Announcements

Oral exams:

Physics: July 13, 14 & August 11, 12.

Math: July 13 & August 23.

Please make a date with me first, then register with office.

Register at least two weeks in advance.

Monday 24th May public holiday also no lecture on 25th May.

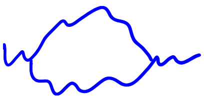
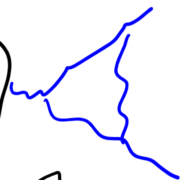
There is a paper to read for May 25 on course homepage

Dyson-Schwinger Eq's and Quantization of gauge theories (Summer '21)

Kreimer, Lect. May 17 2021.

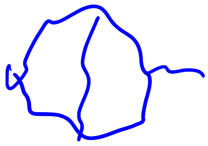
Anatomy of a gauge theory cont'd

Insertion: $B_+^{\cdots}$ are 1-cocycles so $\Delta B_+(\cdot) = B_+(\cdot)\otimes\mathbb{I} + (id\otimes B_+)\Delta$.

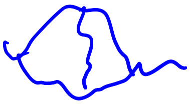
$\frac{1}{2}\, B_+^{\text{graph}}$  $\left(2 \right.$  $\left.\right)$ various orientations

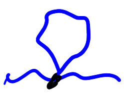
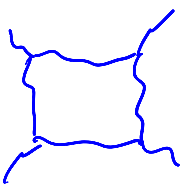
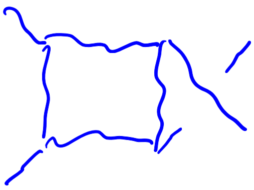
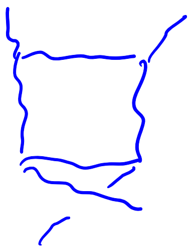
degeneracy of placing

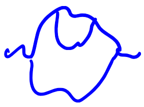
sym fk

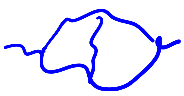
$= \left(\frac{1}{2} \,\middle|\, 1 \,\middle|\, 2 \,\middle|\, \frac{1}{2} \,\middle|\, \frac{1}{3} \,\middle|\, 1 \,\middle|\, 2 \right)$ 

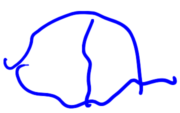
$\frac{1}{\text{ins. places}}$ $\frac{1}{\text{maxf}}$

$= \frac{1}{3}$ 

$\frac{1}{2}\, B_+^{\text{graph}}$  $\Big($  $+$  $+$  $\Big)$

$= \left(\frac{1}{2} \,|\, 1 \,|\, 1 \,|\, 1 \,|\, 1 \,|\, \frac{1}{3} \,|\, 1 \,|\, 2 \right)$ 

$+ \left(\frac{1}{2} \,|\, 1 \,|\, 1 \,|\, 1 \,|\, 1 \,|\, 1 \,|\, 1 \,|\, \frac{1}{3} \,|\, 1 \,|\, 1 \,|\, 1 \,|\, 1 \right)$ 

$= \frac{1}{3}$  $+ \frac{1}{6}$ 

B_+^{graph}  $($  $)$

Indeed, no ramification, just one maximal forest and a single bijection. Similarly, the ghost loop

$$\frac{1}{2}B_+ \left(2\right) = \left(\frac{1}{2}|1|2|\frac{1}{2}|1|1|1\right)\left(\quad + \quad \right)$$
$$= \frac{1}{2}\left(\quad + \quad \right) = \quad . \qquad (103)$$

Finally,

$$\frac{1}{2}B_+ \left(2\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}|\frac{1}{2}|2|\frac{1}{2}|\frac{1}{2}|1|1\right)\left(\quad + \quad \right)$$
$$= \frac{1}{8}\left(\quad + \quad \right)$$
$$= \frac{1}{4}\left(\quad \right). \qquad (104)$$

A single bijection, no ramification and two maximal forests in . This concludes insertions into .

Insertions into $\frac{1}{2}B_+$

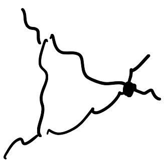
We come to insertions into .

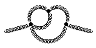
$$\frac{1}{2}B_+ \left(\quad + \quad + \quad \right) = \left(\frac{1}{2}|1|1|1|\frac{1}{3}|1|2\right)$$
$$+ \left(\frac{1}{2}|1|1|1|\frac{1}{3}|1|1\right)$$
$$= \frac{1}{3} \quad + \frac{1}{6} \quad . \qquad (105)$$

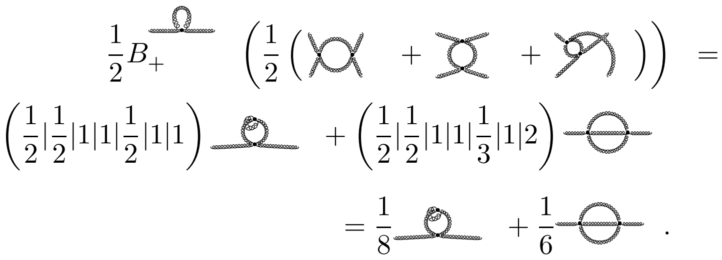
Indeed, has three maximal forests, no ramification as there is only a single insertion place and two of the three bijections lead to it, while one bijection leads to , which also has three maximal forests.

