lattice of periods of the elliptic curves.*

The amplitude is closely related to the regulator in arithmetic algebraic geometry [**38**]*. Let* $conj : M_{\mathbb{C}} \to M_{\mathbb{C}}$ *be the real involution which is the*

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[2] Note that $H^1(E, \mathbb{Q})^\vee \cong H^1(E, \mathbb{Q}(1))$

identity on $M_{\mathbb{R}}$ *and satisfies* $conj(cm) = \bar{c}m$ *for* $c \in \mathbb{C}$ *and* $m \in M_{\mathbb{R}}$. *With notation as above, the extension class* $s(1) - s_F \in H^1(E, \mathbb{C})$ *is well-defined upto an element in* $H^1(E, \mathbb{Q}(2))$ *(i.e. the choice of* $s(1)$*). Since conj is the identity on* $H^1(E, \mathbb{Q}(2))$*, the projection onto the minus eigenspace* $(s(1) - s_F)^{conj=-1}$ *is canonically defined. The regulator is then*

$$\langle \omega, (s(1) - s_F)^{conj=-1} \rangle \in \mathbb{C}. \tag{6.15}$$

Let $E \subset \mathbb{P}^2$ be the elliptic curve defined by the equal mass sunset equation (4.2). Write X, Y, Z for the homogeneous coordinates, and let $x = X/Z, y = Y/Z$. We construct an extension of type (6.13) as follows. The curve passes through the points $(1,0,0), (0,1,0), (0,0,1)$. Let $\rho : P \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at these three points. The inverse image of the coordinate triangle in P is a hexigon we call $\mathfrak{h}$. The union of the three exceptional divisors will be denoted $D = D_1 \cup D_2 \cup D_3$. The elliptic curve lifts to $E \subset P$, meeting each edge of $\mathfrak{h}$ in a single point. The picture is figure 3.

Lemma 6.16. — *The localization sequence*

$$0 \to H^2(P, \mathbb{Q}(1))/\mathbb{Q} \cdot [E] \to H^2(P - E, \mathbb{Q}(1)) \to H^1(E, \mathbb{Q}) \to 0 \tag{6.17}$$

is exact and canonically split as a sequence of $\mathbb{Q}$*-Hodge structures. Here* $[E] \in H^2(P, \mathbb{Q}(1))$ *is the divisor class.*

Proof. — Let $q \in E$ be a point of order 3, e.g. $x = 0, y = -1$. Then q is a flex point, so there exists a line $L \subset \mathbb{P}^2$ with $L \cap E = \{q\}$. Write $S := \{q, (1,0,0), (0,1,0), (0,0,1)\}$. Points in S are torsion on E. It is convenient to work with the dual of (6.17) which is the top row of the following diagram

$$\begin{array}{ccccccc} 0 \to & H^1(E)(1) & \to & H^2(P,E)(1) & \to & H^2(P)^{(0)}(1) & \to 0 \\ & \downarrow & & \downarrow a & & \downarrow 0 & \\ 0 \to & H^1(E-S)(1) & \xrightarrow{b} & H^2(P-D-L, E-S)(1) & \to & H^2(P-D-L)(1) & \to 0 \end{array} \tag{6.18}$$

(Here $H^2(P)^{(0)} := \ker(H^2(P) \to H^2(E).)$ The right hand vertical arrow is zero, so Image$(a) \subset$ Image(b). Thus, the top sequence splits after pushout along $H^1(E) \to H^1(E - S)$. It follows that the dual sequence (6.17) arises from a map of vector spaces

$$H^2(P, \mathbb{Q}(1))/\mathbb{Q}\cdot[E] \to Ext^1_{\text{Hodge St.}}(H^1(E, \mathbb{Q}), \mathbb{Q}(0)) \cong E(\mathbb{C}) \otimes \mathbb{Q}, \tag{6.19}$$

and that the image of this map lies in the subgroup of $E(\mathbb{C}) \otimes \mathbb{Q}$ spanned by 0-cycles of degree 0 supported on S. These 0-cycles are all torsion, so the extension splits. The splitting is cannonical because two distinct splittings would differ by a map of Hodge structures from $H^1(E) \to H^2(P, \mathbb{Q}(1))/\mathbb{Q} \cdot [E]$ and no such map exists. $\square$

As a consequence of the lemma, we have a canonical inclusion $\iota : H^1(E, \mathbb{Q}(-1)) \subset H^2(P - E, \mathbb{Q})$. Consider the diagram where M is defined by pullback

$$\begin{tikzcd} 0 \arrow[r] & H^1(\mathfrak{h} - E \cap \mathfrak{h}) \arrow[r] & H^2(P - E, \mathfrak{h} - E \cap \mathfrak{h}) \arrow[r] & H^2(P - E) \arrow[r] & 0 \\ 0 \arrow[r] & H^1(\mathfrak{h} - E \cap \mathfrak{h}) \arrow[r] \arrow[u, equal] & M \arrow[r] \arrow[u] & H^1(E, \mathbb{Q}(-1)) \arrow[r] \arrow[u, "\iota"'] & 0 \end{tikzcd} \tag{6.20}$$

Dualizing the bottom sequence yields an extension of the form (6.13).

We continue to assume E is the elliptic curve defined by (4.2), and we write ω for the meromorphic two-form which is the integrand in (4.1).

Lemma 6.21. — *The Feynman amplitude coincides with the amplitude* $\langle \omega, s(1) \rangle$ (6.10) *associated to* $M^\vee$ *where* M *is as above. (Here, of course, "coincide" means upto the ambiguity associated to the choice of section* s*.)*

Proof. — ω is a two-form on P with a simple pole on E and no other pole. (This is not quite obvious. The form is given on $\mathbb{P}^2$ and then pulled back to P. It is a worthwhile exercise to check that the pullback does not acquire poles on the exceptional divisors D_1, D_2, D_3.) It therefore represents a class in $F^2H^2(P - E, \mathbb{C})$. From the previous lemma we have $H^2(P - E, \mathbb{Q}) \cong H^1(E, \mathbb{Q}(-1)) \oplus \mathbb{Q}(-1)^3$, so $F^2H^2(P - E, \mathbb{C}) = F^2H^1(E, \mathbb{C}(-1)) = F^2M_{\mathbb{C}}$. (The last identity follows from the bottom line in (6.20).) The dual of (6.20) looks like

$$\begin{tikzcd} 0 \arrow[r] & H_2(P - E) \arrow[r] \arrow[d, "\iota^\vee"] & H_2(P - E, \mathfrak{h} - E \cap \mathfrak{h}) \arrow[r] \arrow[d] & H_1(\mathfrak{h} - E \cap \mathfrak{h}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & H^1(E, \mathbb{Q}(2)) \arrow[r] & M^\vee \arrow[r] & H_1(\mathfrak{h} - E \cap \mathfrak{h}) \arrow[r] & 0. \end{tikzcd} \tag{6.22}$$

