Evaluation techniques for Feynman diagrams

BACHELORARBEIT

zur Erlangung des akademischen Grades
Bachelor of Science
(B. Sc.)
im Fach Physik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät I
Institut für Physik
Humboldt-Universität zu Berlin

von
Marcel Golz
geboren am 07.07.1989 in Cottbus

Betreuung:

1. *Prof. Dr. Dirk Kreimer*
2. *Dr. Oliver Schnetz*

eingereicht am: *21. Mai 2013*

Abstract

This work reviews three techniques used to evaluate Feynman integrals in quantum field theory. After a short exposition of the origin of said integrals we will briefly demonstrate the well-known *integration by parts* and *Gegenbauer polynomial* techniques on the example of the *wheel with three spokes* graph in chapters 2 and 3. In the fourth chapter we will more extensively present a formalism to integrate within an abstract algebra of *polylogarithms*, thereafter using the same graph to demonstrate it. In doing so, we will touch upon various mathematically interesting subjects from algebraic geometry, graph theory and number theory.

Zusammenfassung

Diese Arbeit gibt eine Übersicht über drei Techniken zur Auswertung von Feynman-Integralen in der Quantenfeldtheorie. Nach einer kurzen Erläuterung der Herkunft dieser Integrale werden wir kurz und bündig die wohlbekannten Methoden der *partiellen Integration* und *Gegenbauer-Polynome* am Beispiel des *Rads mit drei Speichen* Graphen demonstrieren. Im vierten Kapitel werden wir ausführlicher einen Formalismus zur Integration innerhalb einer abstrakten Algebra aus *Polylogarithmen* vorstellen und danach den selben Graphen wie zuvor als Beispiel benutzen. Dabei werden uns eine Vielzahl mathematisch interessanter Gegenstände aus algebraischer Geometrie, Graphentheorie und Zahlentheorie begegnen.

Contents

Chapter 1

Introduction

As this work is mainly concerned with the evaluation of integrals arising in quantum field theories, our first step will be a very short review of their basic properties. Starting from the standard canonical quantization process and using the relatively easy example of ϕ^4 theory, we will then introduce the famous Feynman integrals and their graphical representations, the Feynman graphs or diagrams. Following that, we will shortly introduce the formalism of dimensional regularization, which will be needed to tackle divergences in these integrals and Wick rotation which will simplify many calculations. In section 1.2 we will briefly mention some widely used terminology. At last, the Feynman diagram that will serve as our pedagogical example all throughout this work, the wheel with three spokes, will be presented.

1.1 Basics of Quantum Field Theory

1.1.1 Canonical quantization

There are plenty of comprehensive texts on quantum field theory, e.g. [6], and of course this short introduction cannot suffice for the reader to get acquainted with quantum field theory in all its details. Instead it shall serve as a motivation and shed some light on the physical background of the mathematical objects that we will be dealing with.
The starting point for a quantized field theory is usually the classical principle of stationary action, where said action is expressed using the Lagrange function L or the lagrangian density $\mathcal{L}$ respectively.

$$S = \int \mathrm{d}t \; L = \int \mathrm{d}^4 x \; \mathcal{L} \tag{1.1}$$

Here, $\mathcal{L}$ depends only on the field and its derivatives.

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi) \tag{1.2}$$

For the sake of simplicity we will only treat scalar, bosonic fields here.
Having derived the conjugate field

$$\pi = \frac{\delta L}{\delta(\partial_0 \phi)} = \dot{\phi} \tag{1.3}$$

one can now proceed to quantize these fields by postulating them to be operator-valued and have non-vanishing commutation relations. More precisely, they shall satisfy the canonical (equal-time) commutation rules:

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = -[\pi(\mathbf{x},t),\phi(\mathbf{y},t)] = i\delta^3(\mathbf{x}-\mathbf{y}) \tag{1.4}$$

Considering the typical free lagrangian density

$$\mathcal{L}_{free} = \frac{1}{2}\left((\partial_\mu\phi)(\partial^\mu\phi) - m^2\phi^2\right) \tag{1.5}$$

one finds the corresponding equation of motion to be the free Klein-Gordon equation

$$(\partial_\mu\partial^\mu + m^2)\phi = 0. \tag{1.6}$$

If we add an interaction part to the Lagrangian, that is, we describe particles interacting with each other, we have to take a look at the so-called S-Matrix from whose elements we can obtain the transition amplitudes from all initial states to all final states. To find these matrix elements one essentially has to calculate Green functions or correlation functions

$$G_n(x_1,...,x_n) = \langle 0|T\phi(x_1)...\phi(x_n)|0\rangle \tag{1.7}$$

where $|0\rangle$ is the vacuum ground state.

