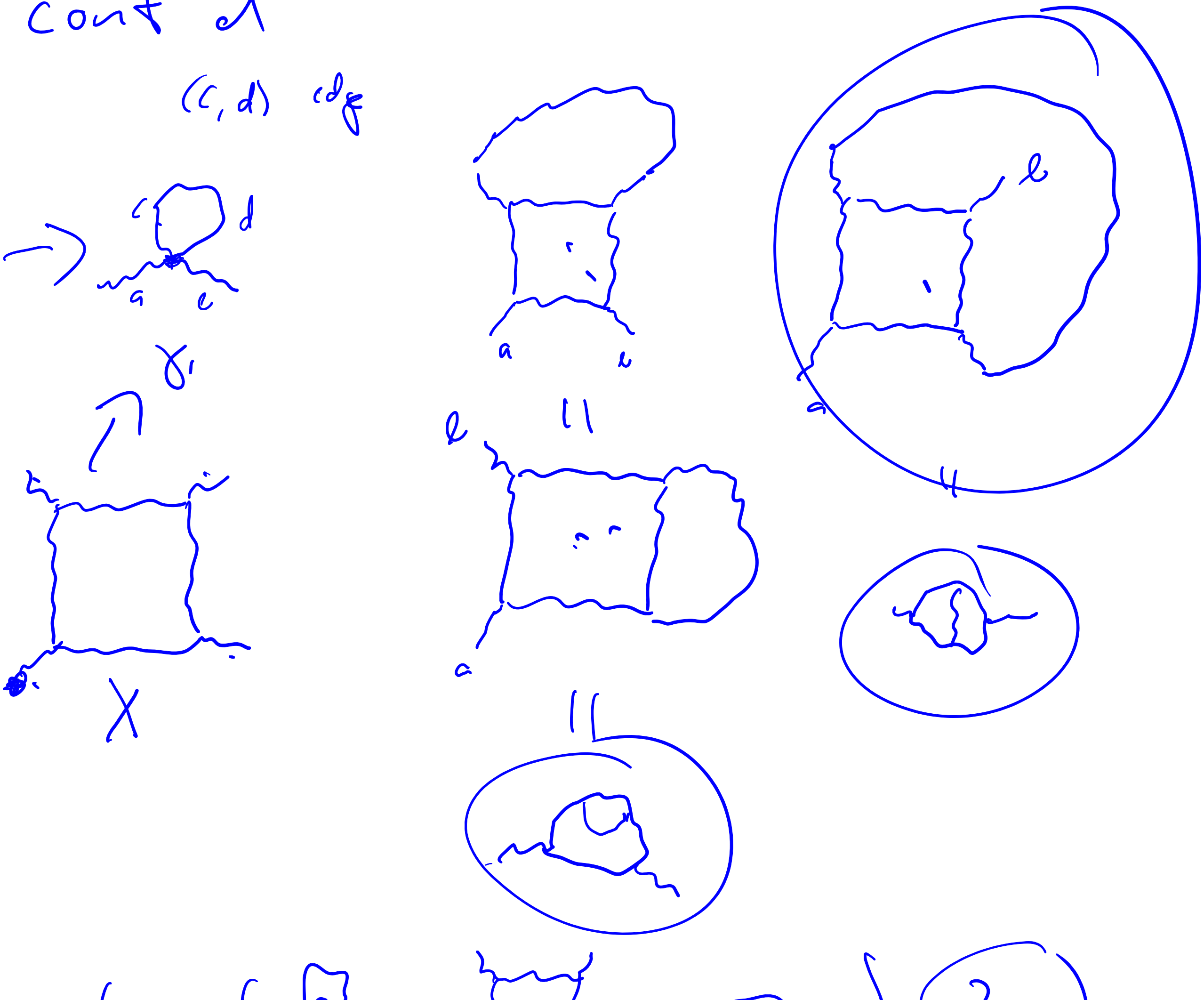
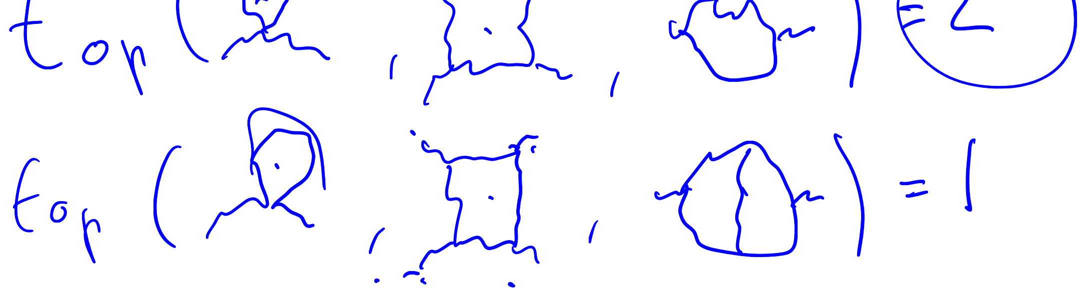
Dyson-Schwinger Eq's and Quantization of gauge theories (Summer '21)

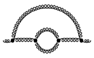
Kreimer, lect. May 11, 2021

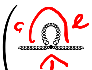
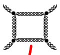
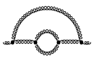
Anatomy of a gauge theory cont'd





straightforward generalization $\mathbf{top}(\gamma_1, X, \gamma)_p$ to products of graphs X, where now p is an appropriate set of chosen residues $\in \gamma_1^{[0]} \cup \gamma_{1,\text{int}}^{[1]}$.

For example if $\gamma =$  , $\gamma_1 =$  , and p the vertex  in γ_1, we have

$\mathbf{top}\Big($  ,  ,  $\Big)_p = 2.$ (45)

By definition, at a given place p,

$$\mathbf{top}(\gamma_1, X, \gamma)_p = \mathbf{top}(\gamma_1, \tilde{X}, \gamma)_p, \tag{46}$$

for all pairs $X, \tilde{X}$ related by a permutation of external legs.

1.6 Ramification in graphs

Above, we have counted the number of ways $\mathbf{top}(\gamma_1, X, \gamma)_p$ how to glue graph(s) X into a chosen single place $p \subset \gamma_1^{[0]} \cup \gamma_{1,\text{int}}^{[1]}$ so as to obtain a given graph γ. Furthermore, there might be various different places $p_i \in \gamma_1$ which provide a bijection for X such that the same γ is obtained.