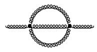
not a vertex

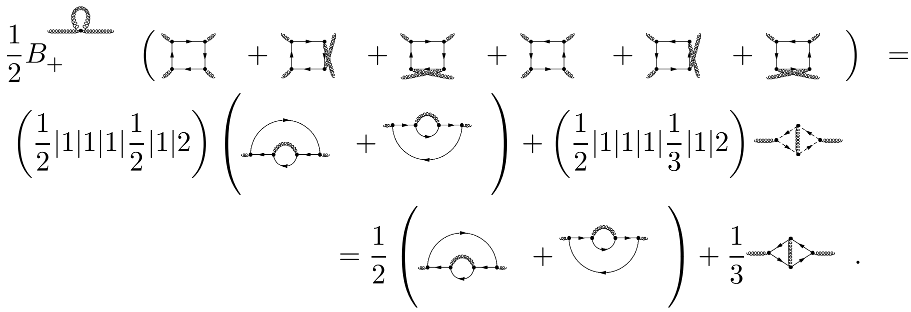
$$\frac{1}{2}B_+ \left(\quad + \quad + \quad + \quad + \quad + \quad \right) =$$
$$\left(\frac{1}{2}|1|1|1|\frac{1}{3}|1|1\right) \quad + \left(\frac{1}{2}|1|1|1|\frac{1}{2}|1|1\right)$$
$$+ \left(\frac{1}{2}|1|1|1|\frac{1}{3}|1|2\right)\left(\quad + \quad \right)$$
$$= \frac{1}{6} \quad + \frac{1}{4} \quad + \frac{1}{3}\left(\quad + \quad + \right). \qquad (106)$$

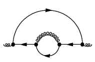
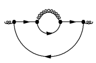
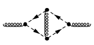


Note that  has three maximal forests and comes from one bijection,  has two maximal forests and comes as well from one bijection, while each of  and  come from two bijections and have three maximal forests.

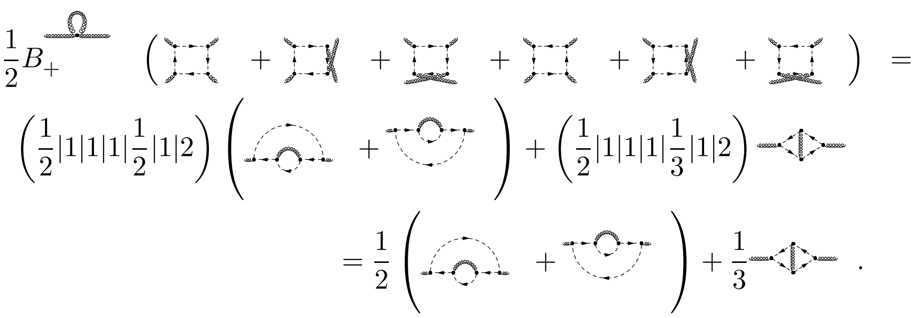
 (107)

This time,  has two maximal forests and comes from one bijection while  has three maximal forests and the two remaining bijections are leading to it.

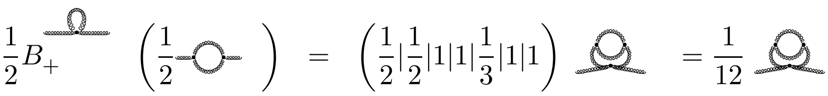
 (108)

Indeed,  and  have both two maximal forests and two bijections leading to them each, while  has three maximal forests and is generated from two bijections.

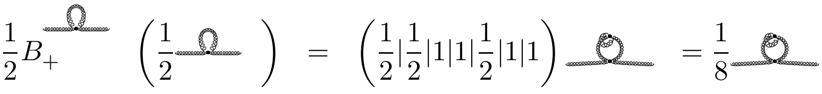
Similarly for ghosts

 (109)

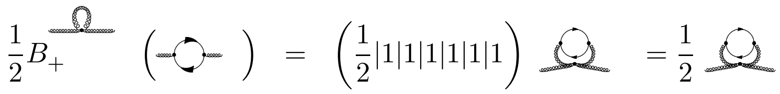
Now insertion of self-energies.

 (110)

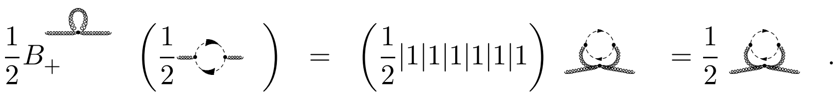
straightforward.

 (111)

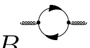
dito. Next,

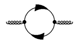
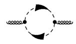

(112)

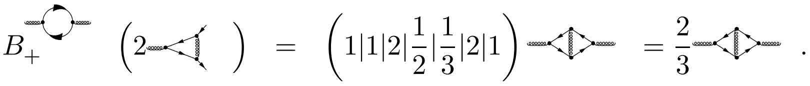
and similar for the ghost-loop


(113)

This concludes insertions into  .