1.1.2 Perturbation theory

In practice it is inevitable to resort to pertubation theory and calculate the expansion of (1.7)

$$G_n(x_1,...,x_n) = \sum_{j=1}^{\infty}\frac{(-i)^j}{j!}\int \mathrm{d}^4y_1\,...\mathrm{d}^4y_j\langle 0|T\phi_{in}(x_1)...\phi_{in}(x_n)\mathcal{L}_{int}(y_1)...\mathcal{L}_{int}(y_j)|0\rangle, \tag{1.8}$$

where $\mathcal{L}_{int}$ is the interaction Lagrangian and ϕ_{in} is the initial state of ϕ in the 'infinite' past, i.e. $\phi(\mathbf{x},t) = U(t,-\infty)\phi_{in}U^{-1}(t,-\infty)$. Through methods that we will not elaborate on here (keywords: Wick's theorem, normal ordering), one finds that the vacuum expectation value in (1.8) can be written as the integrals of so-called *propagators* that typically depend on differences of space-time vectors or, after Fourier transformation, on a momentum 4-vector or sums of these. Graphs

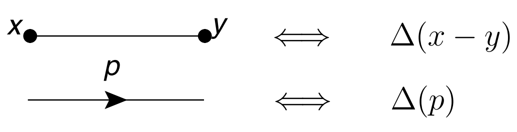


Figure 1.1: Graphical representation of propagators

constructed from edges as shown in Fig. 1.1 are used to visualize equations in quantum field theory and the *Feynman rules* translate graphs into integrals. The rules differ from theory to theory. Typical and simple examples are those for scalar ϕ^4-theory, i.e. the theory with $\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$:

1. For each edge, traversed by momentum k, write a propagator $i(k^2 - m^2 + i\epsilon)^{-2}$
2. For each vertex write a factor $-i\lambda(2\pi)^4\delta^4(p_{in} - p_{out})$ (the delta function conserves momentum)

3. For each closed loop, integrate over the corresponding momentum, i.e. write$\int d^4k$

We will deal with Feynman graphs G characterized by

- number of loops h
- number of internal edges L, numbered by $1 \leq l \leq L$
- number of vertices V, numbered by $1 \leq v \leq V$
- number of external legs E
- the decoration a_l of the edge l, $a_l \in \mathbb{R}^+$ (the power of the corresponding propagator)
- the momenta k associated to each internal edge

The two for this work important classes of Feynman diagrams are *primitive divergent* graphs and *broken primitive divergent* (bpd) graphs. The latter have exactly two external legs, no divergent subgraphs and satisfy $L = 2h + 1$, while the former can be constructed by fixing the broken part of bpd graphs, i.e. connecting the external legs to form a new edge.

1.1.3 Wick rotation

The integrals constructed from the Feynman rules as just presented are only convergent because of the small imaginary part $i\epsilon$. Moreover, the square (of the 2-norm) of space-time or momentum 4-vectors is not positive semidefinite. Both problems can be circumvented by performing a transformation for the time coordinate, namely $t = -i\tau$. The resulting vectors are now euclidean, their square is strictly non-negative and the integration process takes place in euclidean space and, given that it was convergent before, it is convergent even without $i\epsilon$. The result of the calculations in euclidean space can be analytically continued back into Minkowski space to obtain physical results.

1.1.4 Dimensional Regularization

As Feynman integrals tend to diverge for large momenta, one has to find some kind of formalism to either avoid singularities or associate meaningful values to diverging integrals anyway. One possibility is to introduce a cutoff. Another very popular way is the dimensional regularization formalism.
Basically, one analytically continues the diverging integral into D dimensions, where D is an arbitrary complex number. It can be shown that the functional

$$\int d^D p \, f(p) \tag{1.9}$$

with f being any function of the D-dimensional vector p, is in fact well defined and has properties analogous to usual integration. Moreover, for D a real positive integer one retrieves a normal integral.[4]
For our purposes it shall suffice to state some of these properties as far as we will need them [4],[9]:

$\forall\ \alpha, \beta \in \mathbb{C}$:

$$\int \mathrm{d}^D p\ [\alpha f(p) + \beta g(p)] = \alpha \int \mathrm{d}^D p\ f(p) + \beta \int \mathrm{d}^D p\ g(p) \tag{1.10}$$

$$\int \mathrm{d}^D p\ \frac{(p^2)^\alpha}{(p^2 - m^2)^\beta} = \pi^{\frac{D}{2}} (-m^2)^{\frac{D}{2}+\alpha-\beta} \frac{\Gamma(\alpha + \frac{D}{2})\Gamma(\beta - \alpha - \frac{D}{2})}{\Gamma(\frac{D}{2})\Gamma(\beta)} \tag{1.11}$$

$$\int \mathrm{d}^D p\ \frac{1}{(p^2)^\alpha[(p-q)^2]^\beta} = \pi^{\frac{D}{2}} (q^2)^{\frac{D}{2}-\alpha-\beta} \frac{\Gamma(\alpha + \beta - \frac{D}{2})\Gamma(\frac{D}{2} - \alpha)\Gamma(\frac{D}{2} - \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D - \alpha - \beta))} \tag{1.12}$$

$$\int \mathrm{d}^D p\ \frac{\partial f(p)}{\partial p_\mu} = 0 \tag{1.13}$$

In practice and all throughout this work one usually uses $D = 4 - 2\epsilon$. This so-called ϵ-expansion delivers a Laurent series at $\epsilon = 0$. For n-loop integrals the series is at most of degree n (i.e. all coefficients with index smaller than $-n$ are 0) and thus has poles at most of order ϵ^{-n}[9]. These poles encode the divergences of the integral and have to be taken care of through renormalization schemes, but this will not be our concern here.