Let $\mathbf{bij}(\gamma_1, \gamma_2, \gamma)$ be the number of bijections between $\gamma_{2,\text{ext}}^{[1]}$ and adjacent edges of places $p \sim \mathbf{res}(\gamma_2)$ in γ_1 such that γ is obtained.

A graph is described by vertices, edges and relations. For any bijection as above, we understand that the relations in γ_2, together with the relations in γ_1 which remain after removal of a chosen place, and the relations provided by the bijection combine to the relations describing the graph γ.

We let $\{\mathbf{bij}\}(\gamma_1, \gamma_2, \gamma)$ be the set of all such bijections which allow to form γ from γ_1 and γ_2 and write, for each $b \in \{\mathbf{bij}\}(\gamma_1, \gamma_2, \gamma)$,

$$\gamma = \gamma_1 b \gamma_2. \tag{47}$$

We declare $\mathbf{top}(\gamma_1, \gamma_2, \gamma)_p$ to be the number of such bijections restricted to a place p in γ_1.

We have a factorization into the bijections at a given place p, and the distinct bijections which lead to the same result at that place:

$$\mathbf{bij}(\gamma_1, \gamma_2, \gamma) = \mathbf{top}(\gamma_1, \gamma_2, \gamma)\mathbf{ram}(\gamma_1, \gamma_2, \gamma). \tag{48}$$

Here, $\mathbf{ram}(\gamma_1, \gamma_2, \gamma)$ counts the numbers of different places $p \in \gamma_1^{[0]} \cup \gamma_{1,\text{int}}^{[1]}$ which allow for bijections such that

$$\gamma_1 b \gamma_2 = \gamma. \tag{49}$$

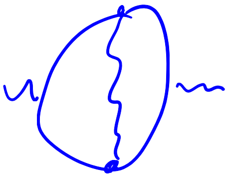
Note that for any two such places $p, \tilde{p}$ we find precisely $\mathbf{top}(\gamma_1, \gamma_2, \gamma)$ such bijections:

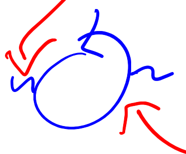
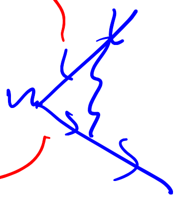
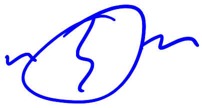
$$\mathbf{top}(\gamma_1, \gamma_2, \gamma) := \mathbf{top}(\gamma_1, \gamma_2, \gamma)_p = \mathbf{top}(\gamma_1, \gamma_2, \gamma)_{\tilde{p}}. \tag{50}$$

One immediately confirms that this number is indeed independent of the place p as we can pair off the bijections at p with the bijections at $\tilde{p}$ for any places p, p', so that the factorization (48) of $\mathbf{bij}(\gamma_1, \gamma_2, \gamma)$ is straightforward.

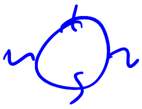
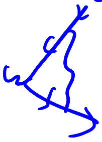
We call this integer $\mathbf{ram}(\gamma_1, \gamma_2; \gamma)$ the ramification index: it counts the degeneracy of inserting a graph at different places - if the ramification index is greater than one, the

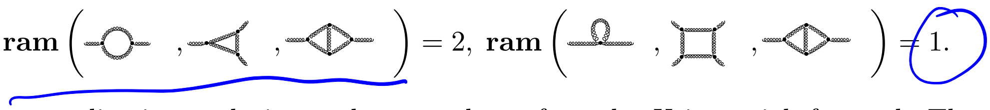
QED



 $\otimes$  $\to$ 

γ_1 γ_2 γ

rank ( ,  , ) = 2 same graph Γ can be obtained from inserting a graph γ_2 into a graph γ_1 at different places. For example



(51)

The generalization replacing γ_2 by a product of graphs X is straightforward. The motivation of the name comes from a comparison with the situation in the study of number fields which will be given in future work.

1.7 pre-Lie structure of graphs

The pre-Lie product we will use is a sum over all bijections and places of graph insertions. Hence it gives the same result for the insertion of any two graphs related by permutation of their external legs. One could formulate the Hopf and Lie structure hence on graphs with amputated external legs, but we will stick with the usual physicists convention and work with Feynman graphs which have external edges.

We define $n(\gamma_1, X, \Gamma)$ as the number of ways to shrink X to its residue (a set of one or more places) in Γ such that γ_1 remains.

We define a bilinear map

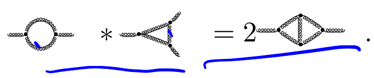
$$\Gamma_1 * \Gamma_2 := \sum_{\Gamma} \frac{n(\Gamma_1, \Gamma_2, \Gamma)}{|\Gamma_2|_\vee} \Gamma. \tag{52}$$

This is a finite sum, as on the rhs only graphs can contribute such that

$$|\Gamma| = |\Gamma_1| + |\Gamma_2|. \tag{53}$$

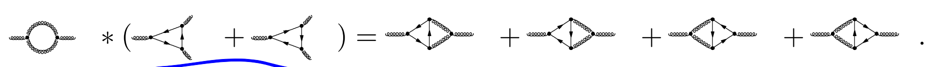
We divide by the number of permutations of external edges $|\Gamma_2|_\vee$ to eliminate the degeneracy in $n(\Gamma_1, \Gamma_2, \Gamma)$, a number which is insensitive to the orientation of edges of Γ_2. Note that for $\Gamma_a \sim_{\text{perm}} \Gamma_b$, we have $\Gamma_1 * \Gamma_a = \Gamma_1 * \Gamma_b$. Here, $\sim_{\text{perm}}$ indicates equivalence upon permutation of external edges.