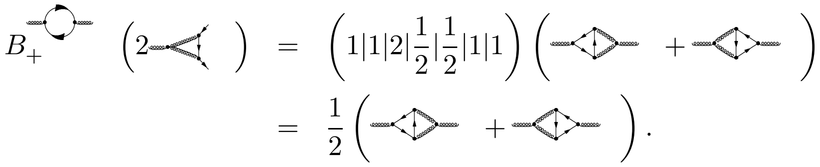
Insertions into B_+  **and B_+** 

It remain the insertions into  and  .


(114)

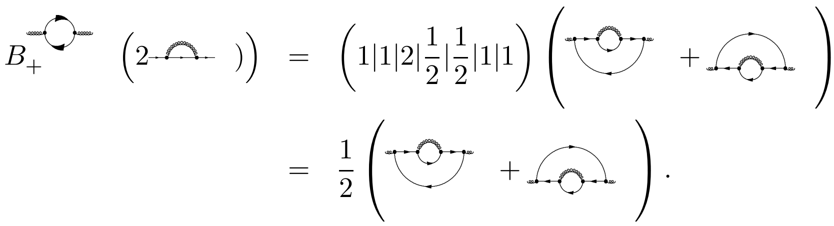
Indeed, there are three maximal forests, a ramification index of two and just a single bijection for each place.

Next


(115)

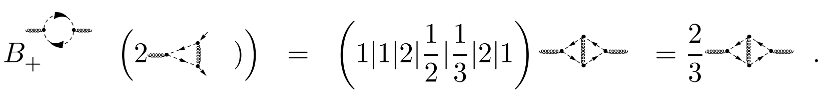
This time we have no ramification and two maximal forests.

Next the self-energy,

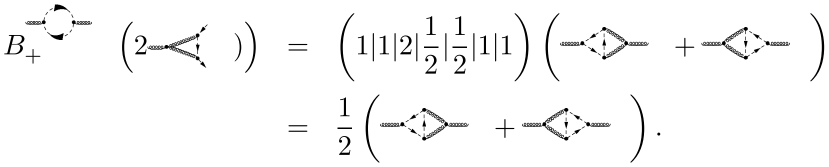

(116)

Again, two maximal forests, single bijection and no ramification.

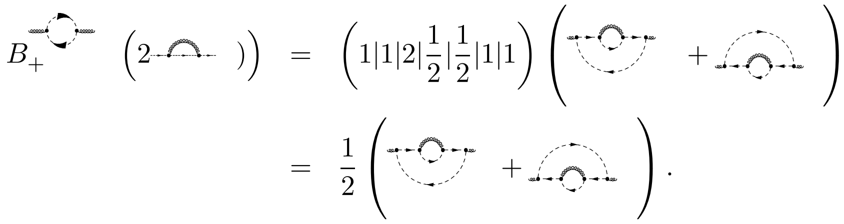
Finally, the ghosts bring nothing new:


(117)

And


(118)

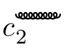
Also,

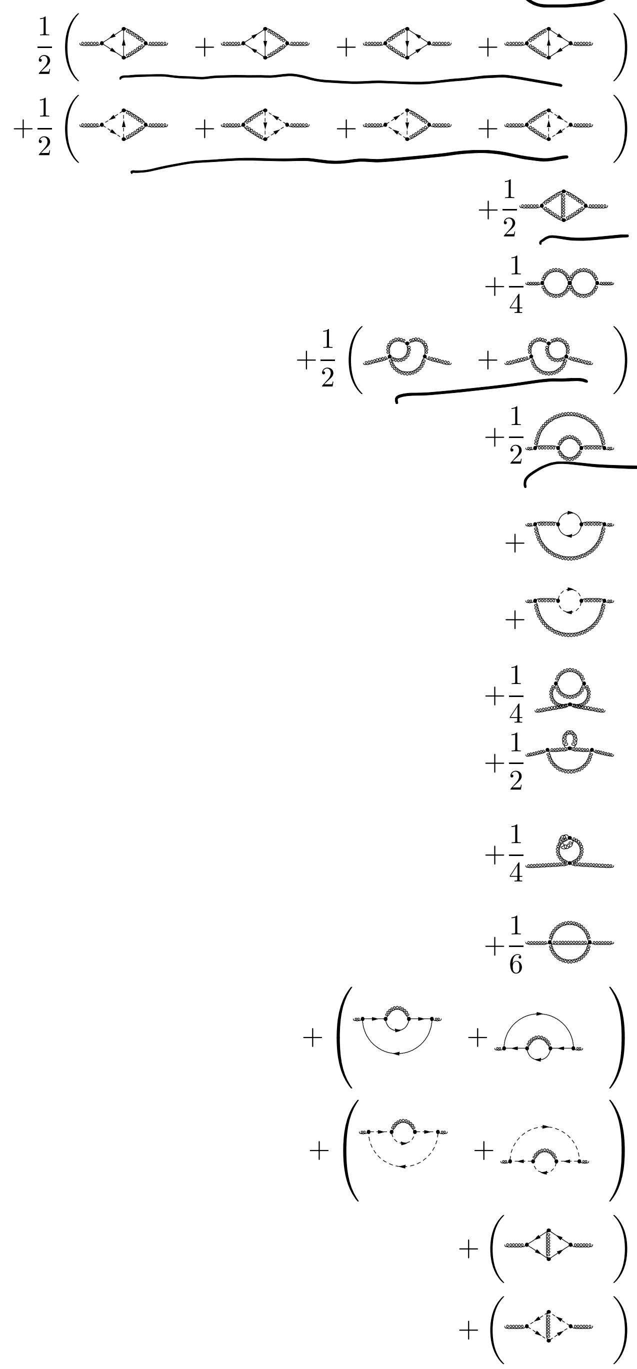


(119)

3.3 Adding up

2-loop contribution for gluon self-energy !?

