Dyson-Schwinger Eq's and Quantization of gauge theories (Summer '21)

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today: Anatomy of a gauge theory

Anatomy of a gauge theory

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Abstract

We exhibit the role of Hochschild cohomology in quantum field theory with particular emphasis on gauge theory and Dyson–Schwinger equations, the quantum equations of motion. These equations emerge from Hopf- and Lie algebra theory and free quantum field theory only. In the course of our analysis we exhibit an intimate relation between the Slavnov-Taylor identities for the couplings and the existence of Hopf sub-algebras defined on the sum of all graphs at a given loop order, surpassing the need to work on single diagrams.

0 Introduction

Over the last seven years many properties of the Hopf algebra structure of renormalization have been investigated, mostly on the mathematical side, with at least one notable exception [1] which showed how to solve a non-linear Dyson–Schwinger equation exactly, as opposed to a mere perturbation expansion.

In this paper, we explore the elementary relations between a perturbative expansion in quantum field theory, the corresponding Hochschild cohomology and the equations of motion in the context of a generic gauge theory. A major feature underlying our analysis is the emphasis on free quantum field theory and locality expressed through Hochschild cohomology. Together they specify the interacting theory. The novel phenomenon we report here is the interplay between the existence of a suitable sub Hopf algebra of perturbation theory and the existence of internal symmetries: the Slavnov–Taylor identities which ensure equality of renormalized couplings are equivalent to the existence of forest formulas indexed just by the loop number, the grading of the usual Hopf algebra of Feynman graphs.

0.1 The structure of Dyson Schwinger equations

If one ultimately wants to address non-perturbative quantum field theory one has to solve the corresponding Dyson–Schwinger equations exactly. This looks prohibitively difficult. But the recent progress with perturbative quantum field theory also points towards methods of solutions for Dyson–Schwinger equations [11, 1]. Let us review the current situation. Detailed references can be found in [13].

Feynman graphs possess a Lie algebra structure which dually governs the structure of renormalization, via the forest formula. The corresponding Hopf algebra furnishes one-cocycles which ensure locality of counterterms. These one-cocycles allow to generate the

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one-particle irreducible Green functions and provide the skeletons underlying the perturbative expansion. The one-cocycles correspond to primitive elements in the linear span of generators of the Hopf algebra and are graded by the loop number. One gets a set of embedded Hopf algebras from a hierarchy of Dyson Schwinger equations based on primitive graphs up to n loops. Underlying this structure is a universal Hopf algebra structure whose generators are given by the sum of all graphs with a given loop number. Let us call this sub Hopf algebra the grading algebra [11, 3]. It is the existence of this grading algebra which allows for the recursions which made the non-perturbative methods of [1] feasible.

The purpose of the present paper is to exhibit this structure for the example of a non-abelian gauge theory. In particular, we exhibit the grading algebra of this theory and show that its existence is equivalent to the existence of the Slavnov–Taylor identities for the couplings.

Dyson–Schwinger equations, illuminating recursive structure in Green functions, are recently on the forefront again in different areas of field theory [14, 15, 16]. We expect that our results are a starting point to understand these phenomena more systematically. In particular, the representations of the rescaling group and hence the role of dilatations in quantum field theory is intimately connected to the role of the Hochschild one-cocycles above, and motivates our quest for understanding their role in non-perturbative quantum field theory.

0.2 Interaction vertices from free quantum fields and locality

We start with a free quantum field theory with free propagators given by the usual requirements of free field theory for fermion and boson fields.

Before we discuss how locality determines the structure of the full Green functions we look at the tree level first, for motivation. We want to emphasize in our approach the known fact that the Feynman rules for tree-level graphs are determined by free quantum field theory (which gives Feynman rules for the edges) and locality (which implies Feynman rules for the vertices).

So let us first remind ourselves how the quest for locality determines the interaction vertices in the context of renormalizable theories, without reference to a classical Lagrangian at this stage.

We exhibit the argument for the example of the QED vertex. Thus we aim to infer the local interaction vertex $v_\mu(q,p)$ for the interaction of a photon, with four-momentum $p-q$, with an e^+e^- pair, with four-momenta p and q.

