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The real radiation antenna functions for $S \to Q\bar{Q}gg$ at NNLO QCD

Werner Bernreuther,[a] **Christian Bogner**[b] **and Oliver Dekkers**[a]

[a] *Institut für Theoretische Physik, RWTH Aachen University, D-52056 Aachen, Germany*

[b] *Institut für Physik, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany*

E-mail: breuther@physik.rwth-aachen.de, bogner@math.hu-berlin.de, dekkers@physik.rwth-aachen.de

Abstract: We analyze, in the antenna subtraction framework, the real radiation antenna functions for processes involving the production of a pair of heavy quarks and two gluons by an uncolored initial state at NNLO QCD. We provide explicit expressions for these functions and discuss their infrared singular behaviour. Our main results are the corresponding integrated antenna functions which are computed analytically. They are expressed in terms of harmonic polylogarithms.

Keywords: QCD, NNLO computations, subtraction methods

Contents

1 Introduction

In this paper we consider the production of a heavy quark antiquark pair, $Q\bar{Q}$, by an uncolored initial state S at order α_s^2, i.e., at next-to-next-to-leading order (NNLO) QCD:

$$S \to Q\ \bar{Q}\ + X\,. \tag{1.1}$$

Reactions of the type (1.1) include $e^+e^- \to \gamma^*, Z^* \to Q\bar{Q}X$, heavy quark-pair production by photon-photon collisions and by the decay of a colorless electrically neutral massive boson of any spin. Our aim is to construct, within the antenna framework [1–3], subtraction terms for regularizing and handling the infrared (IR) divergences that appear in the matrix elements which contribute to (1.1) at NNLO QCD, such that the differential cross sections of (1.1) can be computed to this order. Our paper is a second step in the computation of these NNLO subtraction terms: After having determined the subtraction term for the $Q\bar{Q}q\bar{q}$ final state and its integral over the four-parton phase space in [4], we present here the analogous result for the $Q\bar{Q}gg$ final state.

Before formulating the problem at hand, it seems appropriate to recall the state-of-the-art of the 'subtraction technology' in perturbative QCD. For calculations at NLO QCD the presently most widely used approach is the dipole subtraction method [5–8] and slight modifications thereof [9–14]. A number of computer implementations of this method exist, including those of [11, 13, 15–17]. Other subtraction methods that were worked out at NLO QCD include those of [18–24], and the NLO subtraction within the antenna method [1–3, 25–27].

The infrared structure of 2-loop partonic amplitudes was analyzed in [28–36]. Techniques for handling the IR divergences of the individual contributions to partonic processes at NNLO QCD include the sector decomposition algorithm [37–42], the antenna formalism [3, 4, 43–51], and the subtraction methods [52–62]. Applications to reactions at NNLO QCD include $pp \to H + X$ [63, 64], $pp \to H +$ jet [65], hadronic vector boson production [66–68], $pp \to H + W$ [69], hadronic jet production [70], $H \to b\bar{b}X$ [71], weak decays of heavy quarks [72–75], $e^+e^- \to 2$ jets [76, 77], and $e^+e^- \to 3$ jets [78–81]. A general scheme, based on sector decomposition [37, 38] and phase-space partitioning according to [18], that can be used for massless and massive partons, was presented in

[82, 83] and applied in the computation of the total hadronic $t\bar{t}$ cross section to order α_s^4 [84, 85]. For a related method and its application to $Z \to e^+e^-$, see [86].

At NNLO QCD the antenna method has been worked out completely so far only for processes with massless partons in the final state. Our aim is to fill this gap for reactions of the type (1.1). The theoretical description of the reactions (1.1) to order α_s^2 requires i) the 2-loop amplitudes $S \to Q\bar{Q}$. They are known for $S =$ vector [87, 88], axial vector [89, 90], scalar and pseudoscalar [91]. ii) The tree-level and one-loop amplitudes for $S \to Q\bar{Q}g$. The computation of the one-loop amplitudes is standard; for instance, for e^+e^- annihilation, i.e. $S = \gamma^*, Z^*$, they are given in [92–94]. iii) The tree-level amplitudes $S \to Q\bar{Q}Q\bar{Q}$, $Q\bar{Q}gg$, and $Q\bar{Q}q\bar{q}$, where q denotes a massless quark.

To be specific, let us discuss the ingredients of a calculation of the differential cross section for S decaying into two massive quark jets to order α_s^2. The computation of $d\sigma_{\rm NLO}$ is standard. The contribution of order α_s^2 to the two-jet cross section is given schematically by

$$
\begin{aligned}
d\sigma_{\rm NNLO} \, = & \int_{\Phi_4} \left(d\sigma^{RR}_{\rm NNLO} - d\sigma^{S}_{\rm NNLO}\right) + \int_{\Phi_3} \left(d\sigma^{RV}_{\rm NNLO} - d\sigma^{T}_{\rm NNLO}\right) \\
& + \int_{\Phi_2} d\sigma^{VV}_{\rm NNLO} + \int_{\Phi_4} d\sigma^{S}_{\rm NNLO} + \int_{\Phi_3} d\sigma^{T}_{\rm NNLO} \,.
\end{aligned} \tag{1.2}
$$

Here $d\sigma^{RR}_{\rm NNLO}$, $d\sigma^{RV}_{\rm NNLO}$, and $d\sigma^{VV}_{\rm NNLO}$ denote the contributions from the tree-level amplitudes[1] $S \to Q\bar{Q}q\bar{q}$ and $S \to Q\bar{Q}gg$, the amplitude $S \to Q\bar{Q}g$ to one-loop, and the amplitude $S \to Q\bar{Q}$ to two-loops, respectively. These individual contributions give rise to infrared (IR) divergences. The remaining integrands of (1.2), $d\sigma^{S}_{\rm NNLO}$ and $d\sigma^{T}_{\rm NNLO}$, are the double-real radiation subtraction terms (for $Q\bar{Q}gg$, $Q\bar{Q}q\bar{q}$) and the real-virtual subtraction term, respectively. They have to be constructed such that the integrals over $d\sigma^{RR}_{\rm NNLO} - d\sigma^{S}_{\rm NNLO}$ and over $d\sigma^{RV}_{\rm NNLO} - d\sigma^{T}_{\rm NNLO}$ are finite and can be evaluated numerically. Furthermore, in order to make the cancellation of IR singularities explicit in eq. (1.2), the integrals of these subtraction terms must be computed over the phase-space regions where IR singularities arise. In the antenna subtraction formalism, which we apply in the following, the subtraction terms are constructed from color-ordered matrix elements by exploiting factorization properties of the respective phase-space integrations.

Below, we determine the color-ordered subtraction terms for the $Q\bar{Q}gg$ final state from the matrix element of the process

$$
\gamma^*(q) \to Q(p_1)\; \bar{Q}(p_2)\; + g(p_3)\; g(p_4)\,. \tag{1.3}
$$

We then perform the integrals of the subtraction terms analytically in arbitrary space-time dimensions over the full four-particle phase space. It is worth recalling that these (integrated) antenna functions are not only of relevance for the specific process at hand, but serve also as building blocks for constructing subtraction terms for other processes (1.1) within the antenna formalism. As mentioned above the (integrated) subtraction term for the $Q\bar{Q}q\bar{q}$ final state was already determined in [4]. The remaining subtraction term required in (1.2), the real-virtual subtraction term $d\sigma^{T}_{\rm NNLO}$, can be obtained from the interference of the tree-level and 1-loop matrix element for $\gamma^* \to Q\bar{Q}g$. The integral of this term over the 3-particle phase space can also be done, in arbitrary space-time dimensions, in analytical form and will be given in a future publication [95].

The paper is organized as follows. In Section 2 we determine the two color-ordered antenna functions for (1.3) and discuss their singular limits. In Section 3 we present the results of the integration of these functions over the four-parton phase space in D space-time dimensions. Section 4 contains a summary and outlook.

[1] The IR-finite tree level amplitude $S \to Q\bar{Q}Q\bar{Q}$ is of no concern here.

2 The $Q\bar{Q}gg$ final state and antenna subtraction at NNLO

As mentioned above, we will construct a subtraction term that coincides with the order α_s^2 squared matrix element of

$$S(q) \to Q(p_1)\,\bar{Q}(p_2)\,g(p_3)\,g(p_4) \tag{2.1}$$

in all single and double unresolved limits. The corresponding squared tree-level matrix element, summed over all colors and spins, but excluding the symmetry factor 1/2 for the two gluons in the final state, can be decomposed into color-ordered substructures as follows:

$$\begin{aligned}\left|M^0_{S\to Q\bar{Q}gg}\right|^2 =& \mathcal{N}_0\,(4\pi\alpha_s)^2\left(N_c^2-1\right)\\ &\times\left[N_c\left(\mathcal{M}^0_4(1_Q,3_g,4_g,2_{\bar{Q}})+\mathcal{M}^0_4(1_Q,4_g,3_g,2_{\bar{Q}})\right)-\frac{1}{N_c}\,\tilde{\mathcal{M}}^0_4(1_Q,4_g,3_g,2_{\bar{Q}})\right],\end{aligned} \tag{2.2}$$

where N_c denotes the number of colors. The normalization factor $\mathcal{N}_0$ includes all non-QCD couplings. For the sake of brevity, we have dropped the dependence on the initial state momenta in the leading and subleading-color contributions $\mathcal{M}^0_4$ and $\tilde{\mathcal{M}}^0_4$. Adapting the notation of [3], we use symbolic labels 1_Q, $2_{\bar{Q}}$, 3_g, 4_g for the momenta of the quark, anti-quark and the gluons, respectively. In $\mathcal{M}^0_4(1_Q,k_g,l_g,2_{\bar{Q}})$ the emission of the gluons k, l is ordered, in the sense that there are color-connections between the quark and gluon k, between the gluons k and l, and between gluon l and the antiquark. In the subleading color term $\tilde{\mathcal{M}}^0_4$ both gluons are photon-like, i.e., no non-abelian gluon vertices are involved. Hence, when the two gluons become collinear, this term does not become singular.