The chain of integration in (4.1) can be viewed as a two-chain on $P - E$ with boundary in $\mathfrak{h} - E \cap \mathfrak{h}$. This boundary is the loop which represents the generator of $H_1(\mathfrak{h} - E \cap \mathfrak{h}) = \mathbb{Q}(0)$, so the two-chain gives a splitting of the top line in (6.22). Since $\omega \in M_{\mathbb{C}}$ we can compute the pairing by pushing the chain down to $M^\vee$. But this is how the amplitude (6.10) is defined. $\square$

Finally in this section we discuss briefly the notion of a family of Hodge strutures associated to a family of elliptic curves $f : \mathcal{E} \to X$ where X

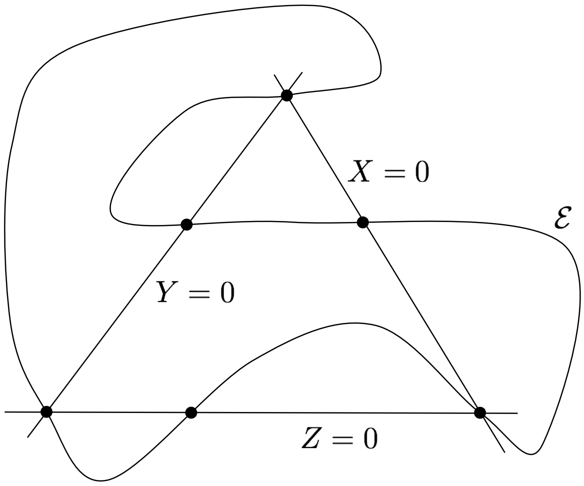

FIGURE 2. The sunset graph E meets the coordinate triangle in six points; three corners and three other points. In the equal mass case, the difference of any two points is six-torsion on the curve.

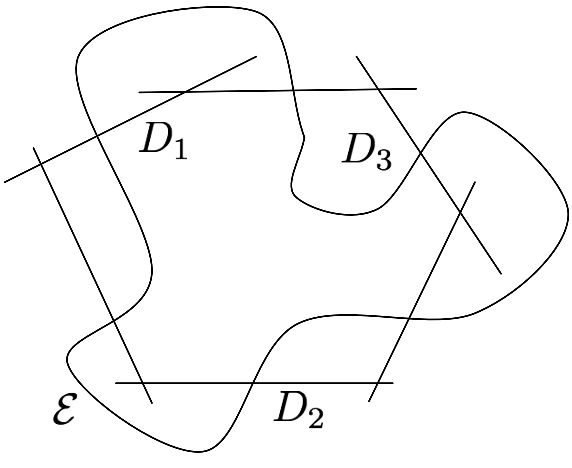

FIGURE 3. After blowup, the coordinate triangle becomes a hexigon in P with three new divisors D_i. E now meets each of the six divisors in one point.

is an open curve. We write $\mathcal{E}_t$ for the individual elliptic fibres. For convenience, we write $\mathcal{V} = \mathcal{V}_{\mathbb{Q}} := R^1 f_* \mathbb{Q}_{\mathcal{E}}(2)$ and $\mathcal{V}_{\mathbb{C}} := \mathcal{V} \otimes \mathbb{C}$. These are rank two local systems of $\mathbb{Q}$ (resp. $\mathbb{C}$) vector spaces on X. We can, of course, also define the $\mathbb{Z}$-local system $\mathcal{V}_{\mathbb{Z}} = R^1 f_* \mathbb{Z}_{\mathcal{E}}(2)$.

The corresponding coherent analytic sheaf $\mathcal{V} \otimes_{\mathbb{Q}} \mathcal{O}_X = \mathcal{V}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_X$ admits a connection, and we may define a twisted de Rham complex which is an exact sequence of sheaves

$$0 \to \mathcal{V}_{\mathbb{C}} \to \mathcal{V}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{d} \mathcal{V}_{\mathbb{C}} \otimes_{\mathbb{C}} \Omega^1_X \to 0. \tag{6.23}$$

The fact that $\mathcal{V}$ is a family of Hodge structures translates into a Hodge filtration $f_* \Omega^1_{\mathcal{E}/X} \subset \mathcal{V} \otimes \mathcal{O}_X$, and in our modular case we will exhibit a section

$$eis \in \Gamma(X, f_* \Omega^1_{\mathcal{E}/X} \otimes \Omega^1_X) \subset \mathcal{V} \otimes \Omega^1_X. \tag{6.24}$$

Remark 6.25. — *The Eisenstein element eis is associated to a modular family, and as such, it is rather special. However, the extension of local systems which it defines via pullback*

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{V}_{\mathbb{C}} & \to & \mathcal{V} \otimes \mathcal{O}_X & \xrightarrow{d} & \mathcal{V} \otimes \Omega^1_X & \to & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle 1 \mapsto eis} & & \\ 0 & \to & \mathcal{V}_{\mathbb{C}} & \to & \mathcal{W} & \to & \mathbb{C}_X & \to & 0 \end{array} \tag{6.26}$$

underlies a family of extensions of Hodge structures. If we tensor this pullback extension with $\mathcal{O}_X$, we get an exact sequence of vector bundles with integrable connections

$$0 \to \mathbb{V} \to \mathbb{W} \to \mathcal{O}_X \to 0 \tag{6.27}$$

The fibres of (6.27) *are of the form* (6.13).

The point is that some analogue of this sequence is available quite generally. It does not depend on the special modular nature of the sunset equal mass case. The amplitude is an algebraic section of $\mathbb{W}$ so the differential equation satisfied by the amplitude will be an inhomogeneous equation associated to the homogeneous equation on $\mathbb{V}$ with source term a rational function on X.