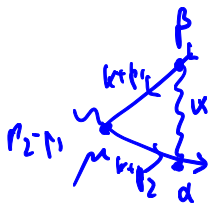
We will infer the Feynman rule for the vertex from the knowledge of the free photon, electron and positron propagator, and from the requirement that we should be able to renormalize by local counterterms.

We start with an Ansatz that at tree-level the sought-after vertex must be of the form

$$v_\mu(q,p) = \sum_{i=1}^{12} a_i f_\mu^{[i]} = a_1\gamma_\mu + a_2\frac{\not{q}q_\mu}{q^2} + \cdots, \tag{1}$$

in accordance with Lorentz covariance, spin representations and dimensions of the free propagators involved. Note that we can assume the coefficients a_i above to be constants: below we are only interested in the behaviour at large momentum transfer at each internal vertex. In that limit, each ratio in

$$\left\{u = \frac{q^2}{q^2+p^2+(p+q)^2}, v = \frac{p^2}{q^2+p^2+(p+q)^2}, w = \frac{(p+q)^2}{q^2+p^2+(p+q)^2}\right\} \tag{2}$$



$= \frac{1}{(2\pi)^4} \int_{-\Lambda}^{\Lambda} V_\beta(p_1, k+p_1) \; D_{\alpha\beta}(k) \; V_\alpha(k+p_2, p_2) \; V_\mu(k+p_2, k+p_1) \; d^4k$

$\sim \frac{1}{k^2}$

12 terms

12 terms

12 terms

$\rightarrow 12^3$ terms.

lim $\Rightarrow$ One loop $\sim \gamma_\mu$ survives.

$\Lambda \to \infty$

ie

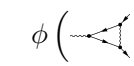
turns to zero or one, so that $a_i(u,v,w)$ turns to a constant indeed.

We can now determine $v_\mu(q,p)$ as follows.

First, we note that free quantum field theory and the requirement for renormalizability provide us with powercounting. We hence construct the 1PI graph with lowest non-vanishing loop order corresponding to our interaction:

This graph is log-divergent. If we demand to have a renormalizable theory, its coefficient of log-divergence must be proportional to the sought after tree-level vertex. This is indeed the well-known clue. By construction, the graph has no subdivergences and hence will evaluate to an expression of the form

$$\phi\left(\quad\right) \sim v_\mu(q,p)\ln\lambda + \text{finite terms}, \tag{3}$$



where we cut-off momentum integrals by λ. The important point is that the only part of this vertex functions which is allowed to diverge with λ is the part proportional to the sought-after tree-level vertex, if we are in a renormalizable theory.

Hence, we can write

$$v_\mu(q,p) \sim \lim_{\lambda\to+\infty} \frac{1}{\ln\lambda}\int_{-\lambda}^{+\lambda} d^4k \left[v_\sigma(q,k)\frac{1}{\not{k}+\not{q}}v_\mu(k+q,k+p)\frac{1}{\not{k}+\not{p}}v_\rho(k,p)D_{\sigma\rho}(k)\right]. \tag{4}$$

Inserting (1) we indeed find that

$$v_\mu(q,p) \sim \gamma_\mu, \tag{5}$$

as was to be expected.

Note that this states the obvious: in a local renormalizable quantum field theory the short-distance singularities can be absorbed by suitable modifications of parameters in the given Lagrangian. Vice-versa, if we make locality the starting point and hence require that short-distance singularities are self-similar to the tree-level terms, we obtain the interaction part of the Lagrangian from this requirement.

There is a cute albeit obvious message hidden here: if we settle on a given set of free fields and demand that they interact in a renormalizable manner, already the lowest loop order scattering processes fix the form of the interaction vertices from the knowledge of free quantum field theory and the requirement of locality, whilst the interaction part of the Lagrangian appears as a derived quantity. In this spirit we will continue to explore what we can learn about QFT, in particular about a gauge theory, starting from free covariances and so-determined interaction terms, and the accompanying one-particle irreducible graphs which go with them. Our guiding principle will still be locality, in its mathematical disguise as Hochschild cohomology.

In particular, we are now interested in formal sums of graphs corresponding to a specific instance of propagation or interaction: the sum of all 1PI graphs which constitute in perturbative QFT the 1PI Green function for that amplitude. The sum over all such 1PI Green functions furnishes then the effective action, by definition. We will study these 1PI Green functions for a non-abelian gauge theory.