1.2 Terminology

1.2.1 Periods, zeta functions and polylogarithms

In the literature one often finds the term *period* used for an evaluated Feynman integral. Periods $\mathcal{P}$ are a class of numbers hierarchically positioned between the algebraic numbers and the complex numbers, i.e. $\overline{\mathbb{Q}} \subset \mathcal{P} \subset \mathbb{C}$.
Don Zagier and Maxim Kontsevich gave an elementary defintion [8]:

Definition 1. *A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.*

A simple example could be

$$\pi = \iint_{x^2+y^2 \leq 1} \mathrm{d}x\mathrm{d}y. \tag{1.14}$$

One especially important function whose values can often be found to be periods of Feynman integrals is the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}, \mathrm{Re}\, s > 1 \tag{1.15}$$

which can be analytically continued to all $s \neq 1$. When concerned with Feynman integrals, s is a positive integer. For higher loop order graphs these *single zeta values* are often not sufficient. They evaluate to *multiple zeta values (MZVs)*, defined by the function

$$\zeta(s_1, ..., s_l) = \sum_{0<n_1<...<n_l} \frac{1}{n_1^{s_1} ... \ n_l^{s_l}}. \tag{1.16}$$

The zeta functions are special cases of the *classical polylogarithm*, defined as

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad \forall\ z, s \in \mathbb{C} \tag{1.17}$$

and the *multiple polylogarithm*

$$\mathrm{Li}_{s_1,\dots,s_l}(z_1, \dots, z_l) = \sum_{0<k_1<\dots<k_l} \frac{z_1^{k_1} \dots\ z_l^{k_l}}{k_1^{s_1} \dots\ k_l^{s_l}}. \tag{1.18}$$

A multiple polylogarithm in one variable is often called *hyperlogarithm*.

1.2.2 The Wheel with three spokes

The integral that we will compute in this work is

$$I = \int \mathrm{d}^4 l\ \mathrm{d}^4 p\ \mathrm{d}^4 q\ \frac{1}{(l^2 + m^2)p^2 q^2 (p-q)^2 (l-p)^2 (q-l)^2} \tag{1.19}$$

where all masses but one are set zero. We can do this because we are interested in the so called *ultraviolet divergences*, i.e. the divergences occurring when the momenta are large. Finite masses would not change these results. We cannot generally nullify all masses though, because this would lead to *infrared divergences* when momenta are 0.
The integral I can be split up into

$$I = \int \mathrm{d}^4 l\ \frac{1}{l^2 + m^2} \cdot I_2 \tag{1.20}$$

$$I_2 = \int \mathrm{d}^4 p\ \mathrm{d}^4 q\ \frac{1}{p^2 q^2 (p-q)^2 (l-p)^2 (q-l)^2} \tag{1.21}$$

which will be useful for our calculations. The corresponding Feynman graphs are

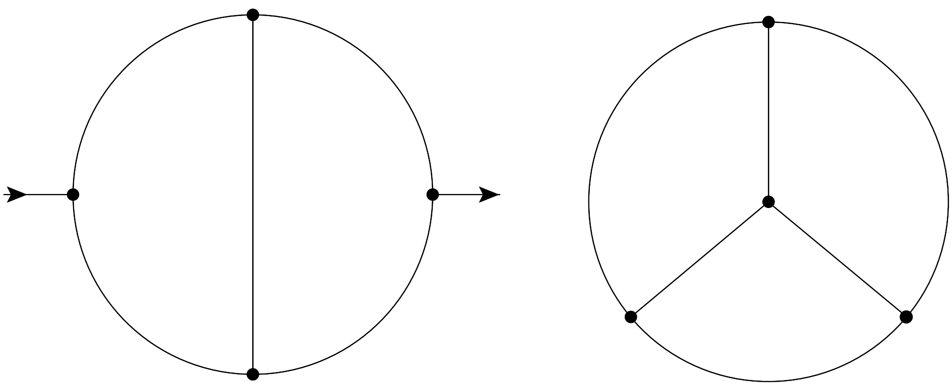


Figure 1.2: The massless two-point two-loop diagram G and the wheel with three spokes $\widetilde{G}$, representing I_2 and I, respectively.

shown in Fig. 1.2.
These specific graphs are interesting and at the same time suitable as an example because they are the graphs with the lowest number of loops that do not evaluate to rational numbers but zeta functions.

Chapter 2

Integration by Parts

The first method we will look at is based on property (1.13) of dimensionally regularized integrals. As already the first step depends highly on the specific integral at hand, application of it is only feasible for easier integrals and there is no general algorithm to compute the result.
We will try to find a function F, such that its derivative can be written in terms of our desired integral (1.21) and integrals of more easily computable graphs.
To save a little space let us call the denominator of (1.21)

$$N \equiv N(p, q, l) \coloneqq p^2 q^2 (p-q)^2 (l-p)^2 (q-l)^2. \tag{2.1}$$

Then let our function be

$$F \equiv F(p, q, l) \coloneqq \frac{(p-q)_\mu}{N}. \tag{2.2}$$

Before we calculate the derivative, let us write down some identities that will be useful [11]:

$$(p-q)^2 = p^2 - 2pq + q^2 \tag{2.3}$$

$$2p(p \pm q) = (p \pm q)^2 - q^2 + p^2 \tag{2.4}$$

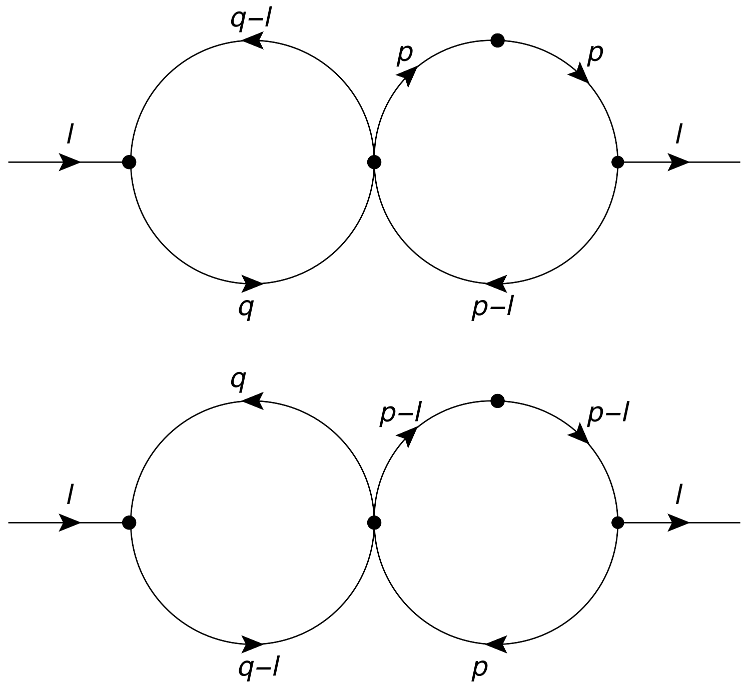
$$2(p-l)(p-q) = (p-l)^2 + (p-q)^2 - (l-q)^2 \tag{2.5}$$

Using the product rule and identities (2.3 - 2.5) we find the derivative of F to be

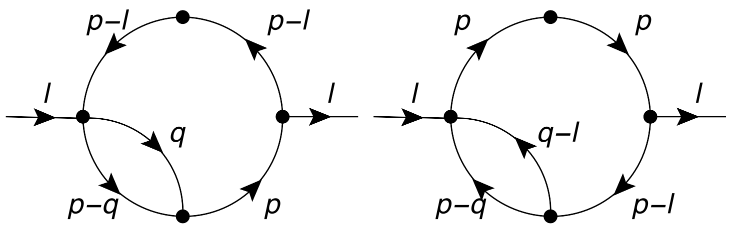
$$\begin{aligned}
\frac{\partial F}{\partial p_\mu} &= \frac{1}{N}\left(\frac{\partial}{\partial p_\mu}(p-q)_\mu\right) + \left(\frac{\partial}{\partial p_\mu}\frac{1}{N}\right)(p-q)_\mu \\
&= \frac{D}{N} + \frac{(p-q)_\mu}{q^2(q-l)^2}\frac{\partial}{\partial p_\mu}\left(\frac{1}{p^2(p-q)^2(l-p)^2}\right) \\
&= \frac{D}{N} - \frac{2(p-q)_\mu p_\mu}{p^2 N} - \frac{2(p-q)_\mu (p-q)_\mu}{(p-q)^2 N} - \frac{2(p-l)_\mu (p-q)_\mu}{(p-l)^2 N}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{D}{N} - \frac{(p-q)^2 + p^2 - q^2}{p^4 q^2 (p-q)^2 (l-p)^2 (q-l)^2} - \frac{2(p-q)^2}{p^2 q^2 (p-q)^4 (l-p)^2 (q-l)^2} \\
&\qquad\qquad\qquad\qquad\qquad - \frac{(p-l)^2 + (p-q)^2 - (q-l)^2}{p^2 q^2 (p-q)^2 (l-p)^4 (q-l)^2} \\
&= \frac{D-4}{N} - \frac{1}{p^4 q^2 (l-p)^2 (q-l)^2} - \frac{1}{p^2 q^2 (l-p)^4 (q-l)^2} \\
&\quad + \frac{1}{p^2 q^2 (l-p)^4 (p-q)^2} + \frac{1}{p^4 (p-q)^2 (l-p)^2 (q-l)^2}.
\end{aligned} \tag{2.6}
$$

Here we can use the fact that the second and third and the fourth and fifth term respectively give the same result when we integrate over internal momenta p and q. This is due to the fact that the corresponding graphs are equivalent (see Fig. 2.1) and momentum conservation in every vertex.



(a) Graphs of the second (up) and third term (low)



(b) Graphs of the fourth (left) and fifth term (right)

Figure 2.1: Visualisation of terms of (2.7)

Thus and with (1.13) we can write

$$
\begin{aligned}
0 &= \int \mathrm{d}^D p \, \mathrm{d}^D q \, \frac{\partial F}{\partial p_\mu} \\
&= \int \mathrm{d}^D p \, \mathrm{d}^D q \, \left(\frac{D-4}{N} - \frac{1}{p^4 q^2 (l-p)^2 (q-l)^2} - \frac{1}{p^2 q^2 (l-p)^4 (q-l)^2} \right. \\
&\qquad\qquad \left. + \frac{1}{p^2 q^2 (l-p)^4 (p-q)^2} + \frac{1}{p^4 (p-q)^2 (l-p)^2 (q-l)^2} \right) \\
&= \int \mathrm{d}^D p \, \mathrm{d}^D q \, \left(\frac{D-4}{N} - \frac{2}{p^4 q^2 (l-p)^2 (q-l)^2} + \frac{2}{p^2 q^2 (l-p)^4 (p-q)^2} \right)
\end{aligned}
\tag{2.7}
$$

Keeping in mind that $D = 4 - 2\epsilon$ and recalling (1.21) we indeed recover our integral I_2 in terms of two other integrals.