If, for example, we consider the sunset graph with unequal masses, we get a somewhat more complicated picture. The extension of coherent sheaves with connections becomes

$$0 \to \mathbb{V} \to \mathbb{W} \to \mathbb{E} \to 0 \tag{6.28}$$

where $\mathbb{E}$ itself contains Kummer extensions of the form

$$0 \to \mathcal{O}_X \to \mathbb{F} \to \mathcal{O}_X \to 0 \tag{6.29}$$

(Kummer means one has a section of $\mathbb{F}$, $e \mapsto 1$, and a rational function f on X such that $e + \log f \cdot 1$ is horizontal for the connection.) The source term in the inhomogeneous equation for the amplitude in this case will involve $\log f$*'s.*

7. The motive for the sunset graph: motivic cohomology

Consider a larger diagram of sheaves

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{V}_{\mathbb{Z}} & \to & \mathcal{V}\otimes\mathcal{O}_X & \to & (\mathcal{V}\otimes\mathcal{O}_X)/\mathcal{V}_{\mathbb{Z}} & \to & 0 \\ & & \downarrow & & \| & & \downarrow{\scriptstyle\delta} & & \\ 0 & \to & \mathcal{V}_{\mathbb{C}} & \to & \mathcal{V}\otimes\mathcal{O}_X & \stackrel{d}{\to} & \mathcal{V}\otimes\Omega^1_X & \to & 0 \end{array} \tag{7.1}$$

The sheaf $(\mathcal{V}\otimes\mathcal{O}_X)/\mathcal{V}_{\mathbb{Z}}$ is the sheaf of germs of analytic sections of a fibre bundle over X with fibre the abelian Lie group $H^1_{Betti}(\mathcal{E}_t, \mathbb{C}/\mathbb{Z}(2))$. Using *motivic cohomology*, we will explicit a family of extensions such that the family of extension classes give a section $mot \in \Gamma(X, \mathcal{V}\otimes\mathcal{O}_X/\mathcal{V}_{\mathbb{Z}})$. We will show that

$$\delta(mot) = eis. \tag{7.2}$$

We will relate mot to the Feynman amplitude viewed as a multivalued function on X, and then we will use eis in order to calculate this function.

$\mathcal{E}/X$ will denote the family of elliptic curves defined by the equal mass sunset equation (4.2). The corresponding family of extensions as in the bottom line of (6.22) determine a family of elements

$$mot_t \in \mathrm{Ext}^1_{HS}(\mathbb{Q}(0), H^1(\mathcal{E}_t, \mathbb{Q}(2)) \cong H^1(\mathcal{E}_t, \mathbb{C}/\mathbb{Z}(2)). \tag{7.3}$$

(Compare to (6.14).) Write $mot \in \Gamma(X, (\mathcal{V}\otimes\mathcal{O}_X)/\mathcal{V}_{\mathbb{Z}})$ for the corresponding section of the bundle of generalized intermediate jacobians.

Lemma 7.4. — *Let $\partial(mot) \in H^1(X, \mathcal{V}_{\mathbb{Q}})$ be the boundary for the top sequence in* (7.1) *tensored with $\mathbb{Q}$. We have $\partial(mot) \in \bigoplus \mathbb{Q}(0) \subset H^1(X, \mathcal{V}_{\mathbb{Q}}) \subset H^2_{Betti}(\mathcal{E}, \mathbb{Q}(2))$.*

Proof. — Consider a compactification

$$\begin{array}{ccc} \mathcal{E} & \to & \overline{\mathcal{E}} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} \\ X & \to & \overline{X} \end{array} \tag{7.5}$$

with $\overline{X}$ a compact Riemann surface and fibres of $\overline{f}$ possibly degenerate. We assume $\overline{\mathcal{E}}$ is a smooth variety, and we write $F = \coprod F_s = \overline{\mathcal{E}} - \mathcal{E}$. (The F_s are possibly singular fibres.) Localization yields an exact sequence of Hodge structures

$$H^2(\overline{\mathcal{E}}, \mathbb{Q}(2)) \to H^2(\mathcal{E}, \mathbb{Q}(2)) \to H_1(F, \mathbb{Q}(0)) \xrightarrow{\mu} H^3(\overline{\mathcal{E}}, \mathbb{Q}(2)) \tag{7.6}$$

In cases of interest to us, the singular fibres F_s will be nodal rational curves or a union of copies of $\mathbb{P}^1$ supporting a loop, so $H_1(F_s, \mathbb{Q}(0)) = \mathbb{Q}(0)$ as a Hodge structure. Since $\overline{\mathcal{E}}$ is smooth and compact, $H^i(\overline{\mathcal{E}}, \mathbb{Q}(2))$ is a pure Hodge structure of weight $i-4$. In particular, the map labeled μ in (7.6) is zero and we get an exact sequence of Hodge structures

$$0 \to \{\text{pure of weight -2}\} \to H^2(\mathcal{E}, \mathbb{Q}(2)) \xrightarrow{r} \coprod_{\text{bad fibres}} \mathbb{Q}(0) \to 0. \tag{7.7}$$

The family of elliptic curves associated to the sunrise diagram with equal masses is *modular*, associated to the modular curve $X = X_1(6)$ (see section 4). In the modular case, a remarkable thing happens [**39**]. The arrow r in (7.7) is canonically split as a map of Hodge structures. This relates to the local system $\mathcal{V}$ above because the Leray spectral sequence for f yields

$$H^1(X, \mathcal{V}) = H^1(X, R^1 f_* \mathbb{Q}_{\mathcal{E}}(2)) \hookrightarrow H^2(\mathcal{E}, \mathbb{Q}(2)). \tag{7.8}$$

The splitting of r in fact embeds $\bigoplus_{\text{bad fibres}} \mathbb{Q}(0) \subset H^1(X, \mathcal{V})$, and the assertion of the lemma is that $\partial(mot)$ lies in the sub Hodge structure $\bigoplus_{\text{bad fibres}} \mathbb{Q}(0)$. The reason for this is that, rather than considering the section of a bundle of generalized intermediate jacobians given by the mot_t, one can work with the *motivic cohomology* of the total space $\mathcal{E}$ of the family.

Motivic cohomology $H^p_M(X, \mathbb{Q}(q))$ is defined using algebraic cycles on products of X with affine spaces. It has functoriality properties (in the case of smooth varieties) which are analogous to Betti cohomology. For $E \subset \mathbb{P}^2$ an elliptic curve, we are interested in

$$H^2_M(E, \mathbb{Q}(2)) \to H^3_M(P, E; \mathbb{Q}(2)). \tag{7.9}$$

Here P is the blowup of $\mathbb{P}^2$ as in figure 3. A sufficient condition for a finite formal sum $\sum_i (C_i, f_i)$ where $C_i \subset P$ is an irreducible curve and f_i is a rational function on C_i to represent an element in $H^3_M(P, E; \mathbb{Q}(2))$ is that firstly the sum $\sum_i (f_i)$ of the divisors of the f_i, viewed as 0-cycles on P should cancel, and secondly that $f_i | C_i \cap E$ should be identically 1. As an example, take the $\mathfrak{h}_i$ to be the irreducible components of the hexagon

$\mathfrak{h} \subset P$. Number the $\mathfrak{h}_i$ cyclically and take f_i having a single zero and a single pole on $\mathfrak{h}_i$ such that the zero of f_i coincides with the pole of f_{i+1}, the unique point $\mathfrak{h}_i \cap \mathfrak{h}_{i+1}$. Since $E \cap \mathfrak{h}_i = \{p_i\}$ is a single point which is not a zero or pole of the f_i, we can scale f_i so $f_i(p_i) = 1$. The resulting formal sum represents an element $T \in H^3_M(P, \mathcal{E}; \mathbb{Q}(2))$.