We hence wish to discuss the formal sums

$$\Gamma^r = \sum g^{2|\Gamma|}\frac{\Gamma}{\text{sym}(\Gamma)}. \tag{6}$$

Here and in the following the superscript r specifies the Green function under consideration, it can be regarded as a collective label for the quantum numbers at the external legs of that function.

Already at this level interesting structure emerges. Our main tool will be the exploration of the Hochschild cohomology of the Hopf algebra structure which comes with such graphs. This Hochschild cohomology provides a mathematical precise formulation of locality [7, 4, 3], and will carry us far in the understanding of the structure of the theory.

0.3 Quantum equations of motion from Hochschild Cohomology

Such Hopf algebras have marvelous properties which they inherit from the universal object $\mathcal{H}_{\rm rt}$ for such algebras: the commutative Hopf algebra of non-planar rooted trees [2].

In that context, it is beneficial to study Dyson–Schwinger equations as formal constructions based on the Hochschild cohomology of such Hopf algebras. Before we justify the connection to Dyson–Schwinger equation through the study of a generic gauge theory below, let us first describe the universal set-up on rooted trees [3].

First, we settle on say a suitable Hopf algebra A which can be $\mathcal{H}_{\rm rt}$ or any suitable sub-Hopf algebra A.

Let A then be any such connected graded Hopf algebra which is free or free commutative as an algebra, and $(B_+^{d_n})_{n\in\mathbb{N}}$ a collection of Hochschild 1-cocycles on it. The Dyson-Schwinger equation is

$$X = \mathbb{I} + \sum_{n=1}^{\infty} \alpha^n w_n B_+^{d_n}(X^{n+1}) \tag{7}$$

in $A[[\alpha]]$. The parameter α plays the role of a coupling constant. The w_n are scalars in k. We decompose the solution

$$X = \sum_{n=0}^{\infty} \alpha^n c_n \tag{8}$$

with $c_n \in A$.

Lemma 1 *The Dyson-Schwinger equation (7) has a unique solution described by $c_0 = \mathbb{I}$ and*

$$c_n = \sum_{m=1}^{n} w_m B_+^{d_m} \left(\sum_{k_1+\ldots+k_{m+1}=n-m,\, k_i\geq 0} c_{k_1} \ldots c_{k_{m+1}} \right). \tag{9}$$

The c_n, coefficients in the n-th term of the perturbative expansion have a very nice property which we will rediscover in quantum field theory:

Theorem 2 *The c_n generate a Hopf subalgebra (henceforth called the grading algebra) of A :*

$$\Delta(c_n) = \sum_{k=0}^{n} P_{n,k} \otimes c_k \tag{10}$$

where the $P_{n,k}$ are homogeneous polynomials of degree $n-k$ in the c_l, $l \leq n$:

$$P_{n,k} = \sum_{l_1+\ldots+l_{k+1}=n-k} c_{l_1} \ldots c_{l_{k+1}}. \tag{11}$$

In particular, the $P_{n,k}$ are independent of the w_n and $B_+^{d_n}$.

In this paper we want to discuss this structure when we pass from the universal object $\mathcal{H}_{\rm rt}$ to the concrete Hopf algebra of say a generic gauge theory. We aim at insights into the non-perturbative structure of QFT and also prepare for new methods of computation in subsequent work. In particular we use the fact that the operadic proof of the above theorem given in [3] extends to our case once the proper insertion maps for graph insertions have been defined. To see the main point, we study first an elementary example.

0.4 A toy Ward identity

Consider the following system of DSEs based on say four Hochschild one-cocycles.

$$\begin{aligned} X_1 &= 1 + \alpha B_+^a(X_3 X_1) + B_+^b(X_2^2) && (12)\\ X_2 &= 1 + \alpha B_+^c(X_3^2) && (13)\\ X_3 &= 1 + \alpha B_+^d(X_3^2). && (14) \end{aligned}$$

