DYSON–SCHWINGER EQUATIONS AND QUANTIZATION OF GAUGE THEORIES (SUMMER '21)

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1. Hopf algebras and DSE

1.1. Hopf algebra of rooted trees. .

H rooted trees

m disjoint union

$\mathbb{I}: \mathbb{Q} \to H \qquad q \to q\mathbb{I}$

$\hat{\mathbb{I}}: H \to \mathbb{Q} \qquad \hat{\mathbb{I}}(X) = 0 \quad \forall X \in Aug$

$\hat{\mathbb{I}}(\mathbb{I}) = 1.$

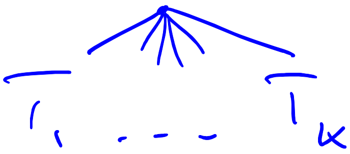
$\Delta: H \to H \otimes H$ co-associative map

$\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}$

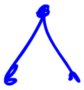
$\Delta(T_1 \ldots T_k) = \Delta(T_1) \ldots \Delta(T_k)$

$$\Delta T = T \otimes \mathbb{I} + \mathbb{I} \otimes T + \sum_{\substack{\text{adm.} \\ \text{cuts } \hat{c}}} P^{\hat{c}}(T) \otimes R^{\hat{c}}(T)$$

$T = B_+(X), \quad B_+: H \to Aug$

$B_+(T_1 \dots T_k) =$ 

$B_+(\mathbb{I}) = \bullet$, $B_+(\bullet) =$ 

$B_+($  $) =$  , $B_+(\bullet\ \bullet) =$ 

$$\Delta \circ B_+(X) = B_+(X)\otimes \mathbb{I} + (\mathrm{id}\otimes B_+)\Delta(X)$$

Thm. Two definitions of the coproduct agree.

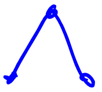
$$\Delta B_+(\bullet\bullet) = B_+(\bullet\bullet)\otimes\mathbb{I} + (\mathrm{id}\otimes B_+)\Delta(\bullet\bullet)$$

$$\underline{\Delta(\bullet\bullet) = \Delta(\bullet)\Delta(\bullet)} = (\bullet\otimes\mathbb{I}+\mathbb{I}\otimes\bullet)^2$$

$$= \bullet\bullet\otimes\mathbb{I} + 2\bullet\otimes\bullet + \mathbb{I}\otimes\bullet\bullet$$

$\Rightarrow \Delta B_+(\bullet\bullet) = B_+(\bullet\bullet)\otimes\mathbb{I} + \bullet\bullet\otimes\bullet + 2\bullet\otimes$ 

$+ \mathbb{I}\otimes \underbrace{B_+(\bullet\bullet)}$ ✓

$\Delta($  $) =$  $\otimes\mathbb{I} + \mathbb{I}\otimes$  $+ 2\bullet\otimes$  $+ \bullet\bullet\otimes\bullet$ ✓

Δ, m, $\mathbb{I}$, $\hat{\mathbb{I}}$,

$\Delta\, T_1 \dots T_k = \Delta T_1 \dots \Delta T_k$ $\Longleftrightarrow$ this is so far a bi-algebra.

For Hopf algebra, we need an antipode, a co-inverse, S.

$$S : H \to H, \quad S(\mathbb{I}) = \mathbb{I},$$

$$S(T_1 \dots T_k) = S(T_1) \dots S(T_k),$$

$$S(T) = -T - \sum_{\substack{\text{adm.}\\ \text{cuts } C_i}} S(P^{C_i}(T))\, R^{C_i}(T).$$

$$S(\bullet) = -\bullet$$

Example:

$$S(\Lambda) = -\Lambda - 2S(\bullet)\,\overset{\bullet}{\underset{\bullet}{\vert}} - S(\bullet\,\bullet)\,\bullet$$

$$= -\Lambda + 2\bullet\overset{\bullet}{\underset{\bullet}{\vert}} - S(\bullet)\,S(\bullet)\,\bullet$$

$$= -\underleftarrow{\Lambda} + \underline{2\bullet\overset{\bullet}{\underset{\bullet}{\vert}}} - \underline{\bullet\,\bullet\,\bullet}$$

other ways of writing S:

$$S(T) = -T - \sum_{\text{all cuts } C_i} (-1)^{|C|}\, P^{C}(T)\, R^{C}(T)$$

$$S(X) = m_H(S \otimes P)\Delta$$

$$P: H \to \mathrm{Aug}$$

1.2. Fixed point equations in H. .

Now, we have a Hopf algebra

$H = H(\mathbb{I}, \overset{1}{\bar{\mathbb{I}}}, m, \Delta, S)$.