Now we indeed confirm that the results adds up to c_2 . Adding up, we indeed find



from (97) + (115)
from (98) + (118)
from (99) + (105)
from (100)
from (100) + (106)
from (101) + (105)
from (102)
from (103)
from (106) + (110)
from (104) + (106)
from (107) + (111)
from (107)
from (108) + (116)
from (109) + (119)
from (108) + (114)
from (109) + (117)

$$+\frac{1}{2}\left(\quad \right) \qquad \text{from (112)}$$

$$+\frac{1}{2}\left(\quad \right). \qquad \text{from (113)}$$

We indeed confirm that the result is

$$\boxed{c_2 = \sum_{\substack{|\Gamma|=2 \\ \mathbf{res}(\Gamma)=}} \frac{\Gamma}{\mathbf{sym}(\Gamma)},} \tag{120}$$

the sum over all graphs at the given loop order, divided by their symmetry factors. This confirms the Hochschild theorem.

Furthermore, we find that

$$\Delta'(c_2) = \left(2c_1 + 2c_1\right) \otimes \frac{1}{2}B_+ \quad (\mathbb{I}) \tag{121}$$

$$+ \left(c_1 + c_1\right) \otimes \frac{1}{2}B_+ \quad (\mathbb{I}) \tag{122}$$

$$+ \left(2c_1 + 2c_1\right) \otimes B_+ \quad (\mathbb{I}) \tag{123}$$

$$+ \left(2c_1 + 2c_1\right) \otimes B_+ \quad (\mathbb{I}). \tag{124}$$

We now impose the Slavnov–Taylor identity, which allows us to write the above as

$$[2c_1^{\mathrm{coupl}} - c_1] \otimes B_+^{1,}, \tag{125}$$

by expanding (80) to order g^2. Vice versa, if we require that the coproduct defines a sub Hopf algebra on the c_j^r, we reobtain the Slavnov–Taylor identities

$$2c_1 + 2c_1 = c_1 + c_1 = 2c_1 + 2c_1 = 2c_1 + 2c_1 . \tag{126}$$

Hence we recover the Slavnov Taylor identities for the couplings from the above requirement. Summarizing, we indeed find

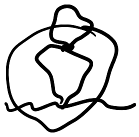
$$\Delta'(c_2) = \left[2c_1^{\mathrm{coupl}} - c_1\right] \otimes c_1 . \tag{127}$$

Note that the above indeed implies

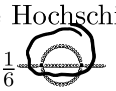
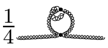
$$bB_+^{1,}\left(\Gamma \; [X_{\mathrm{coupl}}]^2\right) = 0, \tag{128}$$

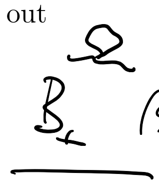
where

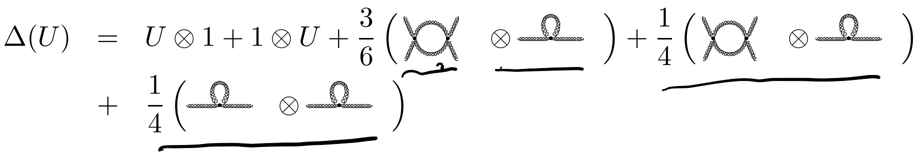
$$B_+^{1,} = \frac{1}{2}B_+ + \frac{1}{2}B_+ + B_+ + B_+ . \tag{129}$$



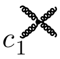
3.4 Hochschild closedness

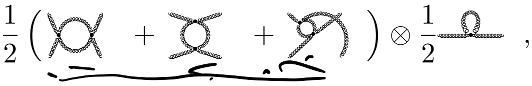
Finally, it is instructive to see how the Hochschild closedness comes about. Working out the coproduct on say the combination $\frac{1}{6}$  $+\frac{1}{4}$  $=: U$ we find





(130)

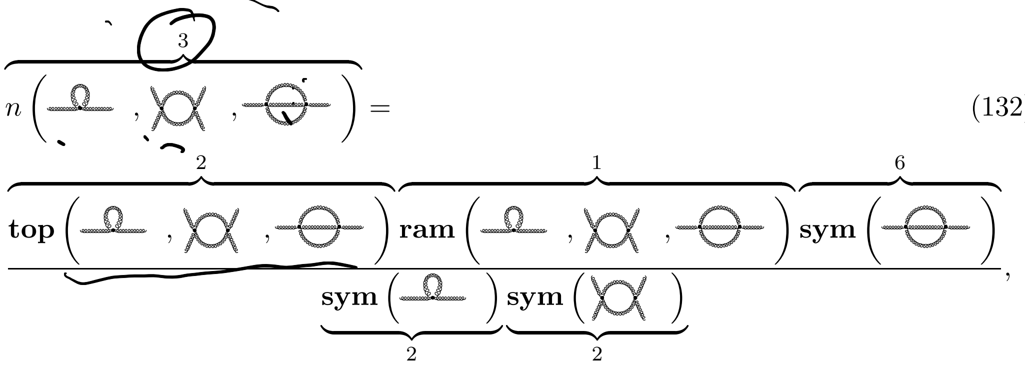
On the other hand, looking at the definition of c_1 , we find a mixed term



(131)

and we now see why we insist on a symmetric renormalization point.