The corresponding contribution to the cross section for 2-jet production may be written as follows:

$$\begin{aligned}d\sigma^{RR,Q\bar{Q}gg}_{\rm NNLO} =& \frac{1}{2}\,\bar{\mathcal{N}}\,(4\pi\alpha_s)^2\left(N_c^2-1\right)d\Phi_4^{(D)}(p_1,p_2,p_3,p_4;q)\,J_2^{(4)}(p_1,p_2,p_3,p_4)\\ &\times\left[N_c\left(\mathcal{M}^0_4(1_Q,3_g,4_g,2_{\bar{Q}})+\mathcal{M}^0_4(1_Q,4_g,3_g,2_{\bar{Q}})\right)-\frac{1}{N_c}\,\tilde{\mathcal{M}}^0_4(1_Q,4_g,3_g,2_{\bar{Q}})\right],\end{aligned} \tag{2.3}$$

where the factor 1/2 is due to Bose symmetry. The factor $\bar{\mathcal{N}}$ contains $\mathcal{N}_0$, the spin averaging factor for the initial state, and the flux factor. The phase-space measure of a n-parton final state in $D = 4-2\epsilon$ dimensions is given by

$$d\Phi_n^{(D)}(p_1,\ldots,p_n;q) = \left(\mu^{4-D}\right)^{n-1}\prod_{i=1}^{n}\frac{d^{D-1}p_i}{(2\pi)^{D-1}2p_i^0}\,(2\pi)^D\,\delta^{(D)}\left(q-\sum_{i=1}^{n}p_i\right), \tag{2.4}$$

where μ is a mass scale. The jet function $J_m^{(n)}$ in (2.3) ensures that only configurations are taken into account where n outgoing partons form m jets.

For later reference, we define the charge and color stripped squared matrix elements $\mathcal{M}^0_2$ and $\mathcal{M}^0_3$ associated with the tree-level squared matrix elements of $S\to Q\bar{Q}$ and $S\to Q\bar{Q}g$, respectively:

$$\left|M_{S\to Q\bar{Q}}\right|^2 = \mathcal{N}_0 N_c\,\mathcal{M}^0_2\big(1_Q,2_{\bar{Q}}\big)+\mathcal{O}(\alpha_s)\;, \tag{2.5}$$

$$\left|M_{S\to Q\bar{Q}g}\right|^2 = \mathcal{N}_0\,(4\pi\alpha_s)\left(N_c^2-1\right)\mathcal{M}^0_3\big(1_Q,3_g,2_{\bar{Q}}\big)+\mathcal{O}\big(\alpha_s^2\big)\;. \tag{2.6}$$

Summation over all spins is understood.

2.1 Antenna subtraction terms

Let us now turn to the subtraction term $d\sigma^{S,\,Q\bar{Q}gg}_{\rm NNLO}$ that must be constructed such that the phase space integration over $d\sigma^{R,\,Q\bar{Q}gg}_{\rm NNLO} - d\sigma^{S,\,Q\bar{Q}gg}_{\rm NNLO}$ becomes finite in $D=4$ space-time dimensions. We decompose this term into a sum of two contributions,

$$d\sigma^{S,\,Q\bar{Q}gg}_{\rm NNLO} = d\sigma^{S,\,a\;Q\bar{Q}gg}_{\rm NNLO} + d\sigma^{S,\,b\;Q\bar{Q}gg}_{\rm NNLO}, \tag{2.7}$$

where $d\sigma_{\rm NNLO}^{S,\,a\;Q\bar{Q}gg}$, $d\sigma_{\rm NNLO}^{S,\,b\;Q\bar{Q}gg}$ cover the singularities due to single-unresolved and double-unresolved configurations, respectively. In analogy to the case of massless quarks [49, 78], these terms are obtained as follows:

$$
\begin{aligned}
d\sigma_{\rm NNLO}^{S,a,Q\bar{Q}gg} &= \frac{1}{2}\bar{\mathcal{N}}\,(4\pi\alpha_s)^2\left(N_c^2-1\right)\,d\Phi_4^{(D)}(p_1,p_2,p_3,p_4;q)\\
&\times\Bigg[N_c\Big(d_3^0(1_Q,3_g,4_g)\,\mathcal{M}_3^0\Big(\widetilde{(13)}_Q,\widetilde{(43)}_g,2_{\bar{Q}}\Big)\,J_2^{(3)}(\widetilde{p_{13}},\widetilde{p_{43}},p_2)\\
&\qquad+d_3^0(2_{\bar{Q}},4_g,3_g)\,\mathcal{M}_3^0\Big(1_Q,\widetilde{(34)}_g,\widetilde{(24)}_{\bar{Q}}\Big)\,J_2^{(3)}(p_1,\widetilde{p_{34}},\widetilde{p_{24}})\\
&\qquad+d_3^0(1_Q,4_g,3_g)\,\mathcal{M}_3^0\Big(\widetilde{(14)}_Q,\widetilde{(34)}_g,2_{\bar{Q}}\Big)\,J_2^{(3)}(\widetilde{p_{14}},\widetilde{p_{34}},p_2)\\
&\qquad+d_3^0(2_{\bar{Q}},3_g,4_g)\,\mathcal{M}_3^0\Big(1_Q,\widetilde{(43)}_g,\widetilde{(23)}_{\bar{Q}}\Big)\,J_2^{(3)}(p_1,\widetilde{p_{43}},\widetilde{p_{23}})\Big)\\
&\qquad-\frac{1}{N_c}\Big(A_3^0(1_Q,3_g,2_{\bar{Q}})\,\mathcal{M}_3^0\Big(\widetilde{(13)}_Q,4_g,\widetilde{(23)}_{\bar{Q}}\Big)\,J_2^{(3)}(\widetilde{p_{13}},p_4,\widetilde{p_{23}})\\
&\qquad+A_3^0(1_Q,4_g,2_{\bar{Q}})\,\mathcal{M}_3^0\Big(\widetilde{(14)}_Q,3_g,\widetilde{(24)}_{\bar{Q}}\Big)\,J_2^{(3)}(\widetilde{p_{14}},p_3,\widetilde{p_{24}})\Big)\Bigg],
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
d\sigma_{\rm NNLO}^{S,b,Q\bar{Q}gg} &= \frac{1}{2}\bar{\mathcal{N}}\,(4\pi\alpha_s)^2\left(N_c^2-1\right)\,d\Phi_4^{(D)}(p_1,p_2,p_3,p_4;q)\\
&\times\Bigg[N_c\Big(A_4^0(p_1,p_3,p_4,p_2)-d_3^0(1_Q,3_g,4_g)\,A_3^0\Big(\widetilde{(13)}_Q,\widetilde{(43)}_g,2_{\bar{Q}}\Big)\\
&\qquad-d_3^0(2_{\bar{Q}},4_g,3_g)\,A_3^0\Big(1_Q,\widetilde{(34)}_g,\widetilde{(24)}_{\bar{Q}}\Big)\Big)\mathcal{M}_2^0\Big(\widetilde{(134)}_Q,\widetilde{(234)}_{\bar{Q}}\Big)\,J_2^{(2)}(\widetilde{p_{134}},\widetilde{p_{234}})\\
&\quad+N_c\Big(A_4^0(p_1,p_4,p_3,p_2)-d_3^0(1_Q,4_g,3_g)\,A_3^0\Big(\widetilde{(14)}_Q,\widetilde{(34)}_g,2_{\bar{Q}}\Big)\\
&\qquad-d_3^0(2_{\bar{Q}},3_g,4_g)\,A_3^0\Big(1_Q,\widetilde{(43)}_g,\widetilde{(23)}_{\bar{Q}}\Big)\Big)\mathcal{M}_2^0\Big(\widetilde{(143)}_Q,\widetilde{(243)}_{\bar{Q}}\Big)\,J_2^{(2)}(\widetilde{p_{143}},\widetilde{p_{243}})\\
&\quad-\frac{1}{N_c}\Big(\tilde{A}_4^0(1_Q,3_g,4_g,2_{\bar{Q}})-A_3^0(1_Q,3_g,2_{\bar{Q}})\,A_3^0\Big(\widetilde{(13)}_Q,4_g,\widetilde{(23)}_Q\Big)\\
&\quad-A_3^0(1_Q,4_g,2_{\bar{Q}})\,A_3^0\Big(\widetilde{(14)}_Q,3_g,\widetilde{(24)}_Q\Big)\Big)\mathcal{M}_2^0\Big(\widetilde{(134)}_Q,\widetilde{(234)}_{\bar{Q}}\Big)\,J_2^{(2)}(\widetilde{p_{134}},\widetilde{p_{234}})\Bigg].
\end{aligned}
\tag{2.9}
$$

The tree-level massive quark-antiquark antenna function A_3^0 and the massive quark gluon antenna d_3^0 were derived in [26]. The four-parton massive quark-antiquark antenna functions A_4^0 and $\tilde{A}_4^0$ govern the color-ordered and non-ordered (photon-like) emission of two gluons between a pair of massive radiator quarks, respectively. They constitute genuine NNLO objects that do not appear in the subtraction procedure at NLO. Their precise definition is given in eqs. (2.17), (A.1), and (A.2) below. The terms $\mathcal{M}_2^0$, $\mathcal{M}_3^0$ are defined in (2.6), (2.5), respectively.

The subtraction terms (2.8) and (2.9) involve redefined on-shell momenta $\widetilde{p_{ij}}$ and $\widetilde{p_{ijk}}$, which are defined by Lorentz-invariant mappings $\{p_i,p_j,p_k\}\to\{\widetilde{p_{ij}},\widetilde{p_{kj}}\}$ and $\{p_i,p_j,p_k,p_l\}\to\{\widetilde{p_{ijk}},\widetilde{p_{ljk}}\}$. For massless final-state partons these mappings have been derived in ref. [78]. As discussed in ref. [49] the same mappings can be applied in the case of massive partons.

An important feature of the antenna subtraction formalism is that the infrared singular structure is governed by the integration over individual antenna functions. The $(m+1)$-parton phase-space measure can be factorized as follows:

$$d\Phi^{(D)}_{m+1}(p_1,\ldots,p_{m+1};\,q) = d\Phi^{(D)}_{m}(p_1,\ldots,\widetilde{p_{ij}},\,\widetilde{p_{kj}},\ldots,p_{m+1};\,q)\cdot d\Phi^{(D)}_{X_{ijk}}(p_i,\,p_j,\,p_k;\,\widetilde{p_{ij}}+\widetilde{p_{kj}}). \quad (2.10)$$

Applying (2.10) to the above subtraction terms and integrating over the momenta not carrying a tilde in eq. (2.8) one obtains

$$\begin{aligned}\int_1 d\sigma^{S,a,Q\bar{Q}gg}_{\text{NNLO}} &= \bar{C}(\epsilon)\,\bar{\mathcal{N}}\,(4\pi\alpha_s)\left(\frac{\alpha_s}{2\pi}\right)\left(N_c^2-1\right)d\Phi_3^{(D)}(p_1,p_2,p_3;q)\,\mathcal{M}^0_3\big(1_Q,3_g,2_{\bar{Q}}\big)\,J_2^{(3)}(p_1,p_3,p_2)\\ &\quad\times\left[\frac{N_c}{2}\Big(\mathcal{D}^0_3(\epsilon;s_{13},m_Q)+\mathcal{D}^0_3(\epsilon;s_{23},m_Q)\Big)-\frac{1}{N_c}\mathcal{A}^0_3(\epsilon,s_{12},m_Q)\right], \qquad (2.11)\end{aligned}$$

where $\bar{C}(\epsilon) = 8\pi^2 C(\epsilon) = (4\pi)^\epsilon e^{-\epsilon\gamma_E}$. Here and below we use the notation $s_{ij} = 2p_i\cdot p_j$ and $s_{ijk} = s_{ij}+s_{ik}+s_{jk}$ for scalar products of momenta. The integrated massive antenna functions $\mathcal{A}^0_3$ and $\mathcal{D}^0_3$ are the integrals of A^0_3 and D^0_3 over appropriate phase-space regions, obtained from a factorization according to eq. (2.10). They were first computed in [26] (see also eqs. (5.15) and (5.17) of [27]).