This element lifts back to an element $S \in H^2_M(\mathcal{E}, \mathbb{Q}(2))$. Indeed, the Milnor symbol

$$S := \{-X/Z, -Y/Z\} \tag{7.10}$$

represents an element in the motivic cohomology of the function field of $\mathcal{E}$. To check that it globalizes, we should check that there are no non-trivial tame symbols at the six points where the fibres $\mathcal{E}_t$ meet the coordinate triangle. By symmetry it suffices to check the points with homogeneous coordinates $(0, -1, 1)$ and $(0, 0, 1)$. Recalling the general formula

$$\mathrm{tame}_x\{f, g\} = (-1)^{ord_x(f) ord_x(g)}(f^{ord_x(g)}/g^{ord_x(f)})(x) \in k(x)^\times \tag{7.11}$$

it is straightforward to check that the tame symbols are both ± 1. Since we are working with coefficients in $\mathbb{Q}$ the presence of torsion need not concern us, and we conclude that S globalizes. The assertion that $S \mapsto T$ in (7.9) amounts to the assertion that S viewed as a Milnor symbol now on P has T as tame symbol. For more on this sort of calculation, one can see [**40**].

We will also need *Deligne cohomology* $H^p_{\mathcal{D}}(X, \mathbb{Q}(q))$, (see the articles of Schneider, Esnault-Viehweg, and Jannsen in [**41**], as well as [**38**]). It sits in an exact sequence

$$\begin{multline} 0 \to \mathrm{Ext}^1_{HS}(\mathbb{Q}(0), H^{p-1}_{Betti}(X, \mathbb{Q}(q))) \to H^p_{\mathcal{D}}(X, \mathbb{Q}(q)) \xrightarrow{b} \\ \mathrm{Hom}_{HS}(\mathbb{Q}(0), H^p_{Betti}(X, \mathbb{Q}(q))) \to 0. \end{multline} \tag{7.12}$$

Motivic and Deligne cohomologies are related by the regulator map

$$reg : H^p_M(X, \mathbb{Q}(q)) \to H^p_{\mathcal{D}}(X, \mathbb{Q}(q)). \tag{7.13}$$

The composition

$$\begin{multline} b \circ reg : H^p_M(X, \mathbb{Q}(q)) \to \mathrm{Hom}_{HS}(\mathbb{Q}(0), H^p_{Betti}(X, \mathbb{Q}(q))) \\ \subset H^p_{Betti}(X, \mathbb{Q}(q)) \end{multline} \tag{7.14}$$

is the Betti realization. Note that the image lands in a sub-Hodge structure of the form $\oplus\mathbb{Q}(0)$.

To finish the proof of the lemma, we consider two diagrams.

$$\begin{array}{ccc} H^2_M(\mathcal{E},\mathbb{Z}(2)) & \to & H^2_M(\mathcal{E}_t,\mathbb{Z}(2)) \\ \downarrow{\scriptstyle reg} & & \downarrow{\scriptstyle reg} \\ H^2_{\mathcal{D}}(\mathcal{E},\mathbb{Z}(2)) & \to & H^2_{\mathcal{D}}(\mathcal{E}_t,\mathbb{Z}(2)) \\ \downarrow{\scriptstyle \text{localize}} & & \uparrow{\scriptstyle \cong} \\ \Gamma(X,(\mathcal{V}\otimes\mathcal{O}_X)/\mathcal{V}_{\mathbb{Z}}) & \xrightarrow{\text{rest. to fibre}} & H^1_{Betti}(\mathcal{E}_t,\mathbb{C}/\mathbb{Z}(2)) \\ & & \downarrow{\scriptstyle \cong} \\ & & \mathrm{Ext}^1_{HS}(\mathbb{Z}(0),H^1_{Betti}(\mathcal{E}_t,\mathbb{Z}(2))). \end{array} \tag{7.15}$$

The arrow that requires some comment here is labeled "localize". A Deligne cohomology class on $\mathcal{E}$ yields by restriction Deligne cohomology classes on the fibres $\mathcal{E}_t$ which are just elements in $H^1_{Betti}(\mathcal{E}_t,\mathbb{C}/\mathbb{Z}(2))$. As t varies, however, these classes are not locally constant. They do not glue to sections of $\mathcal{V}_{\mathbb{C}}/\mathcal{V}_{\mathbb{Z}}$ but rather to $(\mathcal{V}\otimes\mathcal{O}_X)/\mathcal{V}_{\mathbb{Z}}$ as indicated.

The second relevant commutative diagram is

$$\begin{array}{ccc} H^2_M(\mathcal{E},\mathbb{Q}(2)) & \xrightarrow{b} & H^2_{Betti}(\mathcal{E},\mathbb{Q}(2)) \\ \downarrow & & \uparrow{\scriptstyle \text{inject}} \\ \Gamma(X,(\mathcal{V}\otimes\mathcal{O}_X)/\mathcal{V}_{\mathbb{Z}}) & \xrightarrow{\partial} & H^1(X,\mathcal{V}_{\mathbb{Q}}) \end{array} \tag{7.16}$$

Assembling these two diagrams and using that the image of b in (7.16) lies in the subspace of Hodge classes $\bigoplus\mathbb{Q}(0)\subset H^2_{Betti}(\mathcal{E},\mathbb{Q}(2))$, the lemma is proven. $\square$

The fact that $mot\in\Gamma(X,(\mathcal{V}\otimes\mathcal{O}_X)/\mathcal{V}_{\mathbb{Z}})$ comes from motivic cohomology enables us to control how this section degerates at the cusps. This is a slightly technical point and we do not give full detail. The argument is essentially an amalgam of [**41**], p. 47 in the exposé of Esnault-Viehweg, where the regulator of symbols like S (7.10) in $H^1_{Betti}(\mathcal{E},\mathbb{C}/\mathbb{Z}(2))$ is calculated, and the classical computation of the Gauß-Manin connection [**42**]. Let $X\hookrightarrow\overline{X}$ be the compactification, and let $c\in\overline{X}-X$ be a cusp. Let $D^*\subset X$ be a punctured analytic disk around c. Consider a symbol $\{g,h\}$ on the family $\mathcal{E}_{D^*}$. Using [**41**], we can identify the target of the regulator map for the family with the hypercohomology group $\mathbb{H}^1(\mathcal{E}_{D^*},\mathcal{O}^\times_{\mathcal{E}_{D^*}}\xrightarrow{d\log}\Omega^1_{\mathcal{E}_{D^*}/D^*})$. For our purposes it will suffice to calculate the regulator on the open set $\mathcal{E}'_{D^*}=\mathcal{E}_{D^*}-\mathrm{div}(g)-\mathrm{div}(h)$.