For example,



(54)

while



(55)

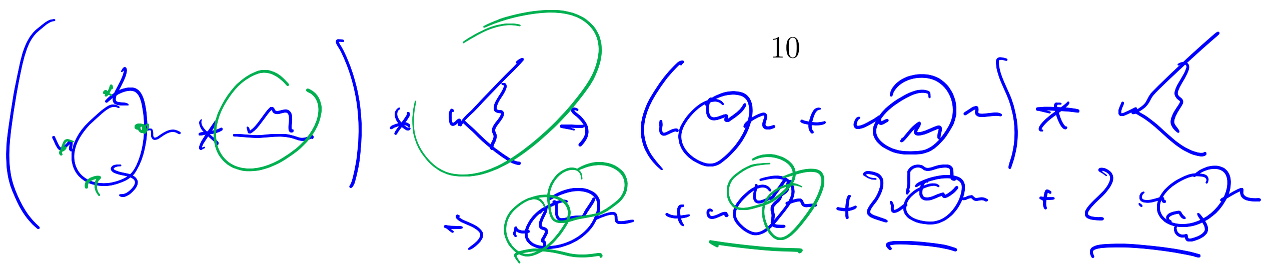
Proposition 3 *This map is pre-Lie:*

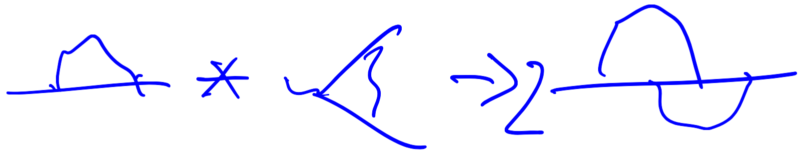
$$(\Gamma_1 * \Gamma_2) * \Gamma_3 - \Gamma_1 * (\Gamma_2 * \Gamma_3) = (\Gamma_1 * \Gamma_3) * \Gamma_2 - \Gamma_1 * (\Gamma_3 * \Gamma_2). \tag{56}$$

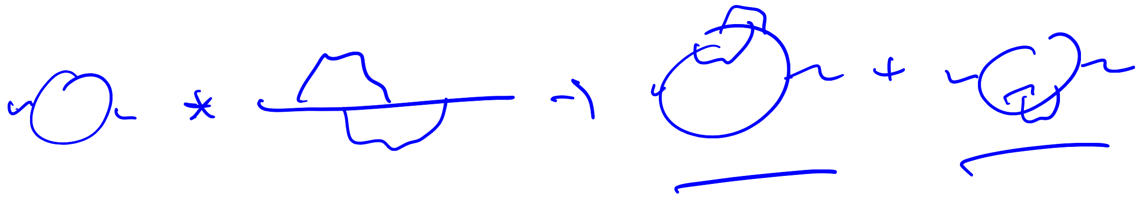
Note that the graphs on the rhs have all the same residue as Γ_1. The proof is analogous to the one in [6]. For a product of graphs X we define similarly

$$\Gamma_1 * X = \sum_{\Gamma} \frac{n(\Gamma_1, X, \Gamma)}{|X|_\vee} \Gamma. \tag{57}$$

This is a straightforward generalization of this map, but certainly not a pre-Lie product in that generality.







1.8 The Lie algebra of graphs

Jacobi: $\sum_{\text{cycl } 1,2,3} [[\Gamma_1,\Gamma_2],\Gamma_3]=0$

We let $\mathcal{L}$ be the corresponding Lie algebra, obtained by antisymmetrizing the pre-Lie product:

$$[\Gamma_1,\Gamma_2] = \Gamma_2 * \Gamma_1 - \Gamma_1 * \Gamma_2. \tag{58}$$

The bracket $[,]$ fulfils a Jacobi identity and we hence get a graded Lie algebra. Note that the terms generated by the Lie bracket involve graphs of different residue.

1.9 The Hopf algebra of graphs

(by Milnor Moore)

Let $\mathcal{H}$ be the corresponding Hopf algebra. Let us quickly describe how it is found. To $\mathcal{L}$, we assign its universal enveloping algebra

$$U(\mathcal{L}) = \bigoplus_{j=0}^{\infty} T(\mathcal{L})^{(j)}, \tag{59}$$

$\mathcal{L}^{(0)} = \mathbb{Q}$

$\mathcal{L}^{(1)} = \mathcal{L}$

$\mathcal{L}^{(2)} = \mathcal{L}\otimes\mathcal{L} \xrightarrow{[\cdot,\cdot]} \mathcal{L}$

where $T(\mathcal{L})^{(j)} = \mathcal{L}^{\otimes j}$ is the j-fold tensorproduct of $\mathcal{L}$. In $U(\mathcal{L})$ we identify

$$\Gamma_1 \otimes \Gamma_2 - \Gamma_2 \otimes \Gamma_1 = [\Gamma_1,\Gamma_2], \tag{60}$$

as usual. We let

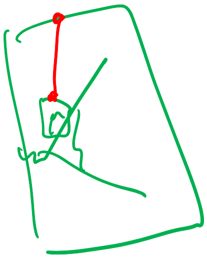
$$\langle \Gamma_1,\Gamma_2\rangle = \begin{cases} 0, & \Gamma_1 \neq \Gamma_2 \\ 1, & \Gamma_1 = \Gamma_2 \end{cases}. \tag{61}$$

Here we understand that entries on the lhs of $\langle\cdot,\cdot\rangle$ belong to the Lie algebra, entries on the rhs to the Hopf algebra.

We compute the coproduct from this pairing requiring

$$\langle [\Gamma_1,\Gamma_2], \Delta(\Gamma)\rangle = \langle \Gamma_1\otimes\Gamma_2 - \Gamma_2\otimes\Gamma_1, \Delta(\Gamma)\rangle, \tag{62}$$

and find the usual composition into subgraphs and cographs



$$\Delta(\Gamma) = \Gamma\otimes 1 + 1\otimes\Gamma + \sum_{\gamma} \gamma\otimes\Gamma/\gamma. \tag{63}$$

The antipode $S:\mathcal{H}\to\mathcal{H}$ is

$$S(\Gamma) = -\Gamma - \sum_{\gamma} S(\gamma)\Gamma/\gamma. \tag{64}$$

The counit $\bar{e}$ annihilates the augmentation ideal as usual [6, 7].

Furthermore, we define $|\Gamma|_{\text{aug}}$ to be the augmentation degree, defined via the projection P into the augmentation ideal. Furthermore, for future use we let $c^r_{k,s}$ be the sum of graphs with given residue r, loop number k and augmentation degree s. $\mathcal{H}_{\text{lin}}$ is the span of the linear generators of $\mathcal{H}$.