The subtraction term for the double unresolved configurations is split into two parts:

$$d\sigma^{S,b,Q\bar{Q}gg}_{\text{NNLO}} = d\sigma^{S,b,1,Q\bar{Q}gg}_{\text{NNLO}} + d\sigma^{S,b,2,Q\bar{Q}gg}_{\text{NNLO}}\,, \qquad (2.12)$$

where $d\sigma^{S,b,2,Q\bar{Q}gg}_{\text{NNLO}}$ contains the terms that involve the four-parton antenna functions A^0_4 and $\tilde{A}^0_4$, while $d\sigma^{S,b,1,Q\bar{Q}gg}_{\text{NNLO}}$ contains all other terms. The subtracted contribution to the differential 2-jet cross section

$$\int_{\Phi_4^{(4)}}\left[d\sigma^{RR,Q\bar{Q}gg}_{\text{NNLO}} - d\sigma^{S,a,Q\bar{Q}gg}_{\text{NNLO}} - d\sigma^{S,b,Q\bar{Q}gg}_{\text{NNLO}}\right]_{\epsilon=0} \qquad (2.13)$$

is by construction IR finite and can be integrated numerically over the four-parton phase space in $D=4$ dimensions.

The splitting (2.12) is convenient because the $\int d\sigma^{S,b,1,Q\bar{Q}gg}_{\text{NNLO}}$ and $\int d\sigma^{S,b,2,Q\bar{Q}gg}_{\text{NNLO}}$ are added to the three-parton and two-parton contribution to $d\sigma_{\text{NNLO}}$, respectively (cf. (1.2)). Hence they are integrated over antenna phase spaces of different parton multiplicities. The respective integrals of (2.12) read

$$\begin{aligned}\int_2 d\sigma^{S,b,2,Q\bar{Q}gg}_{\text{NNLO}} &= \big(\bar{C}(\epsilon)\big)^2\,\bar{\mathcal{N}}\left(\frac{\alpha_s}{2\pi}\right)^2\left(N_c^2-1\right)\,d\Phi_2(p_1,p_2;q)\,\mathcal{M}^0_2\big(1_Q,2_{\bar{Q}}\big)\,J_2^{(2)}(p_1,p_2)\\ &\quad\times\left[N_c\,\mathcal{A}^0_4(\epsilon;s_{12},m_Q)-\frac{1}{2N_c}\,\tilde{\mathcal{A}}^0_4(\epsilon;s_{12},m_Q)\right], \qquad (2.14)\end{aligned}$$

$$\begin{aligned}\int_1 d\sigma^{S,b,1,Q\bar{Q}gg}_{\text{NNLO}} &= -\bar{C}(\epsilon)\,\bar{\mathcal{N}}\,(4\pi\alpha_s)\left(\frac{\alpha_s}{2\pi}\right)\left(N_c^2-1\right)d\Phi_3(p_1,p_2,p_3;q)\,J_2^{(2)}(\widetilde{p_{13}},\widetilde{p_{23}})\\ &\quad\times\mathcal{M}^0_2\Big(\widetilde{(13)}_Q,\widetilde{(23)}_{\bar{Q}}\Big)\,A^0_3\big(1_Q,3_g,2_{\bar{Q}}\big)\\ &\quad\times\left[\frac{N_c}{2}\Big(\mathcal{D}^0_3(\epsilon;s_{13},m_Q)+\mathcal{D}^0_3(\epsilon;s_{23},m_Q)\Big)-\frac{1}{N_c}\mathcal{A}^0_3(\epsilon;s_{12},m_Q)\right]. \qquad (2.15)\end{aligned}$$

The integrated massive four-parton antenna functions $\mathcal{A}^0_4$ and $\tilde{\mathcal{A}}^0_4$, which have not been derived so far, will be given in sec. 3.

2.2 Antenna functions

We derive the antenna functions A^0_4 and $\tilde{A}^0_4$ from the color-ordered tree-level matrix element of the process

$$\gamma^*(q) \to Q(p_1)\,\bar{Q}(p_2)\,g(p_3)\,g(p_4)\,. \qquad (2.16)$$

The respective squared matrix element, summed over spins and colors of the final state but excluding the Bose-symmetry factor 1/2 reads

$$\begin{aligned}\left|M^0_{\gamma^*\to Q\bar{Q}gg}\right|^2 &= (4\pi\alpha)\, e_Q^2\, (4\pi\alpha_s)^2\, \left(N_c^2-1\right) \left|\mathcal{M}^0_{\gamma^*\to Q\bar{Q}}\right|^2 \\ &\quad\times \left[N_c\left(A_4^0(1_Q,\,3_g,\,4_g,\,2_{\bar{Q}}) + A_4^0(1_Q,\,4_g,\,3_g,\,2_{\bar{Q}})\right) - \frac{1}{N_c}\tilde{A}_4^0(1_Q,\,4_g,\,3_g,\,2_{\bar{Q}})\right].\end{aligned} \tag{2.17}$$

The polarizations of γ are summed, but not averaged. e_Q denotes the electric charge of the massive quark in units of the positron charge $e = \sqrt{4\pi\alpha}$ and

$$\left|\mathcal{M}^0_{\gamma^*\to Q\bar{Q}}\right|^2 = 4\left[(1-\epsilon)\, q^2 + 2m_Q^2\right] . \tag{2.18}$$

The antenna functions A_4^0 and $\tilde{A}_4^0$ are given in appendix A.

2.3 Singular limits of the antennae

Before integrating the antenna functions A_4^0 and $\tilde{A}_4^0$, we study their behavior in the single and double unresolved limits. Thereby we verify that the subtraction terms introduced in eqs. (2.8) and (2.9) lead to a finite integral (2.13) over the four-particle phase space. The limiting behavior of the NLO antenna functions was already discussed extensively in the literature [26, 27]. Therefore we restrict ourselves to the respective analysis of the above NNLO antenna functions.

We are not concerned with quasi-collinear limits [6] here, which will be relevant in numerical evaluations only if the squared mass of the quark Q becomes much smaller than kinematic invariants.

2.3.1 Single unresolved limits

In the single unresolved limits, where one gluon becomes soft, the four-parton tree-level antennae A_4^0 and $\tilde{A}_4^0$ factorise as

$$A_4^0\big(1_Q, i_g, j_g, 2_{\bar{Q}}\big) \xrightarrow{i_g\to 0} S(1_Q, i_g, j_g)\, A_3^0\big(1_Q, j_g, 2_{\bar{Q}}\big)\,, \tag{2.19}$$

$$A_4^0\big(1_Q, i_g, j_g, 2_{\bar{Q}}\big) \xrightarrow{j_g\to 0} S\big(2_{\bar{Q}}, j_g, i_g\big)\, A_3^0\big(1_Q, i_g, 2_{\bar{Q}}\big)\,, \tag{2.20}$$

$$\tilde{A}_4^0\big(1_Q, i_g, j_g, 2_{\bar{Q}}\big) \xrightarrow{i_g\to 0} S\big(1_Q, i_g, 2_{\bar{Q}}\big)\, A_3^0\big(1_Q, j_g, 2_{\bar{Q}}\big)\,, \tag{2.21}$$

$$\tilde{A}_4^0\big(1_Q, i_g, j_g, 2_{\bar{Q}}\big) \xrightarrow{j_g\to 0} S\big(1_Q, j_g, 2_{\bar{Q}}\big)\, A_3^0\big(1_Q, i_g, 2_{\bar{Q}}\big)\,, \tag{2.22}$$

where

$$S(i,j,k) = \frac{2s_{ik}}{s_{ij}s_{jk}} - \frac{2m_i^2}{s_{ij}^2} - \frac{2m_k^2}{s_{jk}^2} \tag{2.23}$$

denotes the generalized single-soft eikonal function.

When the two color-connected gluons become collinear, the antenna function A_4^0 behaves as follows:

$$A_4^0\big(1_Q, i_g, j_g, 2_{\bar{Q}}\big) \xrightarrow{i_g\parallel j_g} \frac{1}{s_{34}}\, P_{gg\to g}(z)\, A_3^0\big(1_Q, (ij)_g, 2_{\bar{Q}}\big) + \text{angular}, \tag{2.24}$$

where z is the momentum fraction carried by one of the collinear gluons and

$$P_{gg\to g}(z) = 2\left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right] \tag{2.25}$$

is the spin averaged Altarelli-Parisi splitting function [96]. In (2.24), "angular" indicates the presence of angular terms due to spin correlations. The handling of these terms in the context of the antenna subtraction formalism is discussed in [3, 44].

2.3.2 Double-soft gluon limit

We consider the leading-color antenna A_4^0 in the limit where the momenta p_3 and p_4 of the two gluons become simultaneously soft. This limit is defined by rescaling the gluon momenta by an overall factor λ:

$$p_3 \;\rightarrow\; \lambda p_3 \,, \quad p_4 \;\rightarrow\; \lambda p_4 \,. \tag{2.26}$$

In the limit $\lambda \rightarrow 0$ the quark-antiquark antenna A_4^0 behaves as

$$A_4^0(1_Q, 3_g, 4_g, 2_{\bar{Q}}) \;\longrightarrow\; S(1_Q, 3_g, 4_g, 2_{\bar{Q}})/\lambda^4 + \ldots \,, \tag{2.27}$$

where the ellipses denote less singular contributions. The dominant singular term of $\mathcal{O}(1/\lambda^4)$, the double-soft gluon function S, is given by

$$\begin{aligned}
S(1_Q, 3_g, 4_g, 2_{\bar{Q}}) &= \frac{2s_{12}^2}{s_{13}\,(s_{13}+s_{14})\,s_{24}\,(s_{23}+s_{24})} + \frac{2\,(1-\epsilon)}{s_{34}^2}\left(\frac{s_{13}}{(s_{13}+s_{14})} + \frac{s_{24}}{(s_{23}+s_{24})} - 1\right)^2 \\
&+ \frac{2s_{12}}{s_{34}}\left(\frac{1}{s_{13}s_{24}} + \frac{1}{s_{13}\,(s_{23}+s_{24})} + \frac{1}{s_{24}\,(s_{13}+s_{14})} - \frac{4}{(s_{13}+s_{14})\,(s_{23}+s_{24})}\right) \\
&+ 4m_Q^2\Bigg[-\frac{s_{12}}{s_{13}s_{24}}\left(\frac{1}{(s_{23}+s_{24})\,s_{24}} + \frac{1}{s_{13}\,(s_{13}+s_{14})}\right) + \frac{1}{s_{34}}\Bigg(\frac{1}{(s_{13}+s_{14})^2} + \frac{1}{(s_{23}+s_{24})^2} \\
&- \frac{s_{23}}{s_{13}\,(s_{13}+s_{14})\,s_{24}} - \frac{s_{14}}{s_{13}\,(s_{23}+s_{24})\,s_{24}} - \frac{s_{24}}{s_{13}\,(s_{13}+s_{14})\,(s_{23}+s_{24})} \\
&- \frac{s_{13}}{(s_{13}+s_{14})\,(s_{23}+s_{24})\,s_{24}}\Bigg)\Bigg] + 4m_Q^4\left[\frac{1}{s_{13}^2\,(s_{13}+s_{14})^2} + \frac{1}{s_{13}^2 s_{24}^2} + \frac{1}{(s_{23}+s_{24})^2\,s_{24}^2}\right].
\end{aligned} \tag{2.28}$$

In the massless limit S coincides with the result known from the literature (see e.g. [3]).