$$
\epsilon I_2 = \int \mathrm{d}^D p \, \mathrm{d}^D q \, \left(\frac{1}{p^2 q^2 (l-p)^4 (p-q)^2} - \frac{1}{p^4 q^2 (l-p)^2 (q-l)^2} \right) \tag{2.8}
$$

We can now use (1.12) to calculate these two integrals and receive functions of the 'external' momenta l with gamma function factors depending on ϵ.

$$
\begin{aligned}
&\int \mathrm{d}^D p \, \mathrm{d}^D q \, \frac{1}{p^2 q^2 (l-p)^4 (p-q)^2} \\
&\quad = \int \mathrm{d}^D p \, \pi^{\frac{D}{2}} [p^2]^{\frac{D}{2}-2} \cdot \frac{1}{p^2 (p-l)^4} \cdot \frac{\Gamma(2-\frac{D}{2})\Gamma(\frac{D}{2}-1)\Gamma(\frac{D}{2}-1)}{\Gamma(1)\Gamma(1)\Gamma(D-2)} \\
&\quad = \pi^D [l^2]^{D-5} \cdot \frac{\Gamma(2-\frac{D}{2})\Gamma(\frac{D}{2}-1)\Gamma(\frac{D}{2}-1)}{\Gamma(1)\Gamma(1)\Gamma(D-2)} \\
&\qquad\qquad \times \frac{\Gamma(5-D)\Gamma(D-3)\Gamma(\frac{D}{2}-2)}{\Gamma(3-\frac{D}{2}\Gamma(2)\Gamma(\frac{3}{2}D-5)} \\
&\quad = \pi^{4-2\epsilon} [l^2]^{-1-2\epsilon} \cdot \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \cdot \frac{\Gamma(1+2\epsilon)\Gamma(1-2\epsilon)\Gamma(-\epsilon)}{\Gamma(1+\epsilon)\Gamma(1-3\epsilon)}
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&\int \mathrm{d}^D p \, \mathrm{d}^D q \, \frac{1}{p^4 q^2 (l-p)^2 (p-q)^2} \\
&\quad = \pi^{4-2\epsilon} [l^2]^{-1-2\epsilon} \cdot \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \cdot \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}
\end{aligned}
\tag{2.10}
$$

From here on we will omit the π-factors as they cancel with the (likewise omitted) prefactors of momentum space integrals.

Following the notation of [11] we will make use of the G-function

$$G(\alpha,\beta)=\frac{\Gamma(\alpha+\beta+\epsilon-2)\Gamma(2-\epsilon-\alpha)\Gamma(2-\epsilon-\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(4-2\epsilon-\alpha-\beta)}. \tag{2.11}$$

With this abbreviation our integral I_2 now looks like

$$I_2=\frac{1}{\epsilon}[l^2]^{-1-2\epsilon}\cdot G(1,1)[G(1+\epsilon,2)-G(1,2)]. \tag{2.12}$$

In order to analyse these G-functions a little further we will need the well known Taylor expansion of the gamma function [5]

$$\Gamma(1+z)=\exp\left(-\gamma z+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}z^n\right) \tag{2.13}$$

and its property

$$z\Gamma(z)=\Gamma(1+z) \tag{2.14}$$

The first G-function to examine is $G(1,1)$.

$$\begin{aligned}
G(1,1)&=\frac{1}{\epsilon}\cdot\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1+1-2\epsilon)}\\
&=\frac{1}{\epsilon}\exp\left(\gamma(1-\epsilon)+\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}\left(\epsilon^n+2(-\epsilon)^n-(1-2\epsilon)^n\right)\right) \qquad (2.15)\\
&=:\frac{1}{\epsilon}G(\epsilon)
\end{aligned}$$

We now want to expand $G(\epsilon)$ around $\epsilon=0$. Of course this can be done by any computer algebra program but we will calculate it manually once. At first, we need the derivative

$$\begin{aligned}
\left.\frac{\mathrm{d}G(\epsilon)}{\mathrm{d}\epsilon}\right|_{\epsilon=0}&=\lim_{\epsilon\to 0}\frac{\mathrm{d}G(\epsilon)}{\mathrm{d}\epsilon}\\
&=\lim_{\epsilon\to 0}\left\{G(\epsilon)\cdot\left[-\gamma+\sum_{n=2}^{\infty}\left((-1)^n\frac{\zeta(n)}{n}\right.\right.\right. \qquad (2.16)\\
&\qquad\left.\left.\left.\times\left(n\epsilon^{n-1}-2n(-\epsilon)^{n-1}+2n(1-2\epsilon)^{n-1}\right)\right)\right]\right\}.
\end{aligned}$$

Assuming that all limits actually exist we look at them separately. The G-function itself gives

$$\lim_{\epsilon\to 0}G(\epsilon)=\exp\left(\gamma-\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}\right)=1 \tag{2.17}$$

because it follows from (2.13) for $z=1$ that

$$\gamma=\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}. \tag{2.18}$$

The first two parts of the sum are rather trivial.

$$\lim_{\epsilon\to 0}\sum_{n=2}^{\infty}(-1)^n\zeta(n)\epsilon^{n-1}=0 \tag{2.19}$$

$$\lim_{\epsilon\to 0}\sum_{n=2}^{\infty}(-1)^n\zeta(n)(-\epsilon)^{n-1}=0 \tag{2.20}$$