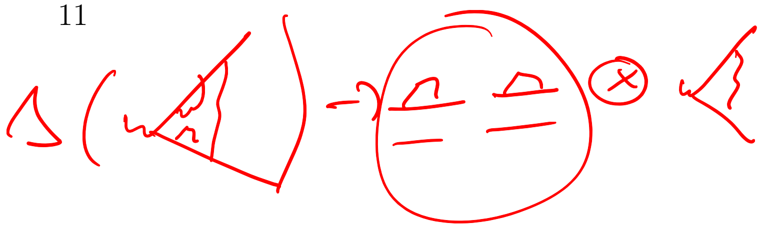
With the Hopf algebra comes its character group, and with it three distinguished objects: the Feynman rules ϕ, the $\bar{R}$ operation

$$\bar{\phi} = m(S_R^{\phi}\otimes\phi P)\Delta, \tag{65}$$

(which is a character with regard to the double structure of Rota–Baxter algebras [5]) and counterterm $-R\bar{\phi}$.

Note that the forgetfulness upon insertion wrt the external legs (46) forces us to work with a symmetric renormalization scheme

$$S_R^{\phi}(\Gamma_1) = S_R^{\phi}(\Gamma_2), \tag{66}$$



We are in business.

for consistency, for all pairs Γ_1, Γ_2 which agree by a permutation of external edges.

Indeed, $\forall\gamma$ and $\Gamma_1 \sim_{\mathrm{perm}} \Gamma_2$,

$$\begin{aligned} 0 &= \gamma * \Gamma_1 - \gamma * \Gamma_2 && (67)\\ &= \bar{\phi}(\gamma * \Gamma_1 - \gamma * \Gamma_2) && (68)\\ &= S_R^{\phi}(\Gamma_1 - \Gamma_2)\phi(\gamma), && (69)\end{aligned}$$

upon using (65) and as $\phi(\gamma * \Gamma_1 - \gamma * \Gamma_2) = 0$, and similarly for all cographs of Γ_1 and Γ_2.

1.10 External structures

In later work it will be useful to disentangle Green functions wrt to their form-factor decomposition. This can be easily achieved by the appropriate use of external structures [6].

γ_n

We hence extend graphs γ to pairs (γ, σ) where σ labels the formfactor and with a forgetful rule

$$\sum_{\sigma_2} (\Gamma_1, \sigma_1) * (\Gamma_2, \sigma_2) := \sum_{\Gamma} \frac{n(\Gamma_1, \Gamma_2, \Gamma)}{|\Gamma_2|_\vee} (\Gamma, \sigma_1). \quad (70)$$

This allows to separate the form-factor decompositions as partitions of unity $1 = \sum_{\sigma_2}$ in computationally convenient ways for which we will use in future work. If we do not sum over σ_2 we can extend our notation to marked graphs as in [6].

2 The theorems

In this section we state the main result. It concerns the role played by the maps $B_+^{k;r}$ to be defined here: they provide the equations of motion, ensure locality, and lead us to the Slavnov–Taylor identities for the couplings.

We start by defining a map

loop number
residue

$$B_+^{k;r} = \sum_{\substack{|\gamma|=k\\ |\gamma|_{\mathrm{aug}}=1\\ \mathrm{res}(\gamma)=r}} \frac{1}{\mathbf{sym}(\gamma)} B_+^{\gamma}, \quad (71)$$

primitive

where B_+^{γ} is a normalized generalization of the pre-Lie insertion into γ defined by requiring $B_+^{k;r}$ to be Hochschild closed. To achieve this, we need to count the maximal forests of a graph Γ. It is the number of ways to shrink subdivergences to points such that the resulting cograph is primitive. To define it more formally, we use Sweedler's notation to write $\Delta(X) = \sum X' \otimes X''$. If $X = \prod \Gamma_i$ is a Hopf algebra element with Γ_i graphs we write

$$\Delta(X) = c(X', X'')\widehat{X'} \otimes \widehat{X''}, \quad (72)$$

which defines scalars $c(X', X'')$. Here, $\widehat{X'}$ and $\widehat{X''}$ are graphs and the section coefficients of the Hopf algebra $c(X', X'')$ are explicitely spelled out.

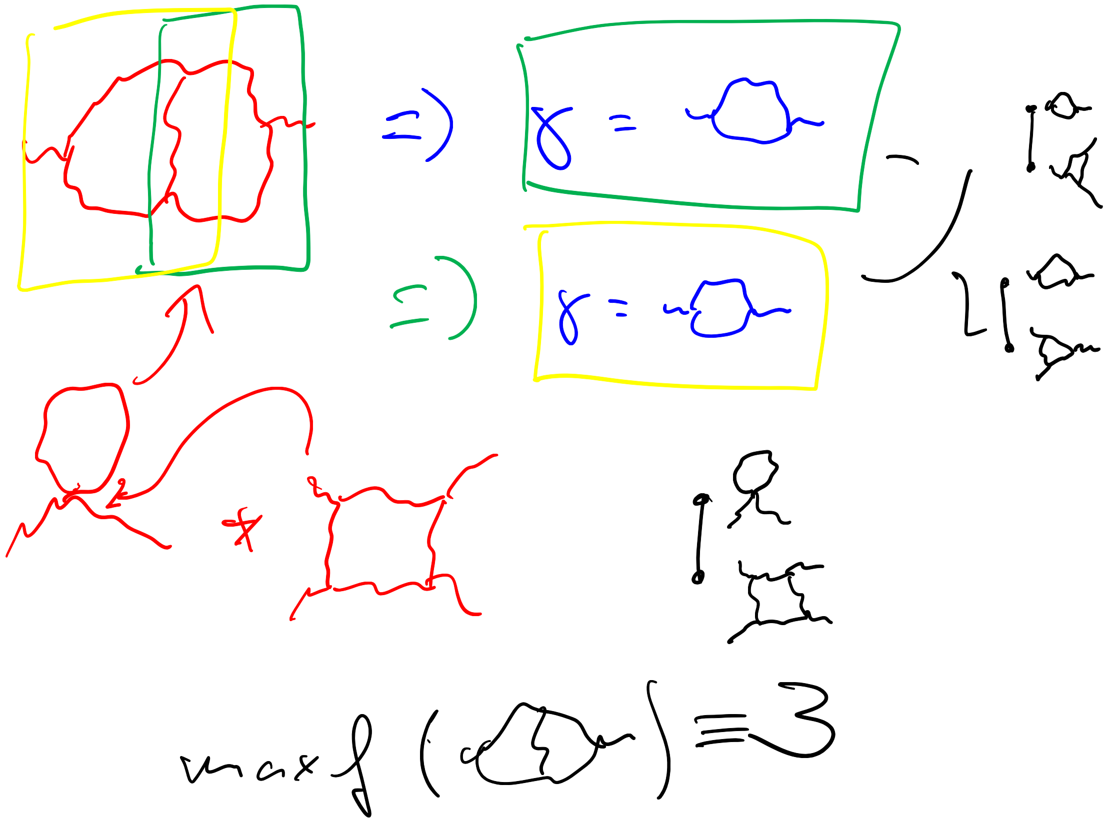
We now set

$$\mathrm{maxf}(\Gamma) = \sum_{|\gamma|_{\mathrm{aug}}=1} \sum c(\Gamma', \Gamma'')\langle\gamma, \Gamma''\rangle. \quad (73)$$