One immediately confirms that imposing the symmetry

$$X_1 X_3 = X_2^2 \qquad (15)$$

in the Hopf algebra is equivalent to giving the sub Hopf algebra, $i \in \{1,2,3\}$,

$$X_i = 1 + \sum_n \alpha^n c_n^{(i)} \qquad (16)$$

$$\Delta(c_n^{(i)}) = \sum_{k=0}^{n} P_{n,k}^{(i)} \otimes c_{n-k}^{(i)}, \qquad (17)$$

where the polynomials $P_{n,k}^{(2)}$ and $P_{n,k}^{(3)}$ are easily determined as in (10). Similarly, upon using the symmetry (15) we find a new equation for X_1

$$X_1 = 1 + \alpha \left[B_+^a + B_+^b \right] (X_1 X_c), \qquad (18)$$

where $X_c := X_3 = X_2^2/X_1$, and all elements $c_n^{(1)}$ are symmetric in exchange of labels a and b. The existence of a sub Hopf algebra on the generators $c_n^{(i)}$ is now straightforwatd to establish as in [3].

We will now focus on the case of a generic non-abelian gauge theory, and exhibit how the Hochschild cohomology of the Hopf algebra of its perturbative expansion, the equations of motion and local gauge symmetry interfere. In particular, we will find a similar situation: the existence of a sub Hopf algebra is equivalent to the existence of relations exhibiting symmetries between Green functions.

To formulate our results we first introduce the pre-Lie algebra of Feynman graphs in this context in the next section.

We then introduce our result and discuss it with the help of a completely worked out two-loop example.

1 Graphs

In this section we first define graphs and the accompanying pre-Lie and Hopf algebras. The material is a straightforward application of previous results to a generic gauge theory.

1.1 The set of graphs

All graphs we consider are built from the following set R of edges and vertices

$$R = \Big\{ \quad , \quad , \quad , \quad , \quad , \quad , \quad \Big\}. \qquad (19)$$

We subdivide into edges and vertices,

$$R_{\mathrm{V}} = \Big\{ \quad , \quad , \quad , \quad \Big\}, \qquad (20)$$

Consider

$$X_1(\alpha) = \underline{\Pi} + \alpha B_+^{a}\left(\overbrace{X_1^2(\alpha)}^{X_1 Q_1}\right) + \alpha B_+^{b}\left(\overbrace{X_1(\alpha) X_2(\alpha)}^{X_1 Q_2}\right),$$

$$X_2(\alpha) = \underline{\Pi} + \alpha B_+^{c}\left(\overbrace{X_2^2(\alpha)}^{X_2 Q_3}\right).$$

$$\Rightarrow X_1(\alpha) = \underline{\Pi} + \alpha \overbrace{(\bullet_a + \bullet_b)}^{c_1^1} + \alpha^2 \overbrace{\left(2 \left[\begin{smallmatrix} a \\ a \end{smallmatrix}\right] + \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] + \left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right] + \left[\begin{smallmatrix} b \\ b \end{smallmatrix}\right] + \left[\begin{smallmatrix} b \\ c \end{smallmatrix}\right]\right)}^{c_1^2} + \dots$$

$$X_2(\alpha) = \underline{\Pi} + \alpha \overbrace{\bullet_c}^{c_2^1} + \alpha^2 \overbrace{\left(2 \left[\begin{smallmatrix} c \\ c \end{smallmatrix}\right]\right)}^{c_2^2} + \dots$$

$$Q_1 = X_1 \qquad Q_2 = X_2 \qquad Q_3 = X_2$$

$$Q_1 \overset{!}{=} Q_2 \overset{!}{=} Q_3 =: Q \qquad X_1(\alpha) = X_2(\alpha)$$

Define an ideal:

Order α contribution in I

$$X_1 - X_2 = I \quad (\Rightarrow) \quad \underbrace{i_1}_{i_2} = \overbrace{\bullet_a + \bullet_b}^{c_1^1} - \overbrace{\bullet_c}^{c_2^1} \in \underset{\cap}{I} \oplus \underset{\cap}{H} + \underset{\cap}{I} \otimes \underset{\cap}{H}$$

$$\tilde{\Delta}(c_2^1 - c_2^2) = 2\, i_1 \otimes \bullet_a + i_1 \otimes \bullet_b + \left.\begin{array}{l} + 2 \cdot c \otimes (\bullet_a + \bullet_b) \\ - 2 \cdot c \otimes \bullet_c \end{array}\right\}$$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} = 2 \cdot c \otimes \underbrace{(\bullet_a + \bullet_b - \bullet_c)}_{i_1} \in \underset{\cap}{H}$$

$$\tilde{\Delta}\, i_2 \subseteq I \otimes H + H \otimes I \qquad i_1 \in I$$

$\Rightarrow$ I is a coideal.