The subleading-color antenna $\tilde{A}_4^0(1_Q, 3_g, 4_g, 2_{\bar{Q}})$ is not color-ordered (and hence symmetric with respect to the interchange of the two gluon momenta p_4 and p_3). Therefore it shows QED-like factorization in the double-soft limit:

$$\tilde{A}_4^0(1_Q, 3_g, 4_g, 2_{\bar{Q}}) \xrightarrow{3_g, 4_g \to 0} S(1_Q, 3_g, 2_{\bar{Q}})\, S(1_Q, 4_g, 2_{\bar{Q}})\,, \tag{2.29}$$

where the single-soft eikonal function is given in (2.23).

3 Integrated antenna functions $\mathcal{A}_4^0$ and $\tilde{\mathcal{A}}_4^0$

In section 2 we outlined the construction of $d\sigma_{NNLO}^{S,\,Q\bar{Q}gg}$ whose integrated counter-part involves the integrated antenna functions $\mathcal{A}_4^0$ and $\tilde{\mathcal{A}}_4^0$. In this section we compute these functions analytically. They are obtained from A_4^0 and $\tilde{A}_4^0$ as follows:

$$\mathcal{A}_4^0(\epsilon; s_{1234}, m_Q) \;=\; \frac{1}{(C(\epsilon))^2}\int d\Phi_{X_{Q\bar{Q}gg}}\, A_4^0(1_Q,\, 3_g,\, 4_g,\, 2_{\bar{Q}})\,, \tag{3.1}$$

$$\tilde{\mathcal{A}}_4^0(\epsilon; s_{1234}, m_Q) \;=\; \frac{1}{(C(\epsilon))^2}\int d\Phi_{X_{Q\bar{Q}gg}}\, \tilde{A}_4^0(1_Q,\, 3_g,\, 4_g,\, 2_{\bar{Q}})\,. \tag{3.2}$$

Here the antenna phase-space measure $d\Phi_{X_{Q\bar{Q}gg}}$ is defined by

$$d\Phi_4^{(D)}(p_1,\, p_2,\, p_3,\, p_4;\, q) = P_2(q^2,\, m_Q^2)\, d\Phi_{X_{Q\bar{Q}gg}}, \tag{3.3}$$

where $d\Phi_4^{(D)}$ is the four-particle phase-space measure defined in eq. (2.4) and P_2 is the integrated two-particle phase space,

$$P_2(q^2,\, m_Q^2) = 2^{-3+2\epsilon}\pi^{-1+\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\left(\frac{\mu^2}{q^2}\right)^{\epsilon}\left(1 - \frac{4m_Q^2}{q^2}\right)^{\frac{1}{2}-\epsilon}. \tag{3.4}$$

First, we rewrite the four-particle phase space measure according to

$$d\Phi_4^{(D)}(p_1,\, p_2,\, p_3,\, p_4;\, q) = \frac{\mu^{12-3D}}{i^4(2\pi)^{3D}}\delta^{(D)}\left(q - \sum_{i=1}^{4} p_i\right)\prod_{i=1}^{4}\frac{d^D p_i}{D_i}, \tag{3.5}$$

with the cut-propagators [63, 97]

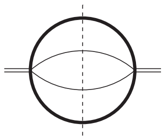
$$\frac{1}{D_i} = 2\pi i\delta^+(p_i^2 - m_Q^2) = \frac{1}{p_i^2 - m_Q^2 + i0} - \frac{1}{p_i^2 - m_Q^2 - i0} \qquad \text{for } i = 1,\, 2, \tag{3.6}$$

$$\frac{1}{D_i} = 2\pi i\delta^+(p_i^2) = \frac{1}{p_i^2 + i0} - \frac{1}{p_i^2 - i0} \qquad \text{for } i = 3,\, 4. \tag{3.7}$$

We introduce six further propagators $D_5, \ldots, D_{10}$ such that we can express all scalar products s_{ij}, s_{ijk} in the integrands A_4^0, $\tilde{A}_4^0$ by the functions $D_1, \ldots, D_{10}$. The set of these ten propagators is linearly dependent. However, each term in the two integrands can be expressed by at most nine or less propagators. The terms can be treated as integrands of cut-integrals, corresponding to four-particle cuts through three-loop propagator-type Feynman graphs. We distribute the terms to appropriate topologies, each given by nine independent propagators, and apply integration-by-parts reduction [98], using the implementation `FIRE` [99] of the Laporta algorithm [100].

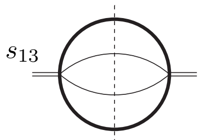
As a result of this reduction we can express the integrated antenna functions $\mathcal{A}_4^0$ and $\tilde{\mathcal{A}}_4^0$ by the following 15 master integrals:

$$T_1(q^2, m_Q^2, \epsilon) =$$



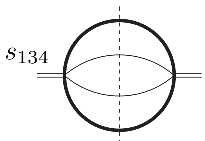
$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q)\,, \tag{3.8}$$

$$T_2(q^2, m_Q^2, \epsilon) =$$



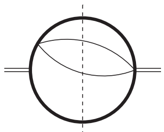
$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q)\; s_{13}\,, \tag{3.9}$$

$$T_3(q^2, m_Q^2, \epsilon) =$$



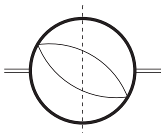
$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q)\; s_{134}\,, \tag{3.10}$$

$$T_4(q^2, m_Q^2, \epsilon) =$$



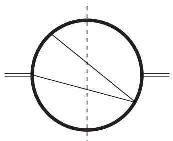
$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q)\, \frac{1}{s_{134}}\,, \tag{3.11}$$

$$T_5(q^2, m_Q^2, \epsilon) =$$

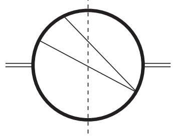


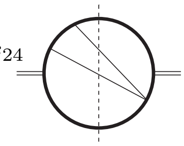
$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q)\, \frac{1}{s_{134}s_{234}}\,, \tag{3.12}$$

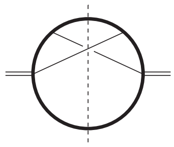
$$T_6(q^2, m_Q^2, \epsilon) =$$

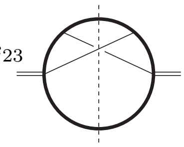


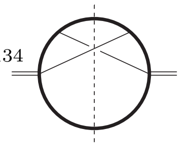
$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q)\, \frac{1}{s_{13}s_{234}}\,, \tag{3.13}$$

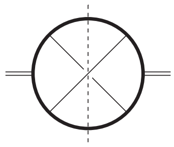
$$T_7(q^2, m_Q^2, \epsilon) =$$

$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q) \, \frac{1}{s_{13} s_{134} s_{234}} \,, \tag{3.14}$$

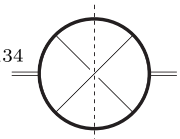
$$T_8(q^2, m_Q^2, \epsilon) = s_{24}$$

$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q) \, \frac{s_{24}}{s_{13} s_{134} s_{234}} \,, \tag{3.15}$$

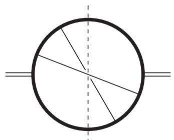
$$T_9(q^2, m_Q^2, \epsilon) =$$

$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q) \, \frac{1}{s_{13} s_{14}} \,, \tag{3.16}$$

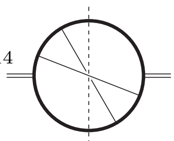
$$T_{10}(q^2, m_Q^2, \epsilon) = s_{23}$$

$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q) \, \frac{s_{23}}{s_{13} s_{14}} \,, \tag{3.17}$$

$$T_{11}(q^2, m_Q^2, \epsilon) = s_{134}$$

$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q) \, \frac{s_{134}}{s_{13} s_{14}} \,, \tag{3.18}$$

$$T_{12}(q^2, m_Q^2, \epsilon) =$$

$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q) \, \frac{1}{s_{13} s_{14} s_{23} s_{24}} \,, \tag{3.19}$$

$$T_{13}(q^2, m_Q^2, \epsilon) = s_{134}$$

$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q) \, \frac{s_{134}}{s_{13} s_{14} s_{23} s_{24}} \,, \tag{3.20}$$

$$T_{14}(q^2, m_Q^2, \epsilon) =$$

$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q) \, \frac{1}{s_{13} s_{23} s_{134} s_{234}} \,, \tag{3.21}$$

$$T_{15}(q^2, m_Q^2, \epsilon) = s_{14}$$

$$= \int d\Phi_4^{(D)}(p_1, p_2, p_3, p_4; q) \, \frac{s_{14}}{s_{13} s_{23} s_{134} s_{234}} \,. \tag{3.22}$$

In these diagrammatic representations bold (thin) lines refer to massive (massless) propagators. The invariants to the left of the cut-diagrams denote numerators of the integrand. The dashed line stands for the four-particle cut and the edges crossed by the dashed lines correspond to the cut-propagators $D_1, \ldots, D_4$. The integrated leading-color antenna $\mathcal{A}_4^0$ can be expressed in terms of the master integrals $T_1, \ldots, T_8$, whereas the computation of the subleadig-color antenna $\tilde{\mathcal{A}}_4^0$ involves the entire set of the above master integrals.

Analytic results for the integrals $T_1, \ldots, T_5$ were already given in [4] in terms of harmonic polylogarithms [101] of argument

$$y = \frac{1-\sqrt{1-\frac{4m_Q^2}{q^2}}}{1+\sqrt{1-\frac{4m_Q^2}{q^2}}}\,. \tag{3.23}$$

The expansion of these expressions near $D=4$ was given in [4] to $\mathcal{O}(\epsilon^2)$ in the case of T_1, T_2 and T_3 and to order $\mathcal{O}(\epsilon)$ in the case of T_4 and T_5. Here, the $\mathcal{O}(\epsilon^3)$-terms of T_1, T_2 and T_3 are also required. We computed them from the results provided in [4] by using the computer programs `HypExp` [102] and `HYPERDIRE` [103].