Note that this counts precisely the ways of shrinking subgraphs to points such that a primitive cograph remains, as it should, using the pairing between the Lie and Hopf algebra and summing over all Lie algebra generators indexed by primitive graphs γ.

What is maxf ?

# of ways I can shrink sub graphs to ~~d~~ obtain a primitive γ.


γ =
γ =
max f ( ) ≡ 3

The same number can by definition be obtained from the section coefficients of the pre-Lie algebra:

$$\mathrm{maxf}(\Gamma) = \sum_{|\gamma|_{\mathrm{aug}}=1} \sum_{|X|=|\Gamma|-|\gamma|} n(\gamma, X, \Gamma), \tag{74}$$

as each maximal forest has a primitive cograph γ and some subdivergences X of loop number $|\Gamma| - |\gamma|$.

We have defined the pre-Lie product so that

$$\gamma * X = \sum_{\Gamma \in \mathcal{H}_{\mathrm{lin}}} \frac{n(\gamma, X, \Gamma)}{|X|_\vee} \Gamma. \tag{75}$$

Now we define

$$B_+^\gamma(X) = \sum_{\Gamma \in \mathcal{H}_{\mathrm{lin}}} \frac{\mathbf{bij}(\gamma, X, \Gamma)}{|X|_\vee} \frac{1}{\mathrm{maxf}(\Gamma)} \frac{1}{[\gamma|X]} \Gamma, \tag{76}$$

for all X in the augmentation ideal. Furthermore, $B_+^\gamma(\mathbb{I}) = \gamma$.

Taking into account the fact that the pre-Lie product is a sum over all labelled composition of graphs and the fact that we carefully divide out the number of possibilities to generate the same graph, we can apply the corresponding results for rooted trees [3]. One concludes in analogy to Theorem 2:

Theorem 4 *(the Hochschild theorem)*

r vertex

$$\Gamma^r \equiv 1 + \sum_{\Gamma \in M_r} \frac{\Gamma}{\mathbf{sym}(\Gamma)} = 1 + \sum_{k=1}^{\infty} g^k \sum_{\substack{|\gamma|=k \\ |\gamma|_{\mathrm{aug}}=1 \\ \mathrm{res}(\gamma)=r}} \frac{1}{\mathbf{sym}(\gamma)} B_+^\gamma(X_\gamma), \tag{77}$$

r edge

where $X_\gamma = \prod_{v \in \gamma^{[0]}} \Gamma^v \prod_{e \in \gamma^{[1]}_{\mathrm{int}}} 1/\Gamma^e$.

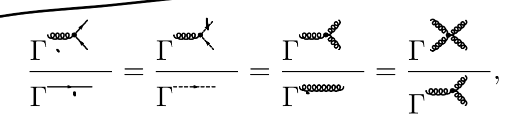
For the next theorem, we have to define the Slavnov–Taylor identities for the coupling. Consider

$$X_{k,r} = \Gamma^r X^k_{\mathrm{coupl}}. \tag{78}$$

We set

$$X_{\mathrm{coupl}} = 1 + \sum_{k=1}^{\infty} g^{2k} c_k^{\mathrm{coupl}}, \tag{79}$$

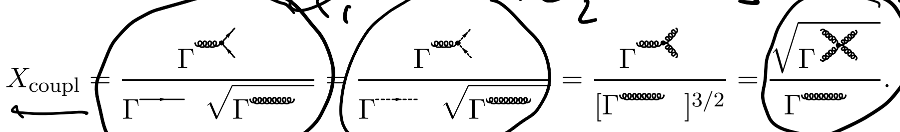
which determines the c_k^{coupl} as polynomials in the c_j^r from the definition of X_{coupl} below. The Slavnov–Taylor identities for the couplings can be written as



(80) $\Leftrightarrow$ coideals

which results in identities in every order in g^2 and leads to define indeed a single coupling X_{coupl} which can be defined in four equal ways, each one corresponding to an interaction monomial in the Lagrangian:

$=Q_1$ $=Q_2$ $=Q_3$ $=Q_4$



(81)

Note that we read this identities as describing the kernel of the counterterm: they hold under the evaluation of the indicated series of graphs by the corresponding character S_R^ϕ.

$$B_+^{\sim}\left(\left(\Gamma^{\sim}\right)^2 \frac{1}{\left(\Gamma^{\sim}\right)^2}\right)$$

$$X^{\sim} = \mathbb{I} - g^2\left(B_+^{\sim}\left(\ \right)\right) - g^2 B_+^{\sim}\left(\Gamma^{\sim}\frac{\Gamma^{\sim}}{\Gamma^{\sim 2}}\right) \quad 2 < Q_3^2 \quad Q_4^2$$

$$\frac{\left(\Gamma^{\sim}\right)^2}{\left(\Gamma^{\sim}\right)^2} = \frac{\left(\Gamma^{\sim}\right)^2}{\left(\Gamma^{\sim}\right)^3}\,\Gamma^{\sim}$$

$$\frac{\Gamma^{\sim}}{\Gamma^{\sim}} = \frac{\Gamma^{\sim}}{\Gamma^{\sim}} \to \Gamma^{\sim} = \Gamma^3\frac{1}{\Gamma^{\sim}}\Gamma^3$$

The following theorem follows on imposing these identities as relations between Hopf algebra elements order by order in g^2. On the other hand, if we assume the following theorem, we would derive the existence of the Slavnov–Taylor identities from the requirement of the existence of the grading sub Hopf algebra furnished by the elements c_k^r.