Let $\{U_j\}$ be an open analytic covering of $\mathcal{E}'_{D^*}$ such that $\log h$ admits a single-valued branch $\log_j h$ on U_j. Let $m_{jk} = (\log_k h - \log_j h)/2\pi i$ on $U_{jk} = U_j \cap U_k$. The regulator applied to $\{g, h\}$ is represented by the Cech cocycle

$$(g^{m_{jk}}, \frac{1}{2\pi i}\log_j h \cdot dg/g) \in C^1(\{U_j\}, \mathcal{O}^\times) \oplus C^0(\{U_j\}, \Omega^1_{\mathcal{E}'_{D^*}/D^*}) \tag{7.17}$$

We want to compute the image of this class under the map δ (7.1). Consider the diagram of complexes with exact columns

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] \\ & \mathcal{O}_{\mathcal{E}'_{D^*}} \otimes \Omega^1_{D^*} \arrow[r, "d"] \arrow[d] & \Omega^1_{\mathcal{E}'_{D^*}/D^*} \otimes \Omega^1_{D^*} \arrow[d, "\cong"] \\ \mathcal{O}^\times_{\mathcal{E}'_{D^*}} \arrow[r, "d\log"] \arrow[d, equal] & \Omega^1_{\mathcal{E}'_{D^*}} \arrow[r, "d"] \arrow[d] & \Omega^2_{\mathcal{E}'_{D^*}} \arrow[d] \\ \mathcal{O}^\times_{\mathcal{E}'_{D^*}} \arrow[r, "d\log"] & \Omega^1_{\mathcal{E}'_{D^*}/D^*} \arrow[r] \arrow[d] & 0 \\ & 0 & \end{tikzcd} \tag{7.18}$$

The cocycle (7.17) represents a one-cohomology class in the bottom complex. The vertical coboundary yields a two-cohomology class in the top complex, which is just the relative de Rham complex shifted and tensored with $\Omega^1_{D^*}$. Note the relative de Rham complex is exactly the bundle $\mathcal{V} \otimes \mathcal{O}_X$ restricted to $D^* \subset X$. In fact, this vertical coboundary is exactly the map δ from (7.1). To calculate, we view the 0-cochain $\frac{1}{2\pi i}\log_j h \cdot dg/g$ as a cochain in the absolute one-forms $\Omega^1_{\mathcal{E}'_{D^*}}$ and apply d

$$d(\frac{1}{2\pi i}\log_j h dg/g) = \frac{1}{2\pi i} dh/h \wedge dg/g \in \Omega^2_{\mathcal{E}'_{D^*}} \cong \Omega^1_{\mathcal{E}'_{D^*}/D^*} \otimes \Omega^1_{D^*}. \tag{7.19}$$

Notice that a log form $dh/h \wedge dg/g$ cannot have more than a first order pole along the components of the fibre over the cusp c. Furthermore, it is clear from (7.19) that

$$\delta(mot) \in \Gamma(\overline{X}, \Omega^1_{\overline{\mathcal{E}}/\overline{X}}(\log \text{cusps}) \otimes \Omega^1_{\overline{X}}). \tag{7.20}$$

We can now rewrite the diagram from (7.1)

$$\begin{tikzcd}
\Gamma(X,(\mathcal{V}\otimes\mathcal{O}_X)/\mathcal{V}_{\mathbb{Z}}) \arrow[r, "\partial"] \arrow[d, "\delta"] & H^1(X,\mathcal{V}_{\mathbb{Z}}) \arrow[d] \\
\Gamma(X,\mathcal{V}\otimes\Omega^1_X) \arrow[r] & H^1(X,\mathcal{V}_{\mathbb{C}}) \\
\Gamma(X,\Omega^1_{\mathcal{E}/X}\otimes\Omega^1_X) \arrow[u, "i"] &
\end{tikzcd} \tag{7.21}$$

We conclude from (7.18) that $\delta(mot)$ lies in the image of i in (7.21) and further that it has at worst logarithmic poles at the cusp. It is known that the space of such sections is spanned by the Eisenstein series eis_ψ discussed in section 8.3.

One other important piece of information we have by virtue of the motivic interpretation of the amplitude concerns the behavior of the Eisenstein section $\delta(mot)$ at the cusps. This is determined by the behavior of the Milnor symbol S (7.10) under the tame symbol mapping

$$tame : H^2_M(\mathcal{E},\mathbb{Q}(2)) \to \coprod_{c\in\overline{X}-X} H^M_1(\mathcal{E}_c,\mathbb{Q}(0)) \tag{7.22}$$

The elliptic curve $\mathcal{E}_t$ in (2.5) has four cusps at $t=0,1,9,\infty$, discussed in section 4. It is elementary to check that the tame symbol $tame(S)$ is trivial for $t=0,1,9$. Indeed, at $t=0$ the curve (4.2) becomes reducible with components $1+X/Z+Y/Z=0$ and $X/Z+Y/Z+XY/Z^2=0$. Since the entries $X/Z, Y/Z$ of the symbol S are not identically $0,\infty$ on either of these components, the tame symbol vanishes. Similarly, at $t=1$ the curve factors as $(X+Z)(Y+Z)(X+Y)$ so again there is no contribution. At $t=9$ the curve has a unique singular point at $X=Z, Y=Z$ and there is no contribution. Finally at $t=+\infty$ the fibre is $XYZ=0$; one has $H^M_1(\{XYZ=0\},\mathbb{Q}(0))=\mathbb{Q}$ and $tame_\infty(S)$ is a generator.