What are suitable
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and

$$R_{\rm E} = \{ \text{} \}. \tag{21}$$

Obviously, $R = R_{\rm E} \cup R_{\rm V}$. We have included edges for the free propagation of the local gauge field, corresponding ghost fields, and fermion fields as the only matter fields. We exclude the discussion of scalar matter fields coupled to gauge fields which deserve a separate discussion in future work. We thus include only the expected vertices in a generic local gauge theory: triple and quartic self-interactions of the gauge field, an interaction of the gauge field with its ghost field and minimal interaction between the gauge and matter fields - $R_{\rm V}$ is determined by $R_{\rm E}$ and locality in the spirit of the argument in the previous section.

We then define one-particle irreducible (1PI) graphs as usual: they remain connected after removal of any one of the internal edges. For such a 1PI graph Γ we have external legs $\Gamma^{[1]}_{\rm ext}$, internal edges $\Gamma^{[1]}_{\rm int}$ and vertices $\Gamma^{[0]}$.

For any 1PI graph Γ we let $\mathbf{res}(\Gamma)$ be the graph when all its internal edges shrink to a point

$$\mathbf{res}(\text{  }) = \text{  }. \tag{22}$$

We call $\mathbf{res}(\Gamma)$ the residue of Γ to emphasize that a graph Γ provides a counterterm $S^{\phi}_{R}(\Gamma)$ which contributes $S^{\phi}_{R}(\Gamma)\phi(\mathbf{res}(\Gamma))$ to the Lagrangian

$$\mathcal{L} = \sum_{r \in R_{\rm E}} 1/\phi(r) + \sum_{r \in R_{\rm V}} \phi(r), \tag{23}$$

where we note that for $r \in R_{\rm E}$ we have $1/\phi(r)$ as the inverse free propagator, as it should be. We extend the notion of a residue of a graph to a product $\Pi_i(\Gamma_i)$ of graphs:

$$\mathbf{res}\,(\Pi_i \Gamma_i) = \Pi_i \mathbf{res}(\Gamma_i). \tag{24}$$

Any element r in R has a superficial degree of divergence (sdd), $w(r)$, given as follows

$$\begin{aligned} w(\text{}) = 1, w(\text{  }) = 1, w(\text{  }) = 2, w(\text{}) = 0, \\ w(\text{  }) = 0, w(\text{  }) = -1, w(\text{  }) = 0. \end{aligned} \tag{25}$$

We introduce the loop number $|\Gamma|$,

$$|\Gamma| = \mathrm{rank}(H_1(\Gamma)), \tag{26}$$

where $H_1(\Gamma)$ is the first homology group of Γ, and

$$\left| \prod_i \Gamma_i \right| = \sum_i |\Gamma_i|. \tag{27}$$

For a 1PI graph Γ we let then its superficial degree of divergence $w(\Gamma)$ be

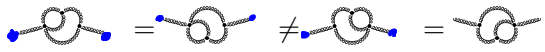
$$w(\Gamma) = -4|\Gamma| + \sum_{p \in \Gamma^{[1]}_{\rm int} \cup \Gamma^{[0]}} w(p). \tag{28}$$

Note that all 1PI graphs which have sdd ≤ 0 have residue in the above finite set R - we are dealing with a renormalizable theory. Here we are mainly interested in the structure of superficially divergent graphs, and hence do not discuss graphs and Green functions which are superficially convergent. For all $r \in R$, we let M_r be the set of graphs such that $\mathbf{res}(\Gamma) = r$.

1.2 Isotopy classes of graphs

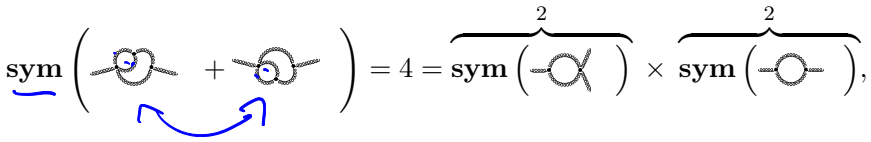
The symmetry factor of a graph Γ, $\mathbf{sym}(\Gamma)$, is defined as usual as the rank of the automorphism group of Γ.