It follows that

the space of maps

G^H_R, $\varphi: H \to R$, R a commutative ring,

such that $\varphi(X_1 X_2) = \varphi(X_1)\varphi(X_2)$

"φ are characters on H with values in R" forms a group G^H_R.

Why group?

group law: $\varphi, \psi \in G^H_R$

$\varphi * \psi = m_R(\varphi \otimes \psi)\Delta$, $\varphi * \varphi \in G^H_R$

so $\varphi * \psi(X_1 X_2) = \varphi * \psi(X_1) \cdot \varphi * \psi(X_2)$ mult. in R

$\varphi^{-1} = \varphi \circ S$

$\mathbb{1}_* * \varphi = \varphi * \mathbb{1}_* = \varphi$

$$\mathbb{1}_* = \mathbb{I} \circ \tilde{\mathbb{I}} .$$

With this as background, we want to consider fixed point eq's in H.

"fixed pt eq's from Hochschild cohomology.

$$B_+ : H \to \mathrm{Aug}$$

remember: B_+ for the Hopf alg. of the ring of polynomials in one variable was the integral operator $\int$, $\int x^n = \frac{1}{n+1} x^{n+1}$.

There is a little de Rham homology associated with it, and "our" B_+ generalizes this.

$$b B_+ = 0, \quad \text{with } b \circ b = 0$$

so that B_+ is a 1-cocycle in Hochschild cohomology.

Foissy proved B_+ is the only 1-cocycle in the Hochschild cohomology of $\mathcal{H}$.

Why is B_+ so interesting?

It allows for fixed pt eqs:

Assume there is a series defined by

$$\underline{X(\alpha)} = \mathbb{I} + \alpha \underline{B_+(\underline{X(\alpha)})}, \leftarrow \text{fixed pt eq}$$

so $X(\alpha) = H[[\alpha]]$

At $\alpha = 0$, $X(\alpha) = \mathbb{I}$

$$X(\alpha) = \underline{\mathbb{I} + \mathcal{O}(\alpha)}$$

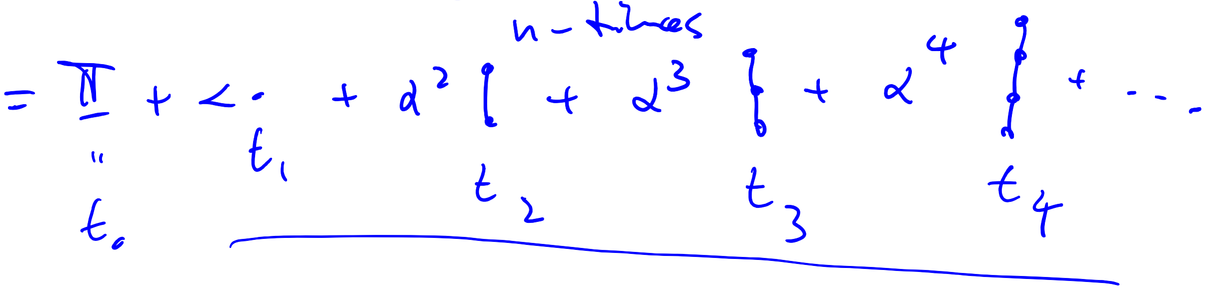
$$\mathbb{I} + \mathcal{O}(\alpha) = \mathbb{I} + \alpha B_+(\mathbb{I})$$

$$= \mathbb{I} + \alpha B_+(\mathbb{I}) = \mathbb{I} + \alpha \bullet$$

$$\Rightarrow X(\alpha) = \mathbb{I} + \alpha \bullet + \underbrace{\alpha^2 B_+(\bullet)}_{\alpha^2 \overset{\bullet}{\underset{\bullet}{|}}} + \mathcal{O}(\alpha^3)$$

$$\Rightarrow \quad X(\alpha) = \mathbb{I} + \alpha B_+(\mathbb{I}) + \alpha^2 B_+ \circ B_+(\mathbb{I}) + \dots$$

$$+ \alpha^n \underbrace{B_+ \circ \dots \circ B_+}_{n\text{-times}}(\mathbb{I})$$



$$\Delta t_n = \sum_{j=0}^{n} t_j \otimes t_{n-j}, \qquad t_0 = \mathbb{I}.$$

This has remarkable properties.

i) $\Delta X(\alpha) = X(\alpha) \otimes X(\alpha)$

"$X(\alpha)$ is group like".