We compute the remaining ten master integrals $T_6,\dots,T_{15}$ with the same techniques that were used in the computation of T_4 and T_5 in [4]. Applying the method of differential equations [104–107], we derive inhomogeneous first order differential equations in the variables $q^2 = s_{1234}+2m_Q^2$ and y for each master integral. In each case, the inhomogeneous part of such a differential equation for a master integral T_i is expressed as linear combination of some of the other master integrals. We analyze these equations order by order in ϵ. If the resulting system of equations decouples and if the inhomogeneous part can be expressed by known results in terms of iterated integrals, we can derive the solution of the corresponding decoupled equation. This involves the y-integration of the inhomogeneous part. Following this procedure we make use of our results for $T_1,\dots,T_5$. These results allow us to begin with those equations where the inhomogeneous part only involves these integrals. Starting from there, we then proceed with the more complicated equations. Using partial integration, partial fraction decomposition and the package `HPL` [108] for the integrations, we obtain analytical results for all the above master integrals in terms of harmonic polylogarithms. The integration constants are fixed by using that the above master integrals vanish at threshold. Explicit expressions for the master integrals can be obtained from the authors upon request.

We then obtain as our main result the integrated antenna functions $\mathcal{A}_4^0$ and $\tilde{\mathcal{A}}_4^0$ in terms of harmonic polylogarithms:

$$\begin{aligned}
\mathcal{A}_4^0(\epsilon; s_{1234}, m_Q) &= \left(s_{1234}+2m_Q^2\right)^{-2\epsilon}\left\{\frac{1}{\epsilon^3}\left[\frac{1}{2}-\left(\frac{1}{2}-\frac{1}{2(1-y)}-\frac{1}{2(1+y)}\right)\mathrm{H}(0;y)\right]\right.\\
&+\frac{1}{\epsilon^2}\left[\left(\frac{19}{12}-\frac{5}{6(1-y)}+\frac{13}{6(1+y)}-\frac{1+2y}{2\left(1+4y+y^2\right)}\right)\mathrm{H}(0;y)+4\mathrm{H}(1;y)\right.\\
&+\left(1-\frac{1}{1-y}-\frac{1}{1+y}\right)\left(\mathrm{H}(1,0;y)-4\mathrm{H}(0,1;y)-3\mathrm{H}(-1,0;y)+\frac{7}{2}\zeta(2)\right)\\
&\left.+\frac{11}{3}+\frac{3y}{1+4y+y^2}\right]+\frac{1}{\epsilon}\left[\left(21-\frac{21}{1-y}-\frac{21}{1+y}\right)\zeta(2)\,\mathrm{H}(-1;y)\right.\\
&+\left(\frac{361}{18}+4\zeta(2)+\frac{\zeta(2)}{2(1-y)^2}-\frac{1201+288\zeta(2)}{72(1-y)}+\frac{\zeta(2)}{2(1+y)^2}\right.\\
&\left.+\frac{95-288\zeta(2)}{72(1+y)}-\frac{2+7y}{\left(1+4y+y^2\right)^2}+\frac{79+248y}{12\left(1+4y+y^2\right)}\right)\mathrm{H}(0;y)\\
&+\left(\frac{88}{3}-7\zeta(2)+\frac{7\zeta(2)}{1-y}+\frac{7\zeta(2)}{1+y}+\frac{24y}{1+4y+y^2}\right)\mathrm{H}(1;y)\\
&+\left(\frac{38}{3}-\frac{20}{3(1-y)}+\frac{52}{3(1+y)}-\frac{4(1+2y)}{1+4y+y^2}\right)\mathrm{H}(0,1;y)\\
&-\left(\frac{10}{3}+\frac{25}{6(1-y)}-\frac{83}{6(1+y)}+\frac{3(1+2y)}{1+4y+y^2}\right)\mathrm{H}(-1,0;y)\\
&-\left(\frac{7}{6}+\frac{5}{6(1-y)}+\frac{5}{6(1+y)}\right)\mathrm{H}(0,0;y)+32\,\mathrm{H}(1,1;y)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{53}{3}+\frac{5}{6(1-y)}-\frac{31}{6(1+y)}+\frac{1+2y}{1+4y+y^2}\right)\mathrm{H}(1,0;y)\\
&+\left(11-\frac{1}{(1-y)^2}-\frac{11}{1-y}-\frac{1}{(1+y)^2}-\frac{11}{1+y}\right)\mathrm{H}(0,-1,0;y)\\
&-\left(19-\frac{1}{(1-y)^2}-\frac{19}{1-y}-\frac{1}{(1+y)^2}-\frac{19}{1+y}\right)\mathrm{H}(0,1,0;y)\\
&+\left(3+\frac{1}{(1-y)^2}-\frac{3}{1-y}+\frac{1}{(1+y)^2}-\frac{3}{1+y}\right)\mathrm{H}(0,0,0;y)\\
&-\left(24-\frac{24}{1-y}-\frac{24}{1+y}\right)\mathrm{H}(-1,0,1;y)+\left(8-\frac{8}{1-y}-\frac{8}{1+y}\right)\mathrm{H}(1,0,1;y)\\
&-\left(32-\frac{32}{1-y}-\frac{32}{1+y}\right)\mathrm{H}(0,1,1;y)-\left(18-\frac{18}{1-y}-\frac{18}{1+y}\right)\mathrm{H}(-1,-1,0;y)\\
&+\left(6-\frac{6}{1-y}-\frac{6}{1+y}\right)\mathrm{H}(-1,1,0;y)+\left(6-\frac{6}{1-y}-\frac{6}{1+y}\right)\mathrm{H}(1,-1,0;y)\\
&-\left(2-\frac{2}{1-y}-\frac{2}{1+y}\right)\mathrm{H}(1,1,0;y)+\frac{1595}{72}+\frac{6(1+4y)}{\left(1+4y+y^2\right)^2}-\frac{2(3-13y)}{1+4y+y^2}\\
&-\left(\frac{85}{6}-\frac{65}{12(1-y)}+\frac{187}{12(1+y)}-\frac{7(1+2y)}{2\left(1+4y+y^2\right)}\right)\zeta(2)\\
&+\left(12-\frac{12}{1-y}-\frac{12}{1+y}+\frac{1}{2(1-y)^2}+\frac{1}{2(1+y)^2}\right)\zeta(3)\Bigg]+F(y)\Bigg\}, \qquad (3.24)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{A}}_4^0(\epsilon;s_{1234},m_Q) = {}&\left(s_{1234}+2m_Q^2\right)^{-2\epsilon}\Bigg\{\frac{1}{\epsilon^2}\Bigg[1-\left(2-\frac{2}{1-y}-\frac{2}{1+y}\right)\mathrm{H}(0;y)\\
&+\left(2-\frac{2}{1-y}-\frac{2}{1+y}+\frac{2}{(1-y)^2}+\frac{2}{(1+y)^2}\right)\mathrm{H}(0,0;y)\Bigg]\\
&+\frac{1}{\epsilon}\Bigg[8\,\mathrm{H}(1;y)-\Bigg(1-\frac{25}{6(1-y)}-\frac{31}{2(1+y)}+\frac{29(1+2y)}{3\left(1+4y+y^2\right)}\\
&+\left(12-\frac{12}{1-y}-\frac{12}{1+y}+\frac{12}{(1-y)^2}+\frac{12}{(1+y)^2}\right)\zeta(2)\Bigg)\mathrm{H}(0;y)\\
&+\left(4-\frac{4}{1-y}-\frac{4}{1+y}\right)\mathrm{H}(1,0;y)-\left(16-\frac{16}{1-y}-\frac{16}{1+y}\right)\mathrm{H}(0,1;y)\\
&+\left(4+\frac{14}{3(1-y)}-\frac{2}{1+y}-\frac{8}{(1-y)^2}+\frac{4+32y}{3\left(1+4y+y^2\right)}\right)\mathrm{H}(0,0;y)\\
&-\left(16-\frac{56}{3(1-y)}-\frac{16}{1+y}+\frac{8+16y}{3\left(1+4y+y^2\right)}\right)\mathrm{H}(-1,0;y)\\
&+\left(16-\frac{16}{1-y}-\frac{16}{1+y}+\frac{16}{(1-y)^2}+\frac{16}{(1+y)^2}\right)\mathrm{H}(0,0,1;y)\\
&+\left(16-\frac{16}{1-y}-\frac{16}{1+y}+\frac{16}{(1-y)^2}+\frac{16}{(1+y)^2}\right)\mathrm{H}(0,-1,0;y)\\
&-\left(4-\frac{4}{1-y}-\frac{4}{1+y}+\frac{4}{(1-y)^2}+\frac{4}{(1+y)^2}\right)\mathrm{H}(0,1,0;y)\\
&+\left(8-\frac{8}{1-y}-\frac{8}{1+y}+\frac{8}{(1-y)^2}+\frac{8}{(1+y)^2}\right)\mathrm{H}(1,0,0;y)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(10-\frac{32}{3(1-y)}-\frac{8}{1+y}+\frac{2}{(1-y)^2}+\frac{2}{(1+y)^2}-\frac{4+8y}{3\left(1+4y+y^2\right)}\right)\mathrm{H}(0,0,0;y)\\
&+\frac{25}{2}+\frac{22y}{1+4y+y^2}+\left(12+\frac{32}{3(-1+y)}-\frac{12}{1+y}-\frac{4+8y}{3\left(1+4y+y^2\right)}\right)\zeta(2)\\
&-\left(8-\frac{8}{1-y}-\frac{8}{1+y}+\frac{8}{(1-y)^2}+\frac{8}{(1+y)^2}\right)\zeta(3)\Bigg]+\tilde{F}(y)\Bigg\}\,. \qquad (3.25)
\end{aligned}
$$

The finite remainders $F(y)$ and $\tilde{F}(y)$ of $\mathcal{O}(\epsilon^0)$ are also given analytically in terms of harmonic polylogarithms up to and including weight four. These expressions are quite long and can be obtained from the authors upon request.

4 Summary and outlook

We addressed the treatment of infrared singularities that arise in the computation of observables, in particular distributions, for processes at NNLO QCD, where a heavy quark-pair is produced by an uncolored initial state. In the framework of the antenna subtraction method, appropriate subtraction terms are construced in terms of universal antenna functions. We constructed the NNLO antenna functions A_4^0, $\tilde{A}_4^0$ for the real-radiation subtraction term that is required for the squared matrix element of the final state that consists of a pair of heavy quarks and two gluons. We discussed the singular limits of this subtraction term and, as our main result, we computed the corresponding integrated antenna functions $\mathcal{A}_4^0$, $\tilde{\mathcal{A}}_4^0$ analytically. Our results include also analytical expressions for a set of master integrals, which we expect to be useful for other applications, too.