Furthermore, we confirm



(132)

as it must by our definitions.

4 Discussion

We have exhibited the inner workings of Hochschild cohomology in the context of the Dyson–Schwinger equations of a generic non-abelian gauge theories. As a first combinatorial exercise we related the Slavnov–Taylor identities for the couplings to the very existence of a sub Hopf algebra which is based on the sum of all graphs at a given loop order. From [1] we know that the existence of this sub Hopf algebra is the first and crucial step towards non-perturbative solutions of such equations. Further steps in that direction are upcoming.

To prepare for this we finish the paper with a short discussion of some further properties of our set-up. This is largely meant as an outlook to upcoming results obtained by combining the Hopf algebra approach to perturbation theory with the structure of Dyson–Schwinger equations.

4.1 Locality and Finiteness

The first result concerns the proof of locality of counterterms and finiteness of renormalizad Hopf algebra. The structure

$$\Gamma^r = 1 + \sum_k g^{2k} B_+^{k;r}(\Gamma^r[X_{\rm coupl}]^k) \tag{133}$$

allows to prove locality of counterterms and finiteness of renormalized Green function via induction over the augmentation degree, involving nothing more than an elementary application of Weinberg's theorem to primitive graphs [7]. It unravels in that manner the source of equisingularity in the corresponding Riemann–Hilbert correspondences [8, 12]. For the DSE equations, this implies that we can define renormalized Feynman rules via the choice of a suitable boundary condition. This leads to an analytic study of the properties of the integral kernels of $\phi(B_+^{k;r}(\mathbb{I}))$ to be given in future work. Furthermore, the sub Hopf algebra of generators c_k^r allows for recursions similar to the ones employed in [1], relating higher loop order amplitudes to products of lower loop order ones. The most crucial ingredient of the non-perturbative methods employed in that paper is now at our disposal for future work.

4.2 Expansions in the conformal anomaly

The form of the arguments $X_{r,k} = \Gamma^r X_{\text{coupl}}^k$ allows for a systematic expansion in the coefficients of the β-function which relates the renormalization group to the lower central series of the Lie algebra $\mathcal{L}$. Indeed, if the β function vanishes X_{coupl} is mapped under the Feynman rules to a constant, and hence the resulting DSE become linear, by inspection. One immediately confirms that the resulting Hopf algebra structure is cocommutative, and the Lie algebra hence abelian [11, 3]. This should relate dilatations in quantum field theory to the representation theory of that lower central series. It will be interesting to compare the results here and more general in [13] with the ones in [15] from this viewpoint.

4.3 Central Extensions

The sub-Hopf algebras underlying the gauge theory theorem remain invariant upon addition of new primitive elements - beyond the one-loop level they obtain the form of a hierarchy of central extensions, which clearly deserves further study. Indeed, if we were to use only $B_+^{1,r}$ instead of the full series of Hochschild one cocycles we would still obtain the same sub Hopf algebra. Thus, this sub Hopf algebra and the structure of the DSEs is universal for a chosen QFT in the sense of [11, 3].

4.4 Radius of convergence

The above structure ensures that the Green functions come as a solution to a recursive equation which naturally provides one primitive generator in each degree. This has remarkable consequences for the radius of convergence when we express perturbation theory as a series in the coefficients c_k^r, upon utilizing properties of generating functions for recursive structures [9].

4.5 Motivic picture

The primitives themselves relate naturally to motivic theory [10]. Each primitive generator is transcendentally distinguished, with the one-loop iterated integral providing the rational seed of the game. The relation to algebraic geometry, motivic theory and mixed Hodge structures coming from QFT as they slowly emerge in [10, 11, 12] are an encouraging sign of the deep mathematical underpinnings of local interacting quantum fields.

Acknowledgments

It is a pleasure to thank Christoph Bergbauer, David Broadhurst, Kurusch Ebrahimi-Fard, Ivan Todorov and Karen Yeats for stimulating discussions.

References

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[3] C. Bergbauer and D. Kreimer, Hopf algebras in renormalization theory: Locality and Dyson-Schwinger equations from Hochschild cohomology, arXiv:hep-th/0506190, to appear.