8. The motive for the sunset graph: Eisenstein Series

We now assume X is a modular curve, i.e. $X\cong\mathbb{H}/\Gamma$ where $\mathbb{H}=\{\tau\in\mathbb{C}\mid\Im\mathrm{m}(\tau)>0\}$ and $\Gamma\subset SL_2(\mathbb{Z})$ is a congruence subgroup. Let

$\tilde{f} : \widetilde{\mathcal{E}} \to \mathbb{H}$ be the pullback of the family of elliptic curves. We have

$$\Omega^1_{\widetilde{\mathcal{E}}/\mathbb{H}} = \mathcal{O}_{\mathbb{H}} dz; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} dz = \frac{dz}{cz+d} \tag{8.1}$$

$$\Omega^1_{\mathbb{H}} = \mathcal{O}_{\mathbb{H}} d\tau; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} d\tau = \frac{d\tau}{(c\tau+d)^2}. \tag{8.2}$$

Let $\psi : \mathbb{Z}^2 \to \mathbb{C}$ be a map such that $\psi(\gamma(a,b)) = \psi(a,b)$ for any $a, b \in \mathbb{Z}$ and any $\gamma \in \Gamma$. Define

$$eis_\psi := \sum_{\substack{(a,b)\in\mathbb{Z}^2 \\ (a,b)\neq(0,0)}} \frac{\psi(a,b) d\tau dz}{(a\tau+b)^3}. \tag{8.3}$$

It is straightforward to check that eis_ψ is invariant under Γ and descends to a section of $\Omega^1_{\mathcal{E}/X} \otimes \Omega^1_X \subset \mathcal{V} \otimes \Omega^1_X$ over $X = \mathbb{H}/\Gamma$.

This is the classical sheaf-theoretic interpretation of Eisenstein series for $SL_2(\mathbb{Z})$ [**43**]. The next step is to determine ψ which we do by studying the constant terms of the q-expansions at the cusps. We have seen in section 4 that for the one-mass sunset graph we have the $X_1(6)$ modular curve, and that the four cusps $t = 0, 1, 9, +\infty$ are mapped by eq. (4.17) to the four cusps for the action of $\Gamma_1(6)$ on $\mathbb{P}^1(\mathbb{Q})$ represented by $\tau = 1/3, 1/2, +i\infty, 0$ respectively . We know the constant term of the q-expansion should vanish for $t = 0, 1, 9$ i.e. for $\tau = +i\infty, 1/3, 1/2$. Indeed, $q = \exp(2\pi i\tau)$ is the parameter at the cusp at $+i\infty$, so $d\tau = \frac{1}{2\pi i} dq/q$ has a pole at the cusp. Thus, the constant term of the q-expansion coincides upto a factor of $2\pi i$ with the residue at the cusp.

Lemma 8.4. — *For $u, v \equiv 0, 1, 2, 3, 4, 5 \mod 6$, define*

$$eis^{u,v}(\tau) = \sum_{(a,b)\equiv(u,v) \bmod 6} \frac{1}{(a\tau+b)^3} \tag{8.5}$$

We have the following assertions about constant terms of q-expansions.
(i) The constant term of the q-expansion at $\tau = \infty$ vanishes unless $u \equiv 0 \mod 6$.
(ii) The constant term of the q-expansion at $\tau = 1/2$ vanishes unless $u + 2v \equiv 0 \mod 6$.
(iii) The constant term of the q-expansion at $\tau = 1/3$ vanishes unless $u + 3v \equiv 0 \mod 6$.
(iv) The constant term of the q-expansion at $\tau = 0$ vanishes unless $v \equiv 0 \mod 6$.

Proof. — The assertion for $\tau \to \infty$ is straightforward because the series (8.5) converges uniformly as $\tau \to +i\infty$ so we can take the limit term by term. The only terms which survive are those with $a = 0$. For the other assertions, we transform the cusp in question to $+i\infty$. For example, taking $\tau' = (\tau - 1)/(2\tau - 1)$ transforms $\tau = 1/2$ to $\tau' = +i\infty$. Substituting the inverse $\tau = (-\tau' + 1)/(-2\tau' + 1)$ yields

$$eis^{u,v}(\tau) = \sum_{(a,b)\equiv(u,v) \bmod 6} \frac{1}{(a(-\tau'+1)/(-2\tau'+1)+b)^3} = \\ (-2\tau'+1)^3 \sum_{(a,b)\equiv(u,v) \bmod 6} \frac{1}{((-a-2b)\tau' + (a+b))^3}. \tag{8.6}$$

This vanishes at $\tau' \to +i\infty$ unless $u + 2v \equiv 0 \mod 6$. The arguments for (iii) and (iv) are similar. □

We consider a series $\sum \psi(a,b)(a\tau + b)^{-3}$. We assume $\psi(a,b)$ only depends on $a, b \mod 6$. Define $\psi^{u,v}(a,b)$ to be 1 if $a \equiv u,\ b \equiv v \mod 6$ and zero otherwise. Write

$$\psi = \sum_{u,v \mod 6} c(u,v)\psi^{u,v}. \tag{8.7}$$

The condition that $eis_\psi(\tau)$ should be invariant under $\Gamma_1(6)$ translates into the requirement $\psi(a,b) = \psi((a,b)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix})$ for $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Gamma_1(6)$. Since

$$(a,b)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \equiv (a, b+\beta a) \mod 6 \tag{8.8}$$

it follows that

$$c(u,v) = c(u, v+\beta u); \quad \forall \beta \in \mathbb{Z}. \tag{8.9}$$

Writing $(a,b) = 6^n(a',b')$ where a', b' are not both divisible by 6, we can write $eis^{0,0}(\tau)$ as a sum with constant coefficients of $eis^{u,v}$ with $(u,v) \neq (0,0)$. Hence we can take $c(0,0) = 0$. We can further simplify by dropping $\psi^{u,v}$ if u, v have a common factor 2 or 3. Further we can assume $c(u,v) = -c(-u,-v)$. We see now from (ii) in lemma 8.4 that terms contributing to the constant term at the cusp 1/2 are $(u,v) = (4,1), (2,5)$. Since $eis^{4,1} = -eis^{2,5}$ and $c(4,1) = -c(2,5)$ there is only one contribution which is a non-zero multiple of $c(4,1)$. It follows that $c(4,1) = c(2,5) = 0$. Similarly the terms which contribute at $\tau = 1/3$ are $(3,1), (3,5)$. Again these are negatives and the same argument implies $c(3,1) = c(3,5) = 0$.