Theorem 5 *(the gauge theory theorem)*

$$i)\ \Gamma^r \equiv 1 + \sum_{\Gamma\in M_r} \frac{\Gamma}{\mathbf{sym}(\Gamma)} = 1 + \sum_{k=1}^{\infty} g^k B_+^{k;r}(X_{k,r}) \tag{82}$$

$$ii)\ \Delta(B_+^{k;r}(X_{k,r})) = \mathbb{I}\otimes B_+^{k;r}(X_{k,r}) + (\mathrm{id}\otimes B_+^{k;r})\Delta(X_{k,r}). \tag{83}$$

$$iii)\ \Delta(c_k^r) = \sum_{j=0}^{k} \mathrm{Pol}_j^r(c)\otimes c_{k-j}^r, \tag{84}$$

where $\mathrm{Pol}_j^r(c)$ *is a polynomial in the variables* c_m^r *of degree* j*, determined as the order* j *coefficient in the Taylor expansion of* $\Gamma^r[X_{\mathrm{coupl}}]^j$.

3 Two-loop Example

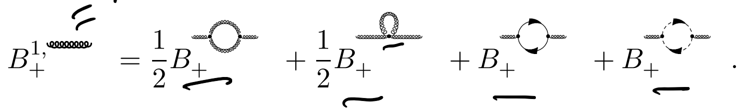
3.1 One-loop graphs

This section provides an instructive example. We consider our non-abelian gauge theory and first list its one-loop graphs, which provide by definition maps from $\mathcal{H}\to\mathcal{H}_{\mathrm{lin}}$.

The maps $B_+^{k,r}$, we claim, furnish the Hochschild one-cocyles and provide the Dyson–Schwinger equations, in accordance with the Hochschild and gauge theorems. We study this for the self-energy of the gauge boson to two-loops. We want in particular exhibit the fact that each such two-loop graph is a sum of terms each lying in the image of such a map and want to understand the role of Hochschild cohomology.

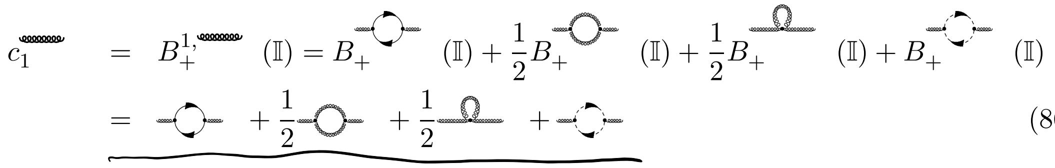
We have for example

$B_+^{k,r}$



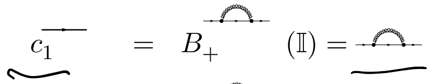
(85)

To find the one-loop graphs we simply have to apply these maps to the unit of the Hopf algebra of graphs, which is trivial:

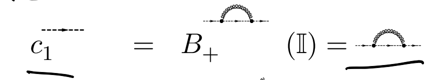


(86)

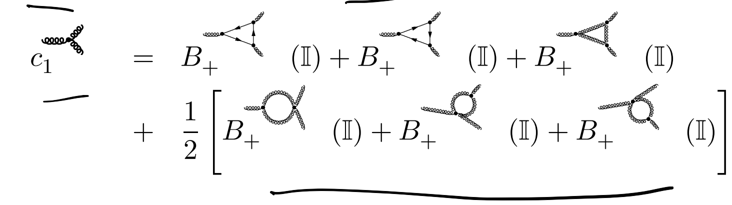
and similarly

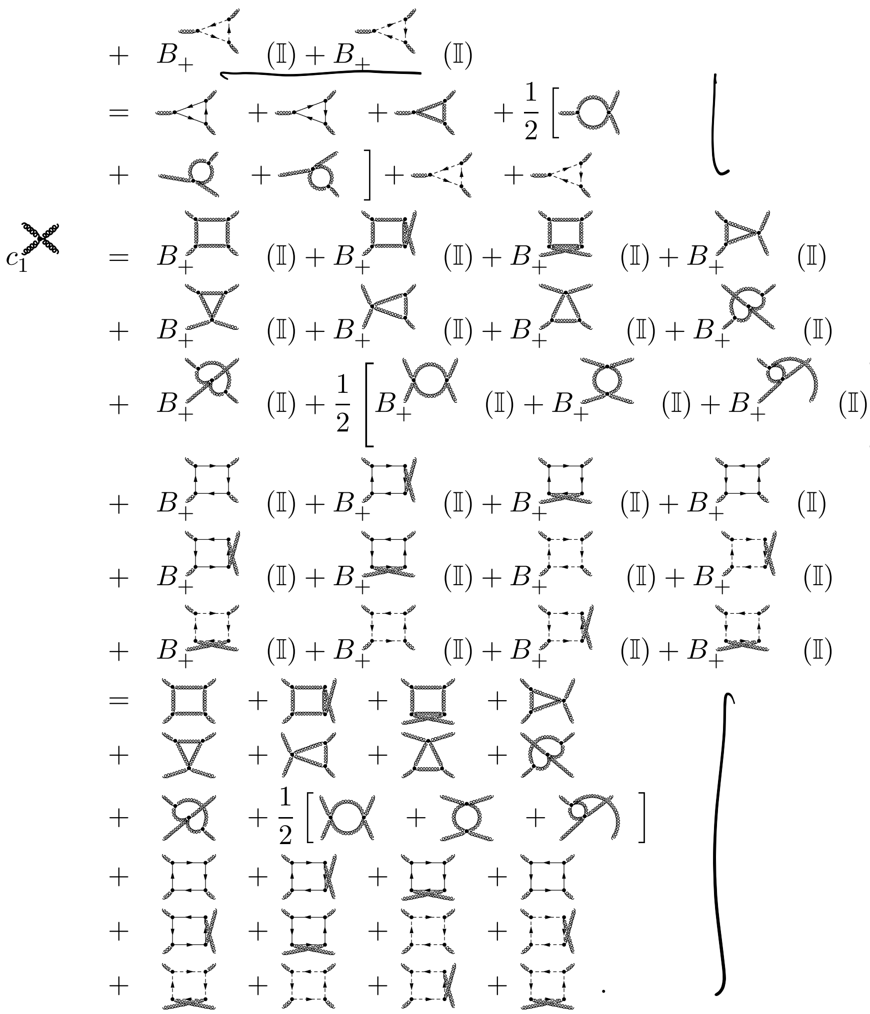


(87)