The antenna functions A_4^0, $\tilde{A}_4^0$ and their integrated versions $\mathcal{A}_4^0$, $\tilde{\mathcal{A}}_4^0$ provide a further step towards the calculation of $d\sigma_{\mathrm{NNLO}}$ for reactions of the type (1.1) within the antenna framework. All building blocks for the real-radiation subtraction terms and their integrated counter-parts are now available. The missing piece, the real-virtual subtraction term $d\sigma^T_{\mathrm{NNLO}}$ (cf. (1.2)), can be constructed from the interference of the tree-level and 1-loop matrix element for $\gamma^* \to Q\bar{Q}g$. Its integral over the 3-particle phase space can also be done in analytical form. We plan to present this result in a future publication [95].

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A Antenna functions A_4^0 and $\tilde{A}_4^0$

For the numerical computation of (2.13) in $D = 4$ dimensions, only the four-dimensional parts of the antenna functions A_4^0 and $\tilde{A}_4^0$ are required. However, the integrated antenna functions $\mathcal{A}_4^0$ and $\tilde{\mathcal{A}}_4^0$, which we computed in sec. 3, must be determined with A_4^0 and $\tilde{A}_4^0$ in D dimensions. The four-dimensional parts of A_4^0 and $\tilde{A}_4^0$ read:

$$
\begin{aligned}
A_4^0(1_Q,3_g,4_g,2_{\bar{Q}}) = \frac{1}{s_{1234}+4m_Q^2}\Big(&-\frac{2}{s_{13}}-\frac{2}{s_{24}}+\frac{3s_{12}}{s_{134}^2}+\frac{3s_{12}}{s_{234}^2}+\frac{5}{s_{134}}+\frac{5}{s_{234}}-\frac{6s_{12}}{s_{13}s_{134}}-\frac{6s_{12}}{s_{234}s_{24}}\\
&-\frac{6s_{12}}{s_{134}s_{24}}-\frac{6s_{12}}{s_{13}s_{234}}+\frac{3s_{23}}{s_{134}^2}+\frac{3s_{14}}{s_{234}^2}-\frac{3s_{23}}{s_{13}s_{134}}-\frac{3s_{14}}{s_{234}s_{24}}+\frac{3s_{13}}{s_{234}^2}+\frac{3s_{24}}{s_{134}^2}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{4s_{12}^2}{s_{13}s_{134}s_{234}} - \frac{4s_{12}^2}{s_{134}s_{234}s_{24}} - \frac{7s_{13}}{s_{134}s_{234}} - \frac{7s_{24}}{s_{134}s_{234}} - \frac{3s_{14}}{s_{13}s_{234}} - \frac{3s_{23}}{s_{134}s_{24}} + \frac{6s_{12}}{s_{13}s_{24}} \\
& + \frac{4s_{12}^2}{s_{13}s_{134}s_{24}} + \frac{4s_{12}^2}{s_{13}s_{234}s_{24}} + \frac{3s_{13}}{s_{134}s_{24}} + \frac{3s_{24}}{s_{13}s_{234}} + \frac{3s_{14}}{s_{13}s_{24}} + \frac{3s_{23}}{s_{13}s_{24}} + \frac{10s_{12}}{s_{134}s_{234}} \\
& + \frac{3s_{12}s_{23}}{s_{13}s_{134}s_{24}} + \frac{3s_{12}s_{14}}{s_{13}s_{234}s_{24}} + \frac{s_{23}^2}{s_{13}s_{134}s_{24}} + \frac{s_{14}^2}{s_{13}s_{234}s_{24}} - \frac{s_{13}}{s_{234}s_{24}} - \frac{s_{24}}{s_{13}s_{134}} \\
& + \frac{2s_{12}^3}{s_{13}s_{134}s_{234}s_{24}} + \frac{3s_{12}s_{13}}{s_{134}s_{234}s_{24}} + \frac{3s_{12}s_{24}}{s_{13}s_{134}s_{234}} - \frac{s_{13}^2}{s_{134}s_{234}s_{24}} - \frac{s_{24}^2}{s_{13}s_{134}s_{234}} + \frac{2s_{12}}{s_{34}^2} \\
& - \frac{2s_{13}}{s_{34}^2} - \frac{2s_{24}}{s_{34}^2} + \frac{2s_{14}}{s_{34}^2} + \frac{2s_{23}}{s_{34}^2} + \frac{2s_{12}s_{13}^2}{s_{134}^2s_{34}^2} + \frac{2s_{12}s_{24}^2}{s_{234}^2s_{34}^2} - \frac{4s_{12}s_{13}}{s_{134}s_{34}^2} - \frac{4s_{12}s_{24}}{s_{234}s_{34}^2} \\
& + \frac{2s_{13}^2}{s_{134}s_{34}^2} + \frac{2s_{24}^2}{s_{234}s_{34}^2} + \frac{2s_{13}^2s_{23}}{s_{134}^2s_{34}^2} + \frac{2s_{14}s_{24}^2}{s_{234}^2s_{34}^2} - \frac{4s_{13}s_{23}}{s_{134}s_{34}^2} - \frac{4s_{14}s_{24}}{s_{234}s_{34}^2} + \frac{4s_{12}s_{13}s_{24}}{s_{134}s_{234}s_{34}^2} \\
& + \frac{2s_{13}^2s_{24}}{s_{134}^2s_{34}^2} + \frac{2s_{13}s_{24}^2}{s_{234}^2s_{34}^2} + \frac{6s_{12}}{s_{13}s_{34}} + \frac{6s_{12}}{s_{24}s_{34}} + \frac{4s_{12}s_{13}}{s_{134}^2s_{34}} + \frac{4s_{12}s_{24}}{s_{234}^2s_{34}} - \frac{6}{s_{34}} \\
& - \frac{12s_{12}}{s_{134}s_{34}} - \frac{12s_{12}}{s_{234}s_{34}} + \frac{6s_{13}}{s_{134}s_{34}} + \frac{6s_{24}}{s_{234}s_{34}} + \frac{2s_{14}}{s_{13}s_{34}} + \frac{2s_{23}}{s_{24}s_{34}} + \frac{4s_{23}}{s_{13}s_{34}} + \frac{4s_{14}}{s_{24}s_{34}} \\
& + \frac{4s_{13}s_{23}}{s_{134}^2s_{34}} + \frac{4s_{14}s_{24}}{s_{234}^2s_{34}} - \frac{7s_{23}}{s_{134}s_{34}} - \frac{7s_{14}}{s_{234}s_{34}} + \frac{2s_{12}^2}{s_{13}s_{234}s_{34}} + \frac{2s_{12}^2}{s_{134}s_{24}s_{34}} \\
& - \frac{s_{13}}{s_{234}s_{34}} - \frac{s_{24}}{s_{134}s_{34}} + \frac{4s_{12}s_{13}}{s_{134}s_{234}s_{34}} + \frac{4s_{12}s_{24}}{s_{134}s_{234}s_{34}} - \frac{2s_{13}^2}{s_{134}s_{234}s_{34}} - \frac{2s_{24}^2}{s_{134}s_{234}s_{34}} \\
& + \frac{2s_{12}s_{14}}{s_{13}s_{234}s_{34}} + \frac{2s_{12}s_{23}}{s_{134}s_{24}s_{34}} + \frac{s_{14}^2}{s_{13}s_{234}s_{34}} + \frac{s_{23}^2}{s_{134}s_{24}s_{34}} + \frac{2s_{12}^2}{s_{13}s_{24}s_{34}} - \frac{8s_{12}^2}{s_{134}s_{234}s_{34}} \\
& + \frac{s_{13}}{s_{24}s_{34}} + \frac{s_{24}}{s_{13}s_{34}} - \frac{2s_{12}s_{13}}{s_{134}s_{24}s_{34}} - \frac{2s_{12}s_{24}}{s_{13}s_{234}s_{34}} + \frac{s_{13}^2}{s_{134}s_{24}s_{34}} + \frac{s_{24}^2}{s_{13}s_{234}s_{34}} \\
& + \frac{2s_{12}s_{14}}{s_{13}s_{24}s_{34}} + \frac{2s_{12}s_{23}}{s_{13}s_{24}s_{34}} + \frac{s_{14}^2}{s_{13}s_{24}s_{34}} + \frac{s_{23}^2}{s_{13}s_{24}s_{34}} + \frac{4s_{13}s_{24}}{s_{134}^2s_{34}} + \frac{4s_{13}s_{24}}{s_{234}^2s_{34}} \\
& - \frac{2s_{13}s_{23}}{s_{134}s_{24}s_{34}} - \frac{2s_{14}s_{24}}{s_{13}s_{234}s_{34}} + \frac{s_{12}s_{34}}{s_{13}s_{134}^2} + \frac{s_{12}s_{34}}{s_{234}^2s_{24}} + \frac{2s_{34}}{s_{13}s_{134}} + \frac{2s_{34}}{s_{234}s_{24}} - \frac{4s_{13}s_{24}}{s_{134}s_{234}s_{34}} \\
& + \frac{s_{23}s_{34}}{s_{13}s_{134}^2} + \frac{s_{14}s_{34}}{s_{234}^2s_{24}} + \frac{3s_{34}}{s_{13}s_{234}} + \frac{3s_{34}}{s_{134}s_{24}} + \frac{6s_{12}s_{34}}{s_{13}s_{134}s_{234}} + \frac{6s_{12}s_{34}}{s_{134}s_{234}s_{24}} - \frac{8s_{34}}{s_{134}s_{234}} \\
& - \frac{3s_{12}s_{34}}{s_{13}s_{134}s_{24}} - \frac{3s_{12}s_{34}}{s_{13}s_{234}s_{24}} - \frac{s_{23}s_{34}}{s_{13}s_{134}s_{24}} - \frac{s_{14}s_{34}}{s_{13}s_{234}s_{24}} + \frac{s_{13}s_{34}}{s_{234}^2s_{24}} + \frac{s_{24}s_{34}}{s_{13}s_{134}^2} \\
& - \frac{3s_{13}s_{34}}{s_{134}s_{234}s_{24}} - \frac{3s_{24}s_{34}}{s_{13}s_{134}s_{234}} - \frac{3s_{34}^2}{s_{13}s_{134}s_{234}} - \frac{3s_{34}^2}{s_{134}s_{234}s_{24}} + \frac{s_{34}^2}{s_{13}s_{134}s_{24}} + \frac{s_{34}^2}{s_{13}s_{234}s_{24}} \\
& + \frac{3s_{12}s_{34}^2}{s_{13}s_{134}s_{234}s_{24}} - \frac{s_{34}^3}{s_{13}s_{134}s_{234}s_{24}} - \frac{4s_{12}^2s_{34}}{s_{13}s_{134}s_{234}s_{24}} \\
& + m_Q^2 \bigg(\frac{12}{s_{134}^2} + \frac{12}{s_{234}^2} - \frac{4}{s_{13}s_{134}} - \frac{4}{s_{234}s_{24}} - \frac{20}{s_{13}s_{234}} - \frac{20}{s_{134}s_{24}} + \frac{28}{s_{134}s_{234}} \\
& - \frac{4(s_{12}+s_{14})^2}{s_{13}s_{234}s_{24}^2} - \frac{4(s_{12}+s_{23})^2}{s_{13}^2s_{134}s_{24}} - \frac{2(2s_{12}+s_{13}+2s_{14})}{s_{234}s_{24}^2} - \frac{2(2s_{12}+2s_{23}+s_{24})}{s_{13}^2s_{134}} \\
& + \frac{8(s_{12}-s_{34})}{s_{13}s_{134}s_{24}} + \frac{8(s_{12}-s_{34})}{s_{13}s_{234}s_{24}} + \frac{8s_{13}^2}{s_{134}^2s_{34}^2} + \frac{8s_{24}^2}{s_{234}^2s_{34}^2} - \frac{16s_{13}}{s_{134}s_{34}^2} - \frac{16s_{24}}{s_{234}s_{34}^2} \\
& + \frac{8}{s_{13}s_{34}} + \frac{8}{s_{24}s_{34}} - \frac{20}{s_{134}s_{34}} - \frac{20}{s_{234}s_{34}} + \frac{8s_{12}-12s_{24}}{s_{13}s_{234}s_{34}} + \frac{8s_{12}-12s_{13}}{s_{134}s_{24}s_{34}} + \frac{8}{s_{34}^2} \\
& - \frac{4s_{12}s_{13}}{s_{134}s_{234}s_{24}s_{34}} - \frac{4s_{12}s_{24}}{s_{13}s_{134}s_{234}s_{34}} - \frac{4s_{23}(s_{12}+s_{23})}{s_{13}s_{134}s_{24}s_{34}} - \frac{4s_{14}(s_{12}+s_{14})}{s_{13}s_{234}s_{24}s_{34}} + \frac{6}{s_{13}s_{24}} \\
& - \frac{4(s_{13}+s_{14})}{s_{234}s_{24}s_{34}} - \frac{4(s_{23}+s_{24})}{s_{13}s_{134}s_{34}} + \frac{4(s_{12}+4s_{13}+s_{23}+s_{24})}{s_{134}^2s_{34}} + \frac{4(s_{12}+s_{13}+s_{14}+4s_{24})}{s_{234}^2s_{34}} \\
& + \frac{4(s_{12}+s_{13}+s_{14}+s_{34})}{s_{234}^2s_{24}} + \frac{4(s_{12}+s_{23}+s_{24}+s_{34})}{s_{13}s_{134}^2} + \frac{8s_{12}-4(s_{14}+s_{23})}{s_{13}s_{24}s_{34}} + \frac{16s_{13}s_{24}}{s_{134}s_{234}s_{34}^2} \\
& + \frac{16(s_{13}+s_{24}-2s_{12})}{s_{134}s_{234}s_{34}} - \frac{2(2s_{12}+2s_{14}+s_{23}+s_{34})}{s_{13}s_{24}^2} - \frac{2(2s_{12}+s_{14}+2s_{23}+s_{34})}{s_{13}^2s_{24}} \\
& + \frac{-20s_{12}+8s_{13}+14s_{34}}{s_{134}s_{234}s_{24}} + \frac{-20s_{12}+8s_{24}+14s_{34}}{s_{13}s_{134}s_{234}} + \frac{8s_{12}^2-14s_{12}s_{34}+8s_{34}^2}{s_{13}s_{134}s_{234}s_{24}} \bigg) \\
& + m_Q^4 \bigg(\frac{16}{s_{13}s_{134}^2} + \frac{16}{s_{234}^2s_{24}} + \frac{4}{s_{13}^2s_{134}} + \frac{4}{s_{234}s_{24}^2} + \frac{4}{s_{13}s_{24}^2} + \frac{4}{s_{13}^2s_{24}} \\
& + \frac{16}{s_{134}^2s_{34}} + \frac{16}{s_{234}^2s_{34}} - \frac{16(s_{12}+s_{14})}{s_{13}s_{234}s_{24}^2} - \frac{16(s_{12}+s_{23})}{s_{13}^2s_{134}s_{24}} - \frac{16}{s_{13}s_{134}s_{234}} - \frac{16}{s_{134}s_{234}s_{24}} \\
& - \frac{16s_{23}}{s_{13}s_{134}s_{24}s_{34}} - \frac{16s_{13}}{s_{134}s_{234}s_{24}s_{34}} - \frac{16s_{14}}{s_{13}s_{234}s_{24}s_{34}} - \frac{16s_{24}}{s_{13}s_{134}s_{234}s_{34}} \\
& + \frac{4(s_{12}+s_{13}+s_{14})}{s_{234}^2s_{24}^2} + \frac{4(s_{12}+s_{23}+s_{24})}{s_{13}^2s_{134}^2} - \frac{12s_{34}}{s_{13}s_{134}s_{234}s_{24}} + \frac{4(s_{12}+s_{14}+s_{23}+s_{34})}{s_{13}^2s_{24}^2} \bigg) \\
& + m_Q^6 \left(\frac{16}{s_{13}^2s_{134}^2} + \frac{16}{s_{13}^2s_{24}^2} + \frac{16}{s_{24}^2s_{234}^2} \right) \bigg) + \mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\tilde{A}_4^0(1_Q,3_g,4_g,2_{\bar{Q}}) \;=\; & \frac{1}{s_{1342}+4m_Q^2}\bigg(-\frac{2}{s_{13}} - \frac{2}{s_{24}} - \frac{1}{s_{23}} - \frac{1}{s_{14}} + \frac{2}{s_{134}} + \frac{2}{s_{234}} + \frac{3s_{12}}{2s_{13}s_{14}} + \frac{3s_{12}}{2s_{23}s_{24}} \\