An element $g \in H$ is grouplike

$$\Leftrightarrow \quad \Delta g = g \otimes g$$

$$\Delta \mathbb{I} = \mathbb{I} \otimes \mathbb{I}$$

Consider an evaluation $\varphi \in G_{\mathbb{R}}^{H}$

$\varphi: H \to \mathbb{R}$ given by

↑character

$$\varphi(B_+(X))(y) = \int_1^y \frac{\varphi(X)(u)}{u}\,du$$

$$\varphi(\cdot)(y) = \int_1^y \frac{\varphi(\underline{\mathbb{I}})(u)}{u}\,du$$

$$\varphi(\underline{\mathbb{I}}\cdot\underline{\mathbb{I}}) \overset{!}{=} \varphi(\underline{\mathbb{I}}) \Rightarrow \varphi(\underline{\mathbb{I}})(u) = 1 \quad \forall u$$

$$\varphi(\cdot)(y) = \int_1^y \frac{1}{u}\,du = \ln y$$

$$\varphi\left(\begin{smallmatrix}\bullet\\ \vert\\ \bullet\end{smallmatrix}\right)(y) = \frac{1}{2}\ln^2 y$$

$$\varphi(t_n)(y) = \frac{1}{n!}\ln^n y$$

$$\Rightarrow \quad \varphi(X(\alpha)) = 1 + \sum_{j=1}^{\infty} \alpha^j \frac{1}{j!}\ln^j y \,,$$

$$\Rightarrow \quad = \exp(\alpha \ln y) = y^\alpha .$$

$$\ln \varphi(X(\alpha)) = \alpha \ln y$$

Group-like-ness is reflected by the

fact that $\varphi(X(\alpha))$ is conformally invariant. (behaves like a power-law, so conformal weight α).

Not a bad idea to consider

$$\ln X(\alpha) = \alpha\,\bullet + \alpha^{2}\frac{1}{2}\big(2\underbrace{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} - \quad \bullet\bullet}_{\rho_2}\big) - \alpha^{3}\frac{1}{3}\underbrace{\Big(\begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix} - \;\bullet\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} + \frac{1}{3}\bullet\bullet\bullet\Big)}_{\rho_3} + \vdots$$

with $\Delta(\rho_i) = \rho_i \otimes \mathbb{I} + \mathbb{I}\otimes \rho_i$

so ρ_i is primitive

$$\Delta\Big(\underbrace{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} - \frac{1}{2}\;\bullet\bullet}_{\rho_2}\Big) = \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\otimes\underline{\mathbb{I}} + \underline{\mathbb{I}}\otimes\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} + \cancel{\bullet\otimes\bullet} - \frac{1}{2}\bullet\bullet\otimes\underline{\mathbb{I}}$$
$$- \frac{1}{2}\underline{\mathbb{I}}\otimes\bullet\bullet - \cancel{\bullet\otimes\bullet}$$

$$= \rho_2\otimes\underline{\mathbb{I}} + \underline{\mathbb{I}}\otimes\rho_2$$

$$\Delta\rho_3 = \underline{\rho_3\otimes\mathbb{I} + \mathbb{I}\otimes\rho_3} + \cancel{\bullet\otimes\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} - \;\otimes\bullet - \bullet\otimes\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix} - \begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\otimes\bullet - \bullet\bullet\otimes\bullet - \bullet\bullet\otimes\bullet\; 3\bullet\otimes\bullet\bullet + 3\bullet\bullet\otimes\bullet}$$