We have now the following information about the $c(i,j)$:

$$
\begin{gathered}
c(1,0)=c(1,1)=c(1,2)=c(1,3)=c(1,4)=c(1,5) \quad \text{(by (8.9))}\\
c(2,0)=c(2,2)=c(2,4)=0 \quad \text{(by divisibility)}\\
c(2,1)=c(2,3)=c(2,5)=0 \quad \text{(by (8.9) and the above)}\\
c(3,1)=c(3,4)=0 \quad \text{(by (8.9) and the above)}\\
c(3,2)=c(3,5)=0 \quad \text{(by (8.9) and the above)}\\
c(3,0)=c(3,3)=0 \quad \text{(by divisibility)}\\
c(4,x)=-c(2,-x)=0\\
c(5,x)=-c(1,0).
\end{gathered} \tag{8.10}
$$

For the character $\psi(n)$ defined in eq. (5.30) we have now proven

Theorem 8.11. — *Define*

$$\psi(a,b)=\psi(a):=\frac{(-1)^{a-1}}{\sqrt{3}}\left(\sin(\frac{\pi a}{3})+\sin(\frac{2\pi a}{3})\right). \tag{8.12}$$

Then for some $c\neq 0$ we have

$$\delta(mot)=c\cdot eis_\psi\in\Gamma(X,\mathcal{V}\otimes\Omega^1_X). \quad (cf.\ (7.1)) \tag{8.13}$$

Proof. — The list of values on the right hand side for $a=0,1,2,3,4,5$ is $0,1,0,0,0,-1$. This list is uniquely determined upto scale by the above conditions. □

Our objective now is to sharpen theorem 8.11 by computing the constant c. The residue of the symbol S (7.10) at $t=+\infty$ is 1, but we should multiply by $1/6$ because the local parameter at the cusp $\tau=0$ is $\exp(2\pi i\tau/6)$. The constant term in the q-expansion of the Eisenstein series is computed in lemma 8.16 below to be $-\frac{\pi^3}{9\sqrt{3}}$. Combining these values, we get

$$\text{Residue}(S)=\frac{1}{6}=\text{Residue}(\delta(mot))=-c\cdot\frac{-\pi^3}{9\sqrt{3}}, \tag{8.14}$$

therefore

$$c=\frac{-6\sqrt{3}}{\pi(2\pi i)^2}. \tag{8.15}$$

With the following lemma we evaluate at the cusp $\tau=0$, the constant term of the Eisenstein series $\sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)\neq(0,0)}}\psi(a,b)/(a\tau+b)^3$ with ψ the character in (5.30)

Lemma 8.16. — *Let $\psi(a,b) = \psi(a)$ in eq.* (5.30) *as in the theorem 8.11. Then the value of the constant term of the Eisenstein series*

$$\sum_{\substack{(a,b)\in\mathbb{Z}^2\\(a,b)\neq(0,0)}} \frac{\psi(a,b)}{(a\tau+b)^3} \tag{8.17}$$

at the cusp $\tau = 0$ is $-\pi^3/(9\sqrt{3})$.

Proof. — Let $\tau' = -1/\tau$. We need to compute

$$C = \lim_{\tau'\to+i\infty}\Big(\sum_{a\equiv 1 \bmod 6} \frac{1}{(-a+b\tau')^3} - \sum_{a\equiv 5 \bmod 6} \frac{1}{(-a+b\tau')^3}\Big) = \\ -2\Big(\sum_{\substack{a\geq 1\\ a\equiv 1 \bmod 6}} a^{-3} - \sum_{\substack{a\geq 1\\ a\equiv 5 \bmod 6}} a^{-3}\Big). \tag{8.18}$$

One can rewrite this constant term in terms of Hurwitz zeta functions $\zeta(s,a) := \sum_{n\geq 0}(n+a)^{-s}$ as

$$C = 2 - \frac{2}{6^3}\left(\zeta(3,\frac{1}{6}) - \zeta(3,-\frac{1}{6})\right). \tag{8.19}$$

The Hurwitz zeta functions that have the following a-expansion

$$\zeta(s,a) = \frac{1}{a^s} + \sum_{n\geq 0}\binom{s+n-1}{n}\zeta(s+n)\,(-a)^n\,. \tag{8.20}$$

Therefore

$$C = -2 - \frac{1}{54}\sum_{n\geq 0}\binom{2n+3}{2n+1}\zeta(2n+4)\left(-\frac{1}{6}\right)^{2n+1}. \tag{8.21}$$

Remarking that

$$\sum_{n\geq 0}\binom{2n+3}{2n+1}\zeta(2n+4)\,(-x)^{2n+1} = -\frac{1}{2x^3} + \frac{\pi^3}{2}\frac{\cos(\pi x)}{\sin(\pi x)^3} \tag{8.22}$$

one obtains that

$$C = -\frac{\pi^3}{9\sqrt{3}}\,. \tag{8.23}$$

□

It remains now to compute the amplitude asscociated to our Eisenstein series $\delta(mot) = 6\sqrt{3}\cdot eis_\psi/\pi$ and check that it agrees with $\mathcal{J}^2_\ominus(t)$ defined in eq. (4.1) modulo periods of the elliptic curve $\mathcal{E}_t$. Consider again the basic exact sequence (6.23). We know from lemma 7.4 that $\partial(\delta(mot)) \in H^1(X, \mathcal{V}_\mathbb{Q}) \subset H^1(X, \mathcal{V}_\mathbb{C})$. It will be convenient to pull back this sequence and work over the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \Im\mathrm{m}(\tau) > 0\}$. Write $V_\mathbb{Z} \subset V_\mathbb{Q} \subset V_\mathbb{C} \subset V \otimes \mathcal{O}_\mathbb{H}$ for the various base extensions of the representation of $\Gamma_1(6)$ underlying the local system $\mathcal{V}$. Let $V_\mathbb{Z} = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$ such that $dz = \tau\varepsilon_1 + \varepsilon_2$. If γ_1, γ_2 is the dual homology basis, this gives

$$\int_{\gamma_1} dz = \tau; \quad \int_{\gamma_2} dz = 1. \tag{8.24}$$

The pairing $H^1(\mathcal{E}_t, \mathbb{Z}) \otimes H^1(\mathcal{E}_t, \mathbb{Z}) \to \mathbb{Z}(-1)$ yields a symplectic form

$$\langle \varepsilon_1, \varepsilon_2\rangle = 2\pi i = -\langle \varepsilon_2, \varepsilon_1\rangle, \tag{8.25}$$

and $\langle \varepsilon_i, \varepsilon_i\rangle = 0$ for $i = 1, 2$. Consider the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & V_\mathbb{C} \arrow[r] & V \otimes \mathcal{O}_\mathbb{H} \arrow[r, "d"] & V \otimes \Omega^1_\mathbb{H} \arrow[r] & 0 \\
0 \arrow[r] & V_\mathbb{C} \arrow[r] \arrow[u, equal] & N \arrow[r] \arrow[u] & \mathbb{C}_\mathbb{H} \arrow[r] \arrow[u, "1\mapsto\delta(mot)"'] & 0
\end{tikzcd} \tag{8.26}$$