& + \frac{s_{12}}{2s_{13}s_{134}} + \frac{s_{12}}{2s_{234}s_{24}} + \frac{s_{12}}{2s_{23}s_{234}} + \frac{s_{12}}{2s_{134}s_{14}} + \frac{2(s_{12}+s_{13}+s_{14})}{s_{234}^2} + \frac{2(s_{12}+s_{23}+s_{24})}{s_{134}^2} \\
& + \frac{2s_{12}^2}{s_{13}s_{14}s_{23}} + \frac{2s_{12}^2}{s_{14}s_{23}s_{24}} + \frac{2s_{12}^2}{s_{13}s_{14}s_{24}} + \frac{2s_{12}^2}{s_{13}s_{23}s_{24}} + \frac{2s_{12}^3}{s_{13}s_{14}s_{23}s_{24}} + \frac{4(s_{12}-s_{34})}{s_{134}s_{234}} \\
& + \frac{2s_{12}^3}{s_{13}s_{134}s_{23}s_{234}} + \frac{2s_{12}^3}{s_{134}s_{14}s_{234}s_{24}} - \frac{2s_{12}+2s_{13}+s_{24}-s_{34}}{s_{14}s_{234}} - \frac{2s_{12}+s_{13}+2s_{24}-s_{34}}{s_{134}s_{23}} \\
& + \frac{7s_{12}s_{13}}{2s_{14}s_{23}s_{234}} + \frac{7s_{12}s_{24}}{2s_{134}s_{14}s_{23}} + \frac{7s_{12}s_{23}}{2s_{13}s_{134}s_{24}} + \frac{7s_{12}s_{14}}{2s_{13}s_{234}s_{24}} \\
& + \frac{3s_{12}s_{14}}{2s_{13}s_{23}s_{234}} + \frac{3s_{12}s_{23}}{2s_{134}s_{14}s_{24}} + \frac{3s_{12}s_{13}}{2s_{14}s_{234}s_{24}} + \frac{3s_{12}s_{24}}{2s_{13}s_{134}s_{23}} \\
& + \frac{s_{12}s_{13}}{2s_{14}s_{23}s_{24}} + \frac{s_{12}s_{14}}{2s_{13}s_{23}s_{24}} + \frac{s_{12}s_{23}}{2s_{13}s_{14}s_{24}} + \frac{s_{12}s_{24}}{2s_{13}s_{14}s_{23}} \\
& + \frac{3(2s_{12}+s_{14}+s_{23})}{s_{13}s_{24}} + \frac{3(2s_{12}+s_{13}+s_{24})}{s_{14}s_{23}} - \frac{s_{13}+s_{14}-s_{34}}{s_{23}s_{234}} - \frac{s_{23}+s_{24}-s_{34}}{s_{134}s_{14}} \\
& - \frac{s_{13}+s_{14}-2s_{34}}{s_{234}s_{24}} - \frac{s_{23}+s_{24}-2s_{34}}{s_{13}s_{134}} + \frac{3s_{12}s_{34}}{2s_{13}s_{134}^2} + \frac{3s_{12}s_{34}}{2s_{134}^2s_{14}} + \frac{3s_{12}s_{34}}{2s_{23}s_{234}^2} + \frac{3s_{12}s_{34}}{2s_{234}^2s_{24}} \\
& + \frac{(s_{13}+s_{14})s_{34}}{s_{23}s_{234}^2} + \frac{(s_{23}+s_{24})s_{34}}{s_{134}^2s_{14}} + \frac{(s_{13}+s_{14})s_{34}}{s_{234}^2s_{24}} + \frac{(s_{23}+s_{24})s_{34}}{s_{13}s_{134}^2} \\
& + \frac{s_{12}s_{23}s_{34}}{2s_{13}s_{134}s_{14}s_{24}} + \frac{s_{12}s_{14}s_{34}}{2s_{13}s_{23}s_{234}s_{24}} + \frac{s_{12}s_{13}s_{34}}{2s_{14}s_{23}s_{234}s_{24}} + \frac{s_{12}s_{24}s_{34}}{2s_{13}s_{134}s_{14}s_{23}} \\
& + \frac{(3s_{12}-s_{13}-2s_{34})s_{34}}{s_{134}s_{234}s_{24}} + \frac{(3s_{12}-s_{24}-2s_{34})s_{34}}{s_{13}s_{134}s_{234}} + \frac{(3s_{12}+s_{13}-s_{34})s_{34}}{s_{134}s_{23}s_{234}} + \frac{(3s_{12}+s_{24}-s_{34})s_{34}}{s_{134}s_{14}s_{234}} \\
& + \frac{s_{12}s_{34}^2}{2s_{13}s_{134}^2s_{14}} + \frac{s_{12}s_{34}^2}{2s_{23}s_{234}^2s_{24}} + \frac{-2s_{12}+s_{13}-2s_{23}+2s_{34}}{s_{134}s_{24}} + \frac{-2s_{12}-2s_{14}+s_{24}+2s_{34}}{s_{13}s_{234}} \\
& + \frac{s_{12}(2s_{12}+s_{34})}{s_{13}s_{134}s_{23}} + \frac{s_{12}(2s_{12}+s_{34})}{s_{14}s_{234}s_{24}} + \frac{s_{12}(2s_{12}+s_{34})}{s_{13}s_{23}s_{234}} + \frac{s_{12}(2s_{12}+s_{34})}{s_{134}s_{14}s_{24}} \\
& + \frac{4s_{12}^2+s_{13}^2-3s_{12}s_{34}-s_{13}s_{34}+s_{34}^2}{s_{14}s_{23}s_{234}} + \frac{4s_{12}^2+s_{24}^2-3s_{12}s_{34}-s_{24}s_{34}+s_{34}^2}{s_{134}s_{14}s_{23}} \\
& + \frac{4s_{12}^2+s_{14}^2-3s_{12}s_{34}-s_{14}s_{34}+s_{34}^2}{s_{13}s_{234}s_{24}} + \frac{4s_{12}^2+s_{23}^2-3s_{12}s_{34}-s_{23}s_{34}+s_{34}^2}{s_{13}s_{134}s_{24}} \\
& + \frac{2s_{12}^3-4s_{12}^2s_{34}+3s_{12}s_{34}^2-s_{34}^3}{s_{134}s_{14}s_{23}s_{234}} + \frac{2s_{12}^3-4s_{12}^2s_{34}+3s_{12}s_{34}^2-s_{34}^3}{s_{13}s_{134}s_{234}s_{24}} \\
& + m_Q^2\bigg(\frac{7}{s_{13}s_{134}} + \frac{7}{s_{234}s_{24}} + \frac{7}{s_{23}s_{234}} + \frac{7}{s_{134}s_{14}} - \frac{7}{s_{13}s_{14}} - \frac{7}{s_{23}s_{24}} \\
& - \frac{12}{s_{134}s_{23}} - \frac{12}{s_{14}s_{234}} - \frac{12}{s_{13}s_{234}} - \frac{12}{s_{134}s_{24}} + \frac{8}{s_{134}^2} + \frac{8}{s_{234}^2} + \frac{16}{s_{134}s_{234}} \\
& - \frac{3s_{12}}{s_{13}s_{134}s_{23}} - \frac{3s_{12}}{s_{14}s_{234}s_{24}} - \frac{3s_{12}}{s_{13}s_{23}s_{234}} - \frac{3s_{12}}{s_{134}s_{14}s_{24}} \\
& + \frac{8}{s_{14}s_{23}} + \frac{8}{s_{13}s_{24}} - \frac{s_{12}}{s_{13}s_{14}s_{23}} - \frac{s_{12}}{s_{14}s_{23}s_{24}} - \frac{s_{12}}{s_{13}s_{14}s_{24}} - \frac{s_{12}}{s_{13}s_{23}s_{24}} \\
& - \frac{4(s_{12}+s_{13})^2}{s_{14}s_{23}^2s_{234}} - \frac{4(s_{12}+s_{24})^2}{s_{134}s_{14}^2s_{23}} - \frac{4(s_{12}+s_{14})^2}{s_{13}s_{234}s_{24}^2} - \frac{4(s_{12}+s_{23})^2}{s_{13}^2s_{134}s_{24}} \\
& - \frac{2(2s_{12}+2s_{13}+s_{14})}{s_{23}^2s_{234}} - \frac{2(2s_{12}+s_{13}+2s_{14})}{s_{234}s_{24}^2} - \frac{2(2s_{12}+2s_{23}+s_{24})}{s_{13}^2s_{134}} - \frac{2(2s_{12}+s_{23}+2s_{24})}{s_{134}s_{14}^2} \\
& + \frac{5s_{12}-8s_{34}}{s_{134}s_{14}s_{23}} + \frac{5s_{12}-8s_{34}}{s_{14}s_{23}s_{234}} + \frac{5s_{12}-8s_{34}}{s_{13}s_{134}s_{24}} + \frac{5s_{12}-8s_{34}}{s_{13}s_{234}s_{24}} + \frac{s_{34}^2}{s_{13}s_{134}^2s_{14}} + \frac{s_{34}^2}{s_{23}s_{234}^2s_{24}} \\
& - \frac{4(s_{12}-2s_{34})}{s_{13}s_{134}s_{234}} - \frac{4(s_{12}-2s_{34})}{s_{134}s_{234}s_{24}} - \frac{4(s_{12}-2s_{34})}{s_{134}s_{14}s_{234}} - \frac{4(s_{12}-2s_{34})}{s_{134}s_{23}s_{234}} \\
& - \frac{2(4s_{12}+s_{13}+s_{14}+s_{34})}{s_{23}s_{234}s_{24}} - \frac{2(4s_{12}+s_{23}+s_{24}+s_{34})}{s_{13}s_{134}s_{14}} - \frac{2(2s_{12}+2s_{14}+s_{23}+s_{34})}{s_{13}s_{24}^2} \\
& - \frac{2(2s_{12}+2s_{13}+s_{24}+s_{34})}{s_{14}s_{23}^2} - \frac{2(2s_{12}+s_{14}+2s_{23}+s_{34})}{s_{13}^2s_{24}} - \frac{2(2s_{12}+s_{13}+2s_{24}+s_{34})}{s_{14}^2s_{23}} \\
& - \frac{s_{12}(4s_{12}+4s_{13}+3s_{34})}{s_{14}s_{23}s_{234}s_{24}} - \frac{s_{12}(4s_{12}+4s_{24}+3s_{34})}{s_{13}s_{134}s_{14}s_{23}} - \frac{s_{12}(4s_{12}+4s_{14}+3s_{34})}{s_{13}s_{23}s_{234}s_{24}} \\
& - \frac{s_{12}(4s_{12}+4s_{23}+3s_{34})}{s_{13}s_{134}s_{14}s_{24}} + \frac{4s_{12}+4s_{13}+4s_{14}+5s_{34}}{s_{23}s_{234}^2} + \frac{4s_{12}+4s_{13}+4s_{14}+5s_{34}}{s_{234}^2s_{24}} \\
& + \frac{4s_{12}+4s_{23}+4s_{24}+5s_{34}}{s_{13}s_{134}^2} + \frac{4s_{12}+4s_{23}+4s_{24}+5s_{34}}{s_{134}^2s_{14}} + \frac{2\left(4s_{12}^2-s_{12}s_{34}+s_{34}^2\right)}{s_{13}s_{14}s_{23}s_{24}} \\
& + \frac{2\left(4s_{12}^2+s_{12}s_{34}+s_{34}^2\right)}{s_{13}s_{134}s_{23}s_{234}} + \frac{2\left(4s_{12}^2+s_{12}s_{34}+s_{34}^2\right)}{s_{134}s_{14}s_{234}s_{24}} + \frac{8s_{12}^2-14s_{12}s_{34}+8s_{34}^2}{s_{134}s_{14}s_{23}s_{234}} \\
& + \frac{8s_{12}^2-14s_{12}s_{34}+8s_{34}^2}{s_{13}s_{134}s_{234}s_{24}}\bigg) \\
& + m_Q^4\bigg(\frac{24}{s_{13}s_{134}^2} + \frac{24}{s_{234}^2s_{24}} + \frac{24}{s_{134}^2s_{14}} + \frac{24}{s_{23}s_{234}^2} + \frac{4}{s_{13}^2s_{134}} + \frac{4}{s_{234}s_{24}^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4}{s_{134}s_{14}^2}+\frac{4}{s_{23}^2s_{234}}+\frac{4}{s_{14}s_{23}^2}+\frac{4}{s_{14}^2s_{23}}+\frac{4}{s_{13}s_{24}^2}+\frac{4}{s_{13}^2s_{24}}\\
&-\frac{16}{s_{13}s_{134}s_{234}}-\frac{16}{s_{134}s_{14}s_{234}}-\frac{16}{s_{134}s_{23}s_{234}}-\frac{16}{s_{134}s_{234}s_{24}}+\frac{12s_{34}}{s_{13}s_{14}s_{23}s_{24}}\\
&-\frac{16s_{12}}{s_{13}s_{134}s_{14}s_{23}}-\frac{16s_{12}}{s_{14}s_{23}s_{234}s_{24}}-\frac{16s_{12}}{s_{13}s_{134}s_{14}s_{24}}-\frac{16s_{12}}{s_{13}s_{23}s_{234}s_{24}}\\
&+\frac{4(s_{12}+s_{13}+s_{14})}{s_{23}^2s_{234}^2}+\frac{4(s_{12}+s_{23}+s_{24})}{s_{134}^2s_{14}^2}+\frac{4(s_{12}+s_{13}+s_{14})}{s_{234}^2s_{24}^2}+\frac{4(s_{12}+s_{23}+s_{24})}{s_{13}^2s_{134}^2}\\
&-\frac{16(s_{12}+s_{13})}{s_{14}s_{23}^2s_{234}}-\frac{16(s_{12}+s_{24})}{s_{134}s_{14}^2s_{23}}-\frac{16(s_{12}+s_{14})}{s_{13}s_{234}s_{24}^2}-\frac{16(s_{12}+s_{23})}{s_{13}^2s_{134}s_{24}}\\
&-\frac{12s_{34}}{s_{13}s_{134}s_{23}s_{234}}-\frac{12s_{34}}{s_{134}s_{14}s_{234}s_{24}}-\frac{12s_{34}}{s_{134}s_{14}s_{23}s_{234}}-\frac{12s_{34}}{s_{13}s_{134}s_{234}s_{24}}\\
&+\frac{4(s_{12}+s_{14}+s_{23}+s_{34})}{s_{13}^2s_{24}^2}+\frac{4(s_{12}+s_{13}+s_{24}+s_{34})}{s_{14}^2s_{23}^2}\\
&+\frac{8(s_{12}+s_{13}+s_{14}+s_{34})}{s_{23}s_{234}^2s_{24}}+\frac{8(s_{12}+s_{23}+s_{24}+s_{34})}{s_{13}s_{134}^2s_{14}}\Bigg)\\
&+m_Q^6\Bigg(\frac{16}{s_{13}^2s_{134}^2}+\frac{16}{s_{234}^2s_{24}^2}+\frac{16}{s_{134}^2s_{14}^2}+\frac{16}{s_{23}^2s_{234}^2}+\frac{16}{s_{14}^2s_{23}^2}+\frac{16}{s_{13}^2s_{24}^2}\\
&+\frac{32}{s_{13}s_{134}^2s_{14}}+\frac{32}{s_{23}s_{234}^2s_{24}}\Bigg)\Bigg)+\mathcal{O}(\epsilon)\,. \qquad \text{(A.2)}
\end{aligned}
$$

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