We consider graphs up to the usual isotopy, for fixed external legs:



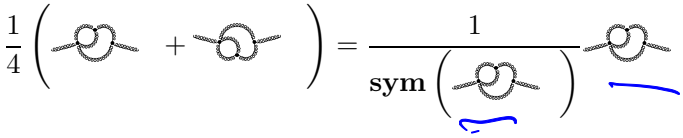
(29)

This plays a role in the study of the pre-Lie structure on graphs below. Indeed, we note that the symmetry factor of a sum of all graphs belonging to an isotopy class is the product of the symmetry factor of the subgraphs times the symmetry factor of the cograph obtained by shrinking the subgraphs. This ensures compatibility of symmetry factors under graph insertions as in the following example.



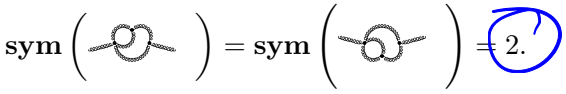
(30)

so that



(31)

with



(32)

1.3 Combinatorial Green functions

We can now speak of the set of superficially divergent 1PI graphs and consider graphs according to their residue and loop number $|\Gamma|$.

We define the formal sums

$$\Gamma^r = 1 + \sum_{\Gamma \in M_r} g^{2|\Gamma|} \frac{\Gamma}{\mathbf{sym}(\Gamma)}, \; r \in R_V, \tag{33}$$

$$\Gamma^r = 1 - \sum_{\Gamma \in M_r} g^{2|\Gamma|} \frac{\Gamma}{\mathbf{sym}(\Gamma)}, \; r \in R_E. \tag{34}$$

We let

$$c_k^r = \sum_{\substack{|\Gamma|=k \\ \mathrm{res}(\Gamma)=r}} \frac{\Gamma}{\mathbf{sym}(\Gamma)}, \; r \in R, \tag{35}$$

be the sum of graphs with given residue $r \in R$ and loop number k. We have $\Gamma^r = 1 + \sum_{k=1}^{\infty} g^{2k} c_k, \; r \in R_V$ and $\Gamma^r = 1 - \sum_{k=1}^{\infty} g^{2k} c_k, \; r \in R_E$.

We call these formal sums Γ^r combinatorial Green functions. Each term on their rhs maps under the Feynman rules to a contribution in the perturbative expansion of the Green functions of our gauge theory. Already the algebraic structure of these combinatorial Green functions is rather interesting. Analytic consequences will be briefly discussed at the end and explored in subsequent work.

Our task in this paper is to acquaint the reader with the structure of these sums, which is amazingly rich even at this elementary combinatorial level.

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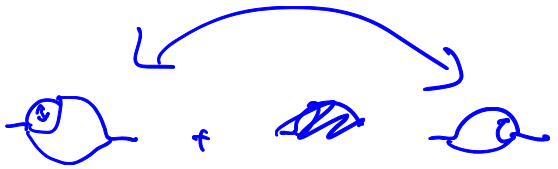
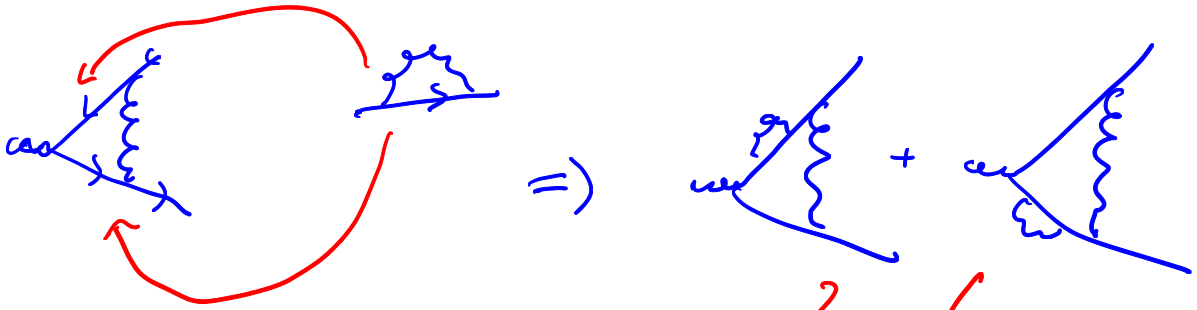