(88)



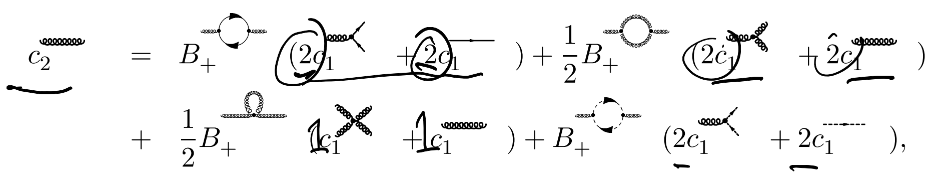


(89)

(90)

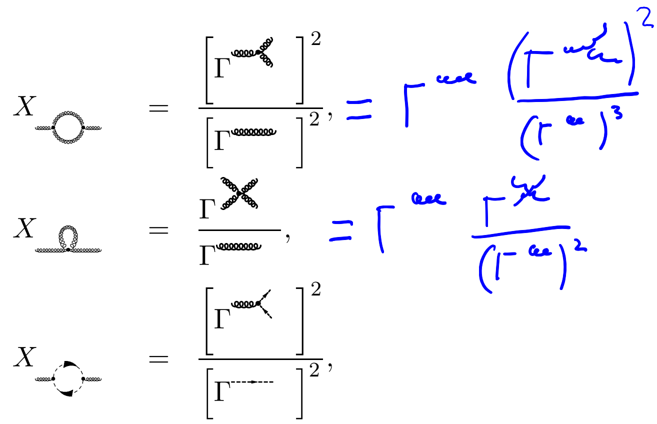
3.2 Two-loop graphs

We now want to calculate



(91)

upon expanding



(92)

(93)

(94)

$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{\psi}(\partial\!\!\!/ - m\mathbb{I} + g A\!\!\!/)\psi$

$\sim g \quad \sim g^2$

g_1^R $\qquad (g_2^R)^2$ $\qquad g_3^R$ $\qquad g_4^R$ $\qquad g_i^R = g \frac{Z}{\sqrt{Z}\, Z}$

$3 = i :$

$$X \;=\; \frac{\left[\Gamma\;\right]^2}{\left[\Gamma\;\right]^2}, \tag{95}$$

to order g^2.

Let us do this step by step. Adding up the contributions, we should find precisely the two-loop contributions to the gauge-boson self-energy, and the coproduct

$$\Delta(c_2) = c_2 \otimes \mathbb{I} + \mathbb{I} \otimes c_2 + [2c_1^{\mathrm{coupl}} - c_1] \otimes c_1 . \tag{96}$$

The minus sign appears on the rhs due to our conventions in (34).

Insertions in $\frac{1}{2}B_+$

Below, we will give coefficients like $\left(\frac{1}{2}|1|2|\frac{1}{2}|\frac{1}{2}|1|1\right)$ in the next equation, where the first entry is the symmetry factor of the superscript γ of B_+^γ, the second entry the symmetry factor of the graphs in the argument X, the third entry the integer weight of that argument, the fourth entry the number of insertion places, the fifth entry the number of maximal forests of the graphs Γ on the rhs, the sixth entry is $\mathbf{top}(\gamma, X, \Gamma)$ and the seventh entry is $\mathbf{ram}(\gamma, X, \Gamma)$. $\Gamma_\gamma[X]$

We start

$$\begin{aligned}\frac{1}{2}B_+\left(2 \;+ 2\;\right) &= \left(\frac{1}{2}|1|2|\frac{1}{2}|\frac{1}{2}|1|1\right)\\ &\times\left(\; + \; + \; + \;\right)\\ &= \frac{1}{4}\left(\; + \; + \; + \;\right)\\ &= \frac{1}{2}\left(\; + \;\right),\end{aligned} \tag{97}$$

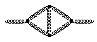
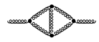
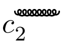
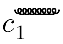
where indeed the symmetry factor for is $1/2$, the symmetry factor for the graphs appearing as argument is 1, and they appear with weight two. We have two three-gluon vertices in , and hence two insertion places. Each graph on the right has two maximal forests, and for each graph the inserted subgraph can be reduced in a unique way to obtain , so the ramification index is one, and the topological weight is unity as well.

Similarly for ghosts

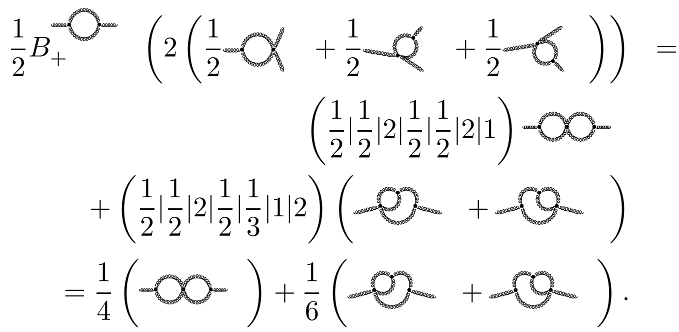
$$\begin{aligned}\frac{1}{2}B_+\left(2 \;+ 2\;\right) &= \left(\frac{1}{2}|1|2|\frac{1}{2}|\frac{1}{2}|1|1\right)\\ &\times\left(\; + \; + \; + \;\right)\\ &= \frac{1}{4}\left(\; + \; + \; + \;\right)\\ &= \frac{1}{2}\left(\; + \;\right).\end{aligned} \tag{98}$$

Next,

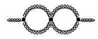
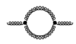
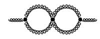
$$\begin{aligned}\frac{1}{2}B_+\left(2\;\right) &= \left(\frac{1}{2}|1|2|\frac{1}{2}|\frac{1}{3}|1|2\right)\\ &= \frac{1}{3} \;.\end{aligned} \tag{99}$$

This is more interesting. There are three maximal forests in  , a one-loop three-point vertex-graph to the left and to the right, and also the four gluon propagators which form the square give a one-loop log-divergent four-point graph. Also, we can form  by inserting the argument into either vertex of  , and hence the ramification index is two. The topological index is 1. Note that the total weight 1/3 of the graph is not its contribution to c_2 . We expect the same graph to be generated from inserting into  , as we will confirm soon. This is generally true: only in the Hochschild closed sum over insertions in all components of c_1 will we see the correct weights emerging.