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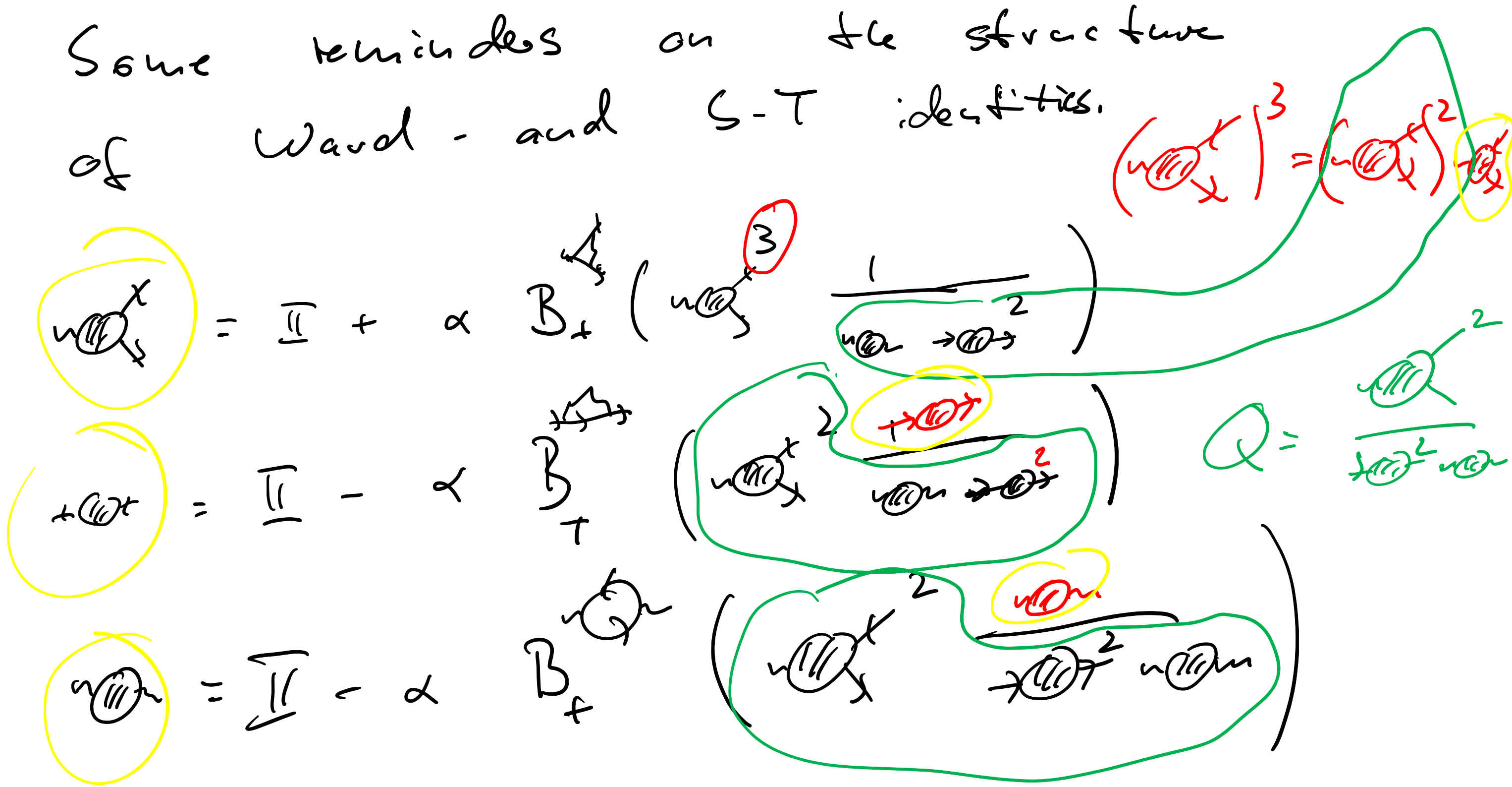
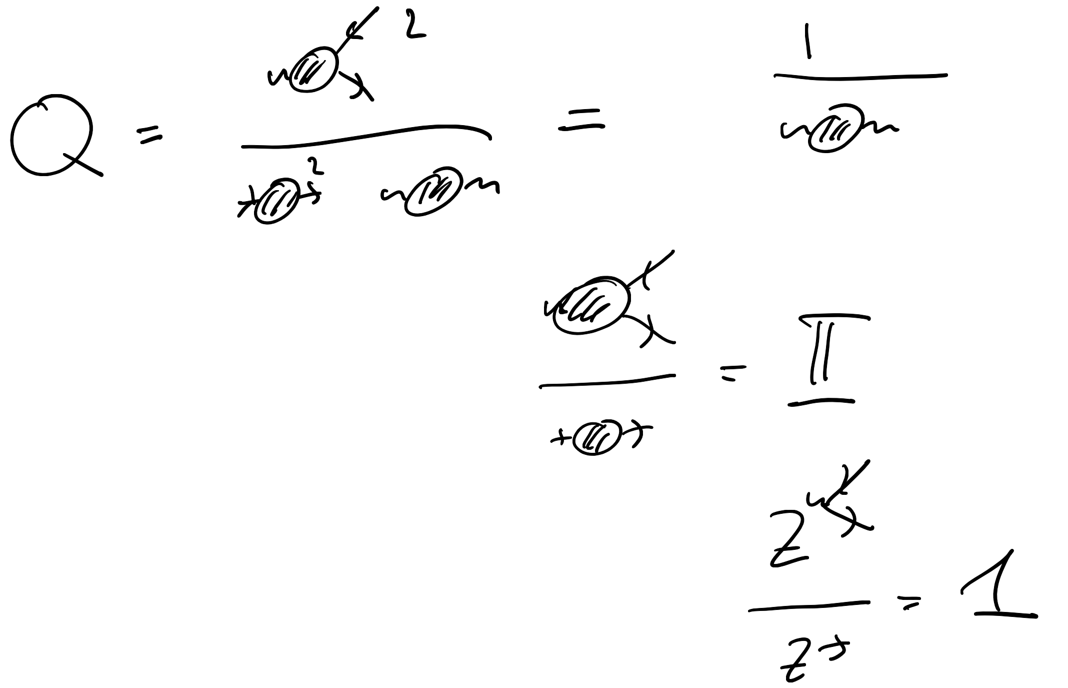
Some reminders on the structure of Ward- and S-T identities.