The third one, alas, is a little more difficult to evaluate.

$$\begin{aligned}
\lim_{\epsilon\to 0}\sum_{n=2}^{\infty}(-1)^n\zeta(n)(1-2\epsilon)^{n-1} &= \lim_{\epsilon\to 0}\frac{1}{1-2\epsilon}\sum_{n=2}^{\infty}\sum_{s=1}^{\infty}\left(\frac{(1-2\epsilon)}{s}\right)^n \\
&= \lim_{\epsilon\to 0}\frac{1}{1-2\epsilon}\sum_{s=1}^{\infty}\frac{\left(\frac{1-2\epsilon}{s}\right)^2}{1-\frac{1-2\epsilon}{s}} \\
&= \lim_{\epsilon\to 0}(1-2\epsilon)\sum_{s=1}^{\infty}\frac{1}{s^2+s(1-2\epsilon)} \\
&= \lim_{\epsilon\to 0}(1-2\epsilon)\frac{\psi(1-2\epsilon+1)+\gamma}{1-2\epsilon} \\
&= \psi(2)+\gamma=1
\end{aligned} \tag{2.21}$$

The order in which the infinite sums were taken could be changed, because as long as $\epsilon>0$ both infinite sums converge absolutely. ψ is the digamma function.[5]
Let us now collect these results and write down the desired Taylor expansion up to order ϵ^1:

$$\begin{aligned}
G(\epsilon) &= \lim_{\epsilon'\to 0}G(\epsilon')+\lim_{\epsilon'\to 0}\frac{\mathrm{d}G(\epsilon')}{\mathrm{d}\epsilon'}\epsilon+\mathcal{O}(\epsilon^2) \\
&= 1+1\cdot(-\gamma+2\cdot 1)\epsilon+\mathcal{O}(\epsilon^2)
\end{aligned} \tag{2.22}$$

from which we immediately find the G-function

$$G(1,1)=\frac{1}{\epsilon}\left(1+\epsilon(2-\gamma)+\mathcal{O}(\epsilon^2)\right). \tag{2.23}$$

In principle, $G(1,2)$ and $G(1+\epsilon,2)$ could be deduced the same way but fortunately there are relations that can be used to relate these G-functions to $G(1,1)$.[11]

$$G(\alpha,\beta)=\frac{(\alpha+\beta-3+\epsilon)(4-\alpha-\beta-2\epsilon)}{(\beta-1)(2-\beta-\epsilon)}G(\alpha,\beta-1) \tag{2.24}$$

$$\begin{aligned}
G(1+\alpha\epsilon,1+\beta\epsilon) = G(1,1)\frac{1}{\alpha+\beta+1}\Big\{&1+(\alpha+\beta)\epsilon+(\alpha+\beta)(\alpha+\beta+2)\epsilon^2 \\
&\Big[(\alpha+\beta)(\alpha+\beta+2)^2-2\zeta(3)(\alpha\beta(\alpha+\beta+3) \\
&+\alpha(\alpha+2)+\beta(\beta+2))\Big]\epsilon^3+\mathcal{O}(\epsilon^4)\Big\}
\end{aligned} \tag{2.25}$$

Our special cases of these equations result in

$$G(1,2) = G(1,1) \cdot \frac{\epsilon(1-2\epsilon)}{-\epsilon} = (2\epsilon - 1)G(1,1) \tag{2.26}$$

$$\begin{aligned} G(1+\epsilon, 2) &= G(1+\epsilon, 1) \cdot \frac{2\epsilon(1-3\epsilon)}{-\epsilon} \\ &= G(1,1)(3\epsilon - 1)\left(1 + \epsilon + 3\epsilon^2 + [9 - 6\zeta(3)]\epsilon^3 + \mathcal{O}(\epsilon^4)\right). \end{aligned} \tag{2.27}$$

After putting everything back together and into equation (2.12) our integral becomes

$$\begin{aligned} I_2 &= \frac{1}{\epsilon}[l^2]^{-1-2\epsilon} \cdot G(1,1)[G(1+\epsilon,2) - G(1,2)] \\ &= \frac{1}{\epsilon^3}[l^2]^{-1-2\epsilon} \cdot G^2(\epsilon)\Big[-1 - \epsilon - 3\epsilon^2 - (9 - 6\zeta(3))\epsilon^3 \\ &\qquad + 3\epsilon + 3\epsilon^2 + 9\epsilon^3 + 1 - 2\epsilon + \mathcal{O}(\epsilon^4)\Big] \\ &= [l^2]^{-1-2\epsilon} \cdot G^2(\epsilon)\left[6\zeta(3) + \mathcal{O}(\epsilon)\right]. \end{aligned} \tag{2.28}$$

Now only the integration over l remains:

$$\begin{aligned} I &= \int \mathrm{d}^D l \, \frac{I_2}{l^2 + m^2} \\ &= (m^2)^{-3\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(3\epsilon)}{\Gamma(2-\epsilon)} G^2(\epsilon)\left[6\zeta(3) + \mathcal{O}(\epsilon)\right] \end{aligned} \tag{2.29}$$

Using again $\Gamma(\epsilon) = \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)$ for ϵ near 0, we are left with

$$I = \frac{6\zeta(3)}{3\epsilon} + \mathcal{O}(\epsilon^0). \tag{2.30}$$

We see that the ϵ-expansion starts with period of the graph divided by the loop number times ϵ. In chapter 4 we will see that this is indeed the case for all primitive divergent graphs.