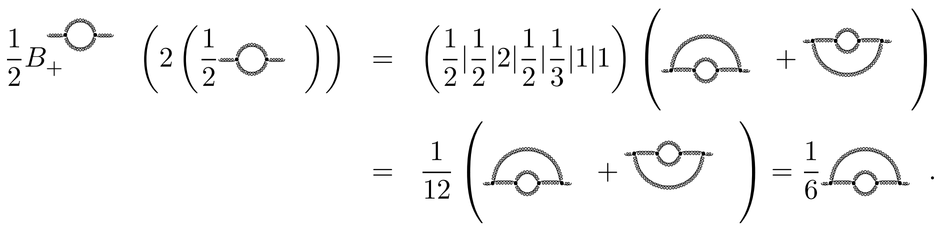
We continue.



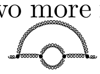
(100)

Here, the first graph  on the rhs has two maximal forests, a ramification index of two as the graph can be obtained by insertion in either vertex of  , and  is generated by one bijection, while the other two graphs have three maximal forests, 1 as a ramification index and 2 as a topological index: there are two different bijections leading to each of them.

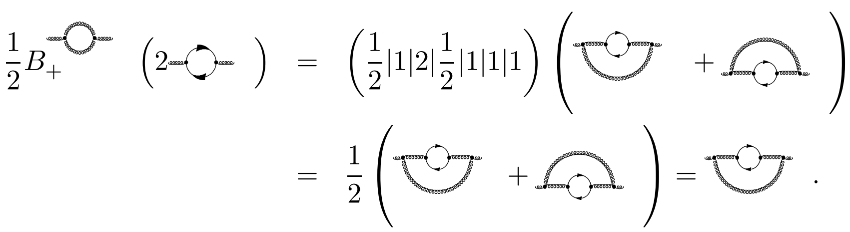
So far we inserted 3-point one-loop vertex corrections. Now we insert propagator corrections.



(101)

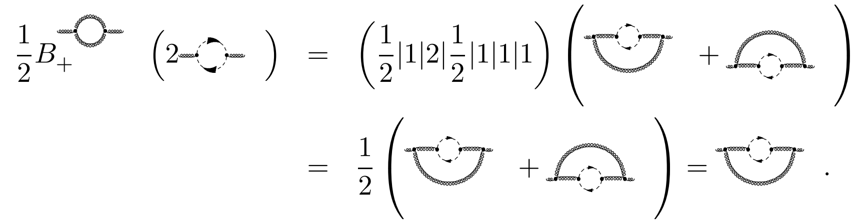
Note that the graph allows for three maximal forests: apart from the inserted one-loop self-energy graph it has two more maximal forests, corresponding to the two one-loop four-point vertex-subgraphs in  , obtained by opening an internal edge in the subgraph.

Next we insert a fermion loop:



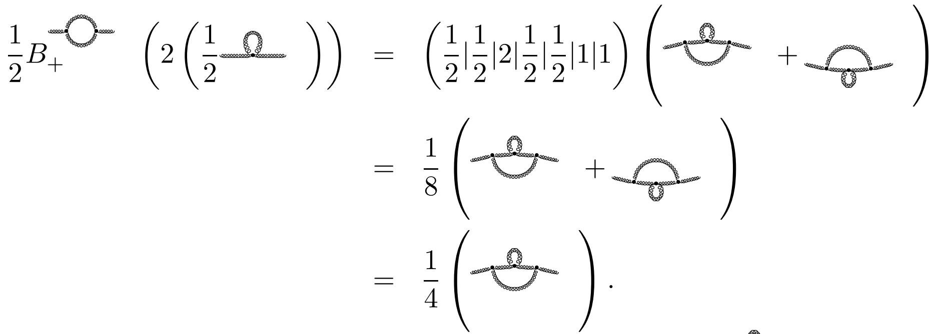
(102)

Indeed, no ramification, just one maximal forest and a single bijection. Similarly, the ghost loop



(103)

Finally,

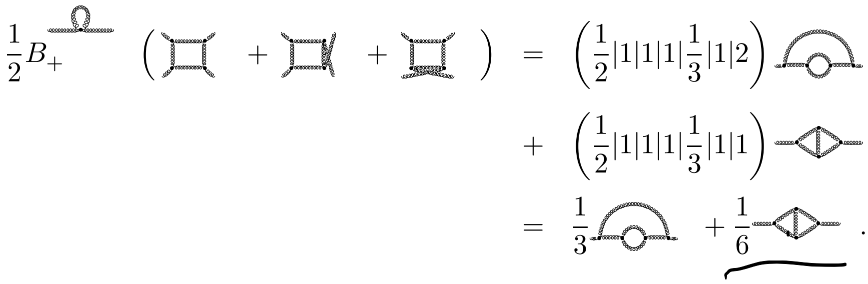


(104)

A single bijection, no ramification and two maximal forests in . This concludes insertions into .

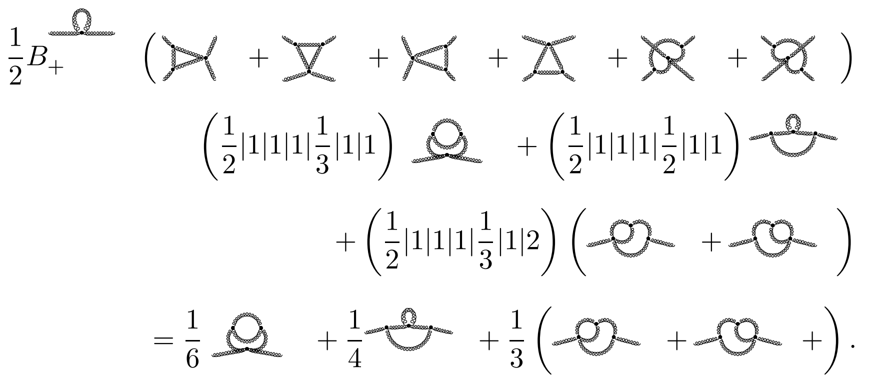
Insertions into $\frac{1}{2}B_+$

We come to insertions into .



(105)

Indeed, has three maximal forests, no ramification as there is only a single insertion place and two of the three bijections lead to it, while one bijection leads to , which also has three maximal forests.



(106)