The idea is we view the bottom sequence in (8.26) as underlying an extension of variations of Hodge structure over $\mathbb{H}$ and we compute the amplitude in the usual way (6.10) by lifting (ie. integrating) $\delta(mot) = (6\sqrt{3}/\pi)eis_\psi$ and pairing the resulting element in $V \otimes \mathcal{O}_\mathbb{H}$ with ω. We can write

$$\omega = \varpi_r dz = \varpi_r(\tau\varepsilon_1 + \varepsilon_2)\,. \tag{8.27}$$

Here ϖ_r is the real period as in equation (4.15). Formally, we find

$$\text{amplitude} = \\ \frac{6\sqrt{3}}{\pi(2\pi i)^2}\varpi_r \left\langle \tau\varepsilon_1 + \varepsilon_2, \int \sum_{(a,b)\neq(0,0)} \frac{\psi(a)(\tau\varepsilon_1 + \varepsilon_2)d\tau}{(a\tau + b)^3} \right\rangle = \\ \frac{12i\sqrt{3}}{(2\pi i)^2}\varpi_r \Big(\tau \int \sum_{(a,b)\neq(0,0)} \frac{\psi(a)d\tau}{(a\tau + b)^3} - \int \sum_{(a,b)\neq(0,0)} \frac{\psi(a)\tau d\tau}{(a\tau + b)^3}\Big). \tag{8.28}$$

We do the integration term by term substituting

$$\int \frac{d\tau}{(a\tau + b)^3} = \frac{-1}{2a(b + a\tau)^2}; \quad \int \frac{\tau d\tau}{(a\tau + b)^3} = -\frac{b + 2a\tau}{2a^2(b + a\tau)^2} \tag{8.29}$$

Since $\psi(0) = 0$, this yields

$$\text{amplitude} = (6i\sqrt{3}/(2\pi i)^2)\varpi_r \sum_{\substack{a\neq 0\\ b\in\mathbb{Z}}} \frac{\psi(a)}{a^2(a\tau+b)}. \tag{8.30}$$

Note there is a convergence issue here since the sum is not absolutely convergent. We will treat this sum using the "Eisenstein summation" regularization following [**44**, eq. (14) on page 13], and we write

$$\lim_{N\to+\infty} \sum_{n=-N}^{N} \frac{1}{\tau+n} = \pi i \frac{q+1}{q-1}; \quad q = \exp(2\pi i\tau). \tag{8.31}$$

Substituting in eq. (8.30)

$$\text{amplitude} = (\frac{-6\pi\sqrt{3}}{(2\pi i)^2})\varpi_r \sum_{a\neq 0} \frac{\psi(a)}{a^2} \frac{q^a+1}{q^a-1} \tag{8.32}$$

Since both $\psi(a)$ and $\frac{q^a+1}{q^a-1}$ are odd as functions of a, we can write this as

$$\text{amplitude} = \frac{12\pi\sqrt{3}}{(2\pi i)^2}\varpi_r \sum_{\substack{a\in\mathbb{Z}\\ a\neq 0}} \frac{\psi(a)}{a^2} \frac{1}{1-q^a}. \tag{8.33}$$

(Here of course we assume $|q| \leq 1$. If $|q| > 1$ we can write down a similar expression involving q^{-1}.) Comparing with the expression for the sunset integral in (5.28) we find the relation

$$\text{amplitude} = \mathcal{J}^2_{\ominus}(t) + \text{periods}\,. \tag{8.34}$$

Remark 8.35. — *The above match between the Feynman integral* $\mathcal{J}^2_{\ominus}(t)$ *and the motivic amplitude is achieved for a value of the constant c in* (8.15) *including an extra factor of* $1/(2i\pi)^2$. *This factor means that one must use for* $d\tau dz$ *the corresponding algebraic coordinate when computing the residue of the Eisenstein series. Presumably, this extra factor of* $(2i\pi)^2$ *originates from the twist of* 2 *in the definition*

$$\mathcal{V}_{\mathbb{Q}} = R^1 f_*(\mathbb{Q}_{\mathcal{E}}(2)) \tag{8.36}$$

which shifts the rational structure by $(2\pi i)^2$.

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A
Elliptic Dilogarithm

In this appendix we recall the main properties of the elliptic dilogarithms following [**27, 46, 47**].

Starting from the Bloch-Wigner dilogarithm

$$\begin{aligned} D(z) &= \Im\mathrm{m}(\mathrm{Li}_2(z) + \log|z|\log(1-z)) \\ &= \frac{1}{2i}\left(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar z) + \frac{1}{2}\log(z\bar z)\log\left(\frac{1-z}{1-\bar z}\right)\right), \end{aligned} \tag{A.1}$$

this function is univalued real analytic on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$, and continuous on $\mathbb{P}^1(\mathbb{C})$. This function satisfies the following relations

$$D(e^{i\theta}) = Cl_2(\theta) = \sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}, \qquad \theta\in\mathbb{R} \tag{A.2}$$

$$D(z^2) = 2\,(D(z) + D(-z))\,. \tag{A.3}$$

We have the following six relations [**47**]

$$\begin{aligned} D(z) &= -D(\bar z) = D(1-z^{-1}) = D((1-z)^{-1}) \\ &= -D(z^{-1}) = -D(1-z) = -D(-z(1-z)^{-1})\,. \end{aligned} \tag{A.4}$$

The $D(z)$ function satisfies

$$dD(z) = \log|z|d\arg(1-z) - \log|1-z|d\arg(z)\,. \tag{A.5}$$

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SPENCER BLOCH, 5765 S. Blackstone Ave., Chicago, IL 60637, USA
E-mail : spencer_bloch@yahoo.com

PIERRE VANHOVE, Institut des Hautes Etudes Scientifiques, Le Bois-Marie, 35 route de Chartres, F-91440 Bures-sur-Yvette, France
Institut de Physique Théorique,, CEA, IPhT, F-91191 Gif-sur-Yvette, France, CNRS, URA 2306, F-91191 Gif-sur-Yvette, France • *E-mail :* pierre.vanhove@cea.fr