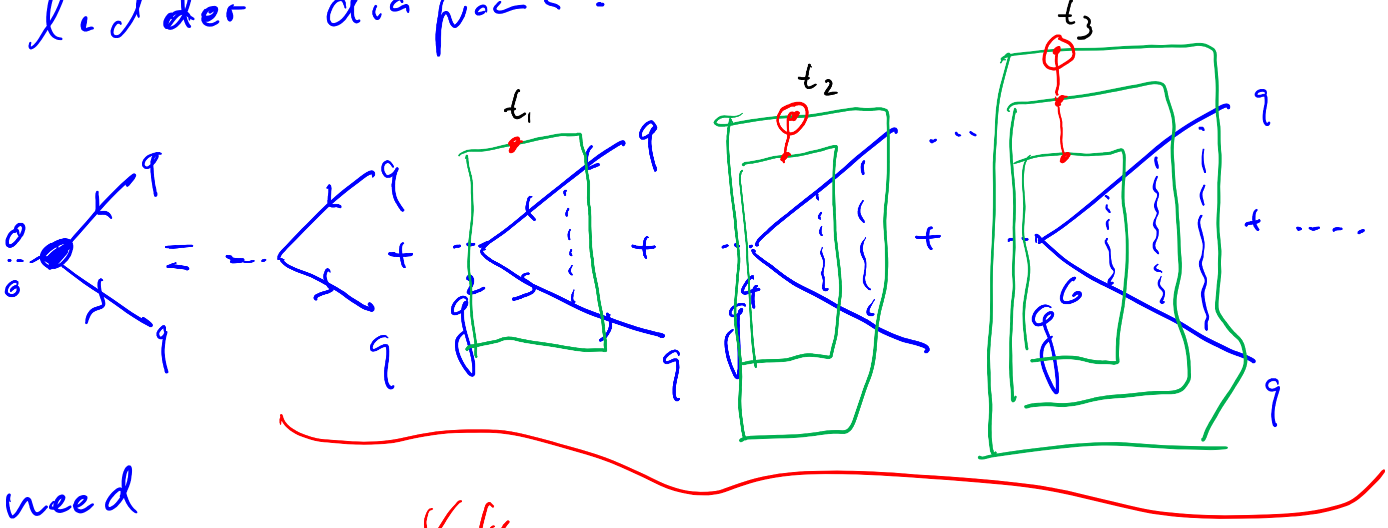
So that gives a connection between conformal invariance and structure of fixed pt eq^n.

1.3. Hopf algebra of graphs. .

Introduce Hopf algebra of graphs.

Famous example is ladder diagrams.

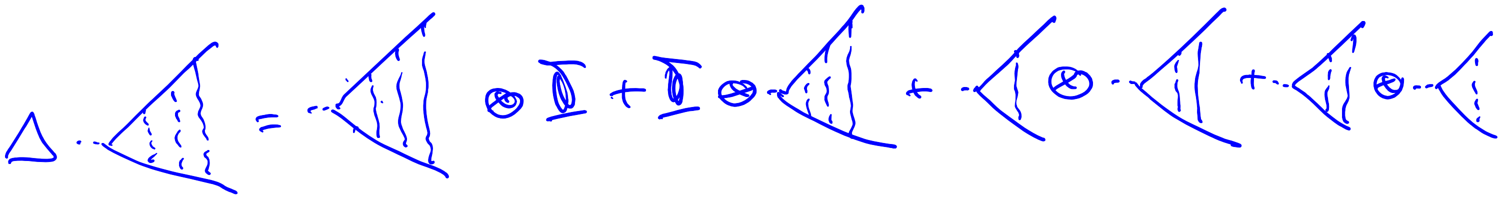
What do I get when summing ladder diagrams:



need

Hopf algebra Yalk structure H_{FG}

$$\Delta t_3 = t_3 \otimes \mathbb{I} + \mathbb{I} \otimes t_3 + t_1 \otimes t_2 + t_2 \otimes t_1$$

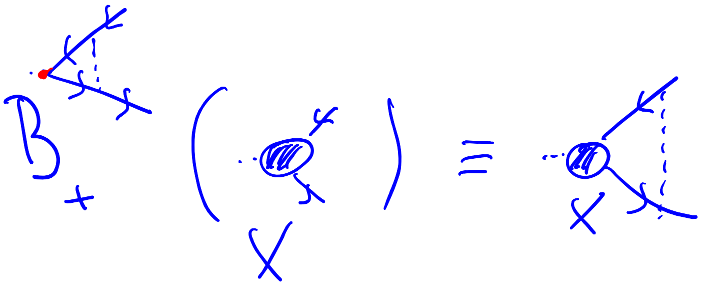


$$\Delta G = G \otimes \mathbb{I} + \mathbb{I} \otimes G + \sum_{g \subsetneq G} g \otimes G/g$$