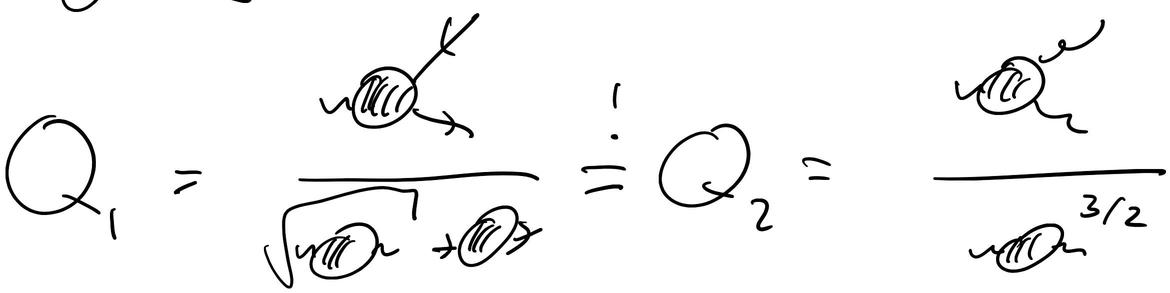


Co ideal structure in QED:



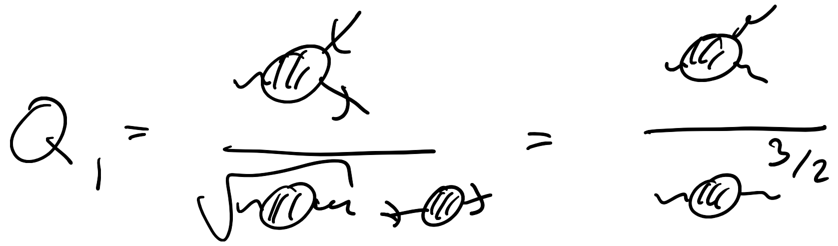
(for long. photon)

in QCD:

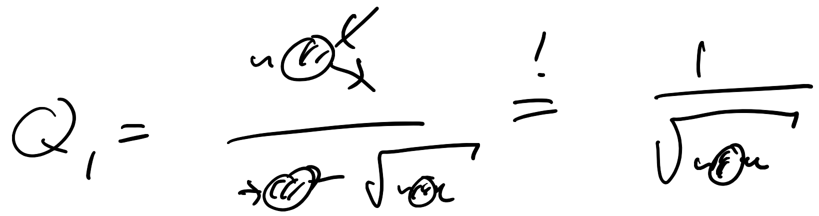


S-T: $Q_1 = Q = Q_2$

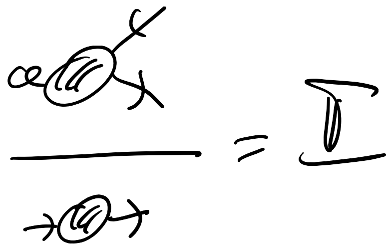
Big question: can we impose QED co-ideal on QCD as well?



and



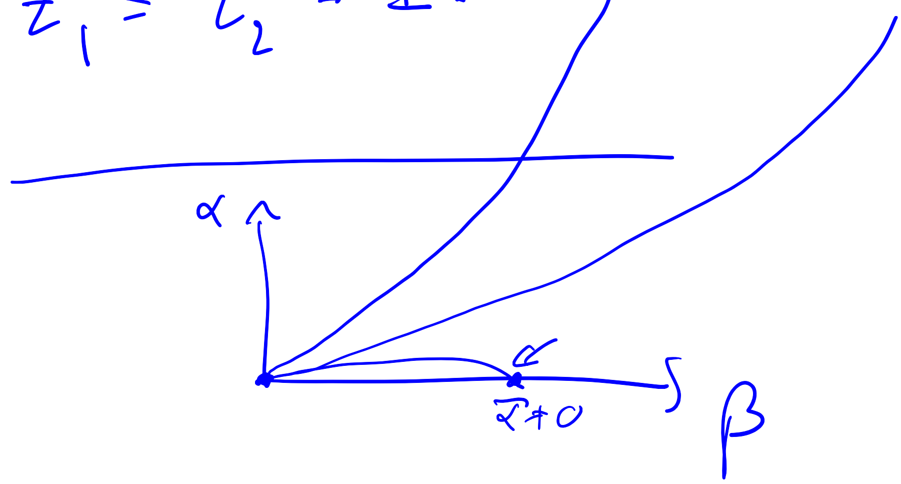
so that



in QCD as well?