Chapter 3

Expansion in Gegenbauer Polynomials

Another method is to expand dimensionally regularized integrals in terms of Gegenbauer polynomials. Developed by Chetyrkin et. al. [3] it is also known as the *Gegenbauer polynomial x-space technique.*
It makes use of the relation

$$\frac{1}{(x_1 - x_2)^{2\lambda}} = \frac{1}{x_1^{2\lambda}} \sum_{n=0}^{\infty} C_n^{\lambda}(\hat{x}_1 \cdot \hat{x}_2) \left(\frac{|x_2|}{|x_1|}\right)^n, \qquad |x_1| > |x_2|,\ \hat{x}_i = \frac{x_i}{|x_i|},\ \lambda = \frac{D}{2} - 1 \tag{3.1}$$

which can be deduced from the Gegenbauer polynomial's generating function.
The x_i are the usual D-dimensional vectors and the C_n^{λ} are the Gegenbauer polynomials. For $\lambda = \frac{1}{2}$ they reduce to Legendre polynomials and for $\lambda = 0$ and $\lambda = 1$ they are Chebyshev polynomials of the first and second kind respectively. Furthermore they satisfy [3],[9]

$$\begin{gathered}\int \mathrm{d}\hat{x}_2\, C_n^{\lambda}(\hat{x}_1 \cdot \hat{x}_2) C_m^{\lambda}(\hat{x}_2 \cdot \hat{x}_3) = \frac{\lambda}{n + \lambda} \delta_{nm} C_n^{\lambda}(\hat{x}_1 \cdot \hat{x}_3) \\ C_n^1(1) = n + 1\end{gathered} \tag{3.2}$$

Later we will need to change to spherical coordinates where the new measure will be

$$\mathrm{d}^D k = S_{D-1} \mathrm{d}|k|\, |k|^{D-1} \mathrm{d}\hat{k} = \frac{\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} (k^2)^{1-\epsilon} \mathrm{d}k^2 \mathrm{d}\hat{k} \tag{3.3}$$

The expansion formula (3.1) gives a hint at the origin of the name of the method: It relies on the propagators in the integral depending only on the difference of at most two vectors. While this is always the case in position space, integrals of non-planar diagrams in momentum space cannot be rearranged in such a form. Here, we will only calculate the convergent integral I_2 for which it suffices to work in $D = 4$ dimensions. In that special case it is possible to remain in momentum space as the relation $\lambda = \frac{D}{2} - 1$ is satisfied.

We have seen that the expansion formula (3.1) depends on the absolute values of the vectors, so if we want to integrate three momenta we will have to calculate $3! = 6$ different cases. Fortunately (1.19) is symmetric so all calculations give the

same result and we only need to do one.
Let $|p| < |q| < |l|$. By applying (3.1) and changing to spherical coordinates we find

$$I_2 = \int \mathrm{d}^D p\, \mathrm{d}^D q\, \frac{1}{p^2 q^2 (p-q)^2 (l-p)^2 (q-l)^2} \tag{3.4}$$

$$\begin{aligned}\frac{1}{6} I_2 &= \int_0^{l^2} \mathrm{d}q^2 \int_0^{q^2} \mathrm{d}p^2 \int \mathrm{d}\hat{p}\, \mathrm{d}\hat{q}\, \frac{1}{q^2 l^4} \\ &\quad \times \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_k^1(\hat{p}\cdot\hat{q}) C_m^1(\hat{p}\cdot\hat{l}) C_n^1(\hat{q}\cdot\hat{l}) \left(\frac{p}{q}\right)^k \left(\frac{p}{l}\right)^m \left(\frac{q}{l}\right)^n\end{aligned} \tag{3.5}$$

Now we can evaluate the angular integrations by applying (3.2):

$$\begin{aligned}\frac{1}{6} I_2 &= \int_0^{l^2} \mathrm{d}q^2 \int_0^{q^2} \mathrm{d}p^2 \int \mathrm{d}\hat{q}\, \frac{1}{q^2 l^4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m+1} C_n^1(\hat{q}\cdot\hat{l}) C_m^1(\hat{q}\cdot\hat{l}) \left(\frac{p^2}{ql}\right)^m \left(\frac{q}{l}\right)^n \\ &= \int_0^{l^2} \mathrm{d}q^2 \int_0^{q^2} \mathrm{d}p^2\, \frac{1}{q^2 l^4} \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} C_n^1(\hat{l}\cdot\hat{l}) \left(\frac{p^2}{l^2}\right)^n \\ &= \int_0^{l^2} \mathrm{d}q^2 \int_0^{q^2} \mathrm{d}p^2 \sum_{n=0}^{\infty} \frac{1}{n+1} [q^2]^{-1} [p^2]^n [l^2]^{-n-2}\end{aligned} \tag{3.6}$$

The radial integration is then easily done:

$$\begin{aligned}\frac{1}{6} I_2 &= \int_0^{l^2} \mathrm{d}q^2 \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} [q^2]^n [l^2]^{-n-2} \\ &= \sum_{n=0}^{\infty} \frac{[l^2]^{-1}}{(n+1)^3} = [l^2]^{-1} \zeta(3)\end{aligned} \tag{3.7}$$

Chapter 4

Iterated Integration in Parametric Space

The third method to be presented is a little more sophisticated and will require several steps and exposition of relatively complicated mathematical objects. We will basically follow the approach of F.C.S. Brown who developed this technique in 2009 [2], delving a little deeper into some parts that are probably not obvious on first sight while omitting other parts that are not too important for us as our intention is simply to calculate the period of the wheel with three spokes again.
Our first step will be to introduce the parametric representation of Feynman integrals and explain the necessary operations to cast it into a form that will subsequently serve our needs. After that we will present the theory underlying the special functions called *polylogarithms* as far as necessary for our purposes.
Having laid the foundation we will then explain Brown's algorithm and the connection between graph polynomials, polylogarithms and the zeta function and finally obtain the result $6\zeta(3)$ once again by applying the algorithm to the wheel with three spokes.