2 inscriptions ? ✓

1.4 Insertion places

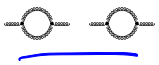
Each graph Γ has internal edges $\in \Gamma^{[1]}_{\text{int}}$ and vertices $\in \Gamma^{[0]}$. We call (subsets of) those edges and vertices places of Γ. Note that each place provides adjacent edges: for a vertex these are the edges attached to it, while each interior point of an edge defines two edges adjacent edges for it: the two pieces of the edge on both sides of that point. In such places, other graphs can be inserted, using a bijection between the external edges of those graphs and the adjacent edges provided by these very places.

The first thing we need to do is to count the number of insertion places with respect to the graphs to be inserted. Let $X = \prod_i \Gamma_i$ be a disjoint union of graphs to be inserted. Let us introduce variables a_r for all $r \in R$. To X we assign the monomial

$$x = \prod_i a_{\text{res}(\Gamma_i)}. \tag{36}$$

This monomial defines integers $n_{X,s}$ for all $s \in R$ by setting

$$x = \prod_{s \in R} a_s^{n_{X,s}}. \tag{37}$$

For example, if $X =$  , then

$$n_{X,\,\cdots} = 2,\ n_{X,s} = 0,\ s \neq \cdots\ . \tag{38}$$

Furthermore, to a graph Γ assign the function b_Γ, and integers $m_{\Gamma,s}$ by

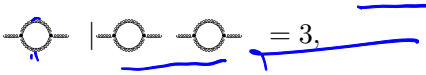
$$b_\Gamma := \prod_{v \in \Gamma^{[0]}} a_v \prod_{e \in \Gamma^{[1]}_{\text{int}}} \frac{1}{1-a_e} = \prod_{s \in R_V} a_s^{m_{\Gamma,s}} \prod_{e \in R_E} \frac{1}{[1-a_e]^{m_{\Gamma,e}}}. \tag{39}$$

Then, we define the number of insertion places for X in Γ, denoted $\Gamma|X$, by

$$\Gamma|X := \prod_{s \in R_V} \binom{m_{\Gamma,s}}{n_{X,s}} \prod_{e \in R_E} \frac{\partial^{n_{X,e}} \frac{1}{[1-a_e]^{m_{\Gamma,e}}}(0)}{n_{X,e}!}. \tag{40}$$

A few examples:

 (41)

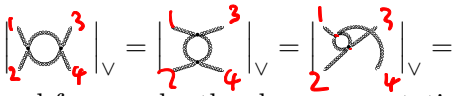
 (42)

as, for the last case,

$$\partial_{a_{\cdots}}^2 \frac{1}{[1-a_{\cdots}]^2}(0) = 6. \tag{43}$$

1.5 Permutation of external edges

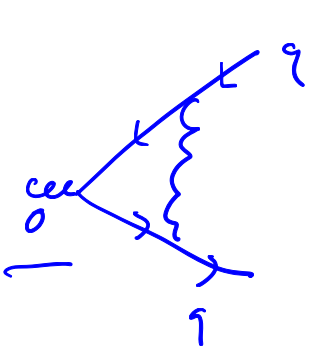
We call $|\Gamma|_\vee$ the number of distinct graphs Γ which are equal upon removal of the external edges. For example

 (44)

Such graphs can be obtained from each other by a permutation of external edges.

Furthermore for graphs $\gamma_1, \gamma_2, \gamma$ we let $\mathbf{top}(\gamma_1, \gamma_2, \gamma)_p$ be the number of bijections between $\gamma^{[1]}_{2,\text{ext}}$ and a chosen place $p = \mathbf{res}(\gamma_2) \in \gamma_1^{[0]} \cup \gamma^{[1]}_{1,\text{int}}$ such that γ is obtained. This counts the number of ways to glue γ_2 into a chosen place $\in \gamma_1$ to obtain γ. This has a

$|\cdot|_V$ counts the number of different graphs with fixed external edges compared to amputated n.



$\sim A(q^2)\gamma_\mu + B(q^2)\frac{\not{q} q_\mu}{q^2} = V_\mu(q^2)$