Preliminary answer: old work by Adler, Baker-Johnson-Willey: $\exists$ a covariant linear gauge in QED such that

$Z_1 = Z_2 = 1$. Usually: $Z_i = 1 + \sum_{j=1}^{\infty} \alpha^j z_{ij} \neq 0$



In QCD, one (I believe) can construct a Wilson covariant gauge such that Z^{WL}

$$= \frac{Z_1}{Z_2} = 1$$

$$\parallel$$

$$Z^{\to}$$

Usually, you have

$$\frac{Z^{\text{WL}}}{Z^{\to}\sqrt{Z^{\text{cusp}}}} = \frac{Z^{\text{cusp}}}{(Z^{\text{cusp}})^{3/2}}$$

This is a strong claim.

Assume you have $N=4$ SYM and assume it is indeed conformal. ($\beta = 0$)

Beta function:

$$\frac{\partial}{\partial \ln \mu^2} \frac{Z^{\text{WL}}(\mu^2)}{Z^{\to}(\mu^2)\sqrt{Z^{\text{cusp}}(\mu^2)}}\Bigg|_{\ln\mu^2=0}$$

Assume $= 1$

$$\Rightarrow \quad \frac{\partial}{\partial \ln(\mu^2)} \frac{1}{\sqrt{Z^{\text{cusp}}(\mu^2)}}\Bigg|_{\ln\mu^2=0} = 0.$$

if $\beta = 0 \approx$ SYM $N=4$.

$$\Rightarrow \frac{\partial}{\partial \ln \mu^2} \Sigma^{ca}\left(\frac{q^2}{\mu^2}\right)\Big|_{\ln \mu^2 = 0} = 0$$

$$\Rightarrow q^2 \frac{\partial}{\partial q^2} \Sigma = 0.$$

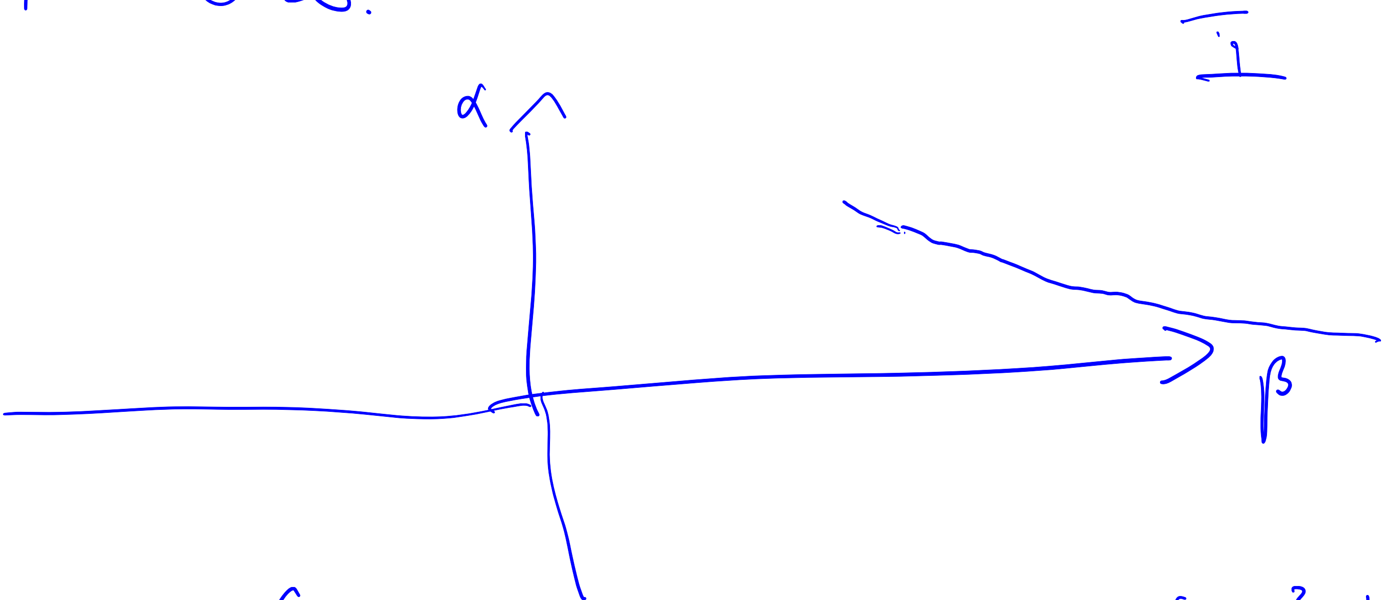
$\Rightarrow$ Σ^{ca} is a rational function of q^2.

$$\Rightarrow \mathrm{Im}\left(\Sigma^{ca}(q^2)\right) = 0.$$

$\Rightarrow$ There is no scattering.

$\Rightarrow$ $\mathcal{N}=4$ SYM is a ~~f~~ uniformly equivalent to a free theory.

Shirkov wrote a paper on running cpl in QCD.



$\hookrightarrow$ $\alpha(\ln q^2) \to 0$ for $q^2 \to +\infty$

Try to analytically cont. $\alpha(\ln q^2)$ for small q^2.

Impossible due to RGE and boundary condition fixes α_{QED} at Λ_{QED}.

$$\frac{1}{1-\alpha\ln\frac{\cdots}{\Lambda_{QED}}}$$

Shirkov:

non (q^2)

Ojima, Kugo, Nakanishi

Clay Math Inst.