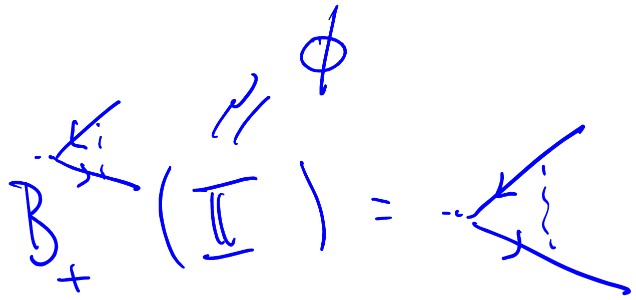
and indeed with this Δ,

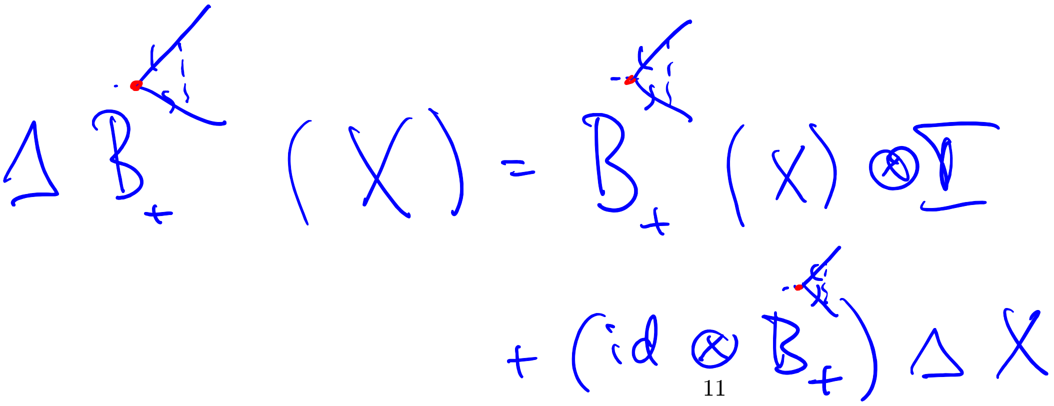
$$(\Delta\otimes \mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta$$ so this will give a Hopf algebra H_{FG}.

Toy B_+:

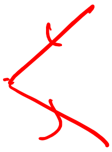


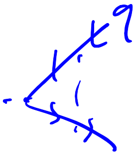
Then

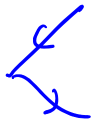




$$\Delta B_+(X) = B_+(X)\otimes\mathbb{I} + (\mathrm{id}\otimes B_+)\Delta X$$

$\mathrm{Yuk}(g^2) =$  $+ \; g^2 B_+$  $\left(\mathrm{Yuk}(g^2)\right)$

$\phi_R\Big($  $\Big) = \; \ldots\ldots$

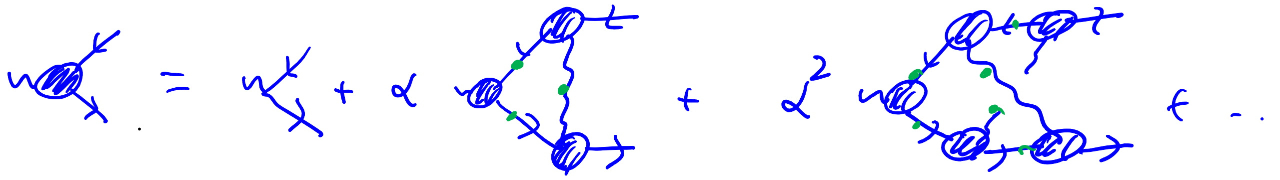
$\phi_R\left(\mathrm{Yuk}(g^2)\right)(q) =$  $+ \; g^2 \int \ldots \phi_R\left(\mathrm{Yuk}(g^2)\right)(k^2)\, d^4k$

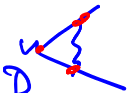
That is the ~~B~~ first true Dyson-Schwinger eq for "the ladder ~ vertex function"

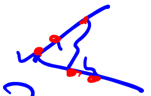
True question: what is the general non-linear situation?

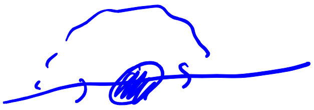
For example

$$X_2(\alpha) = \mathbb{I} + \alpha B_+\left(X_2^2(\alpha)\right) \quad ?$$




B_+ (?)


B_+ (?)



$$\sim \underline{X(\alpha) = \mathbb{I} - \alpha B_+\left(\frac{1}{X(\alpha)}\right)}$$

$$\downarrow$$

$$\boxed{\boxed{f(\alpha)}\, e^{-\frac{1}{\alpha}}} \longrightarrow \boxed{\sum c_{ijk}}\, \alpha^j L^k f_2\, e^{-\frac{2}{\alpha}}$$

$$\sum_{i,j,k} \underline{c_{i,j,k}}\, x^k e^{-\frac{j}{x}}\, \underline{\ln^i(-x)}$$

$c_{0,0,k}$ determine the ordinary pert. th.

$c_{i,j,k}$ come as <u>derivatives</u> of $\underline{c_{0,0,k}}$

1.4. **combinatorial DSE.** .

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HUMBOLDT U. BERLIN