4.1 Parametric Representation of Feynman Integrals

Besides momentum and position space, Feynman integrals can also be written in *parametric space*, i.e. one can transform them so that they depend only on some artificial *Feynman parameters* α_i while the momentum or position space integrals have been dealt with. This reshaping of the integrals is called *Schwinger trick* and makes use of the simple mathematical identity

$$\frac{i}{\Delta} = \int_0^\infty \mathrm{d}\alpha \, e^{i\alpha\Delta}, \qquad \Delta \in \mathbb{C}, \operatorname{Im}\Delta > 0. \tag{4.1}$$

Alternatively one could use

$$\frac{1}{\Delta} = \int_0^\infty \mathrm{d}\alpha \, e^{-\alpha\Delta}, \qquad \Delta \in \mathbb{C}, \operatorname{Re}\Delta > 0 \tag{4.2}$$

which is satisfied by any physical propagator as long as the integral has been Wick rotated into euclidean space earlier.

Remark 1. *To be precise, one should differ between parametric space, obtained by applying the Schwinger trick to momentum space integrals and dual parametric space, obtained by applying it in position space.*

Assuming Δ to be of the form $p^2+i\epsilon$ or k^2 one finds (4.1) and (4.2) to be gaussian integrands

$$\int \mathrm{d}p \, \frac{i}{p^2+i\epsilon} = \int \mathrm{d}p \int_0^\infty \mathrm{d}\alpha \; e^{i\alpha(p^2+i\epsilon)} = \int_0^\infty \mathrm{d}\alpha \, \sqrt{\frac{\pi}{i\alpha}}, \tag{4.3}$$

$$\int \mathrm{d}k \, \frac{1}{k^2} = \int \mathrm{d}k \int_0^\infty \mathrm{d}\alpha \; e^{-\alpha k^2} = \int_0^\infty \mathrm{d}\alpha \, \sqrt{\frac{\pi}{\alpha}}. \tag{4.4}$$

This can be generalized from one dimension to 4 space-time dimensions. (We will only give the one for euclidean space here. The Minkowski one is again very similar with imaginary factors.)

$$\int \mathrm{d}^4k \, \frac{1}{k^2} = \int \mathrm{d}^4k \int_0^\infty \mathrm{d}\alpha \; e^{-\alpha k^2} = \int_0^\infty \mathrm{d}\alpha \, \frac{\pi^2}{\alpha^2} \tag{4.5}$$

Even further generalization to arbitrary loop numbers, decorations and dimensions is possible. The most general integral corresponding to a graph G is

$$I(G) = \int \prod_{i=1}^{h} \mathrm{d}^D k_i \; \prod_{l}^{L} \frac{1}{(\Delta_l)^{a_l}} \tag{4.6}$$

where we integrate L propagators (corresponding to internal lines) over the h internal momenta (corresponding to loops or cycles). If we let momentum conservation confine the domain of integration we can also write $I(G)$ in terms of L integrals, one for each momentum k_l flowing through the edges l.

$$I(G) = \int \prod_{l=1}^{L} \mathrm{d}^D k_l \; \frac{1}{(k_l^2)^{a_l}} \tag{4.7}$$

Using the Schwingertrick on (4.7) then gives

$$I(G) = \frac{\pi^{h\frac{D}{2}}}{\prod_{l=1}^{L}\Gamma(a_l)} \int_0^\infty \prod_{l=1}^{L} \mathrm{d}\alpha_l \; \frac{\alpha_l^{a_l-1}\exp(-\frac{\mathcal{V}_G}{U_G})}{U_G^{\frac{D}{2}}} \tag{4.8}$$

where U_G and $\mathcal{V}_G$ are polynomials in the α_l.

Remark 2. *In the literature the step from (4.7) to (4.8) is said to be well known although it involves some rather non-trivial or at least tedious calculations and most authors treat only special cases like $a_l = 1 \; \forall l, D = 4$ [6] or leave the origin of these polynomials unclear [12]. We have also implicitly assumed the graph to be massless here and will do so from now on. A detailed account of the derivation of (4.8) can be found in appendix B.*

U_G is known as the graph polynomial or Kirchhoff polynomial of a graph G. Often U_G and $\mathcal{V}_G$ are also called the first and second Symanzik polynomial. They are defined by the equations

$$U_G = \sum_T \prod_{l \notin T} \alpha_l \tag{4.9}$$

$$\mathcal{V}_G = \sum_S (q^S)^2 \prod_{l \notin S} \alpha_l = V_G q^2 \tag{4.10}$$

where T is a *spanning tree* of G, $S = T_1 \cup T_2$ is the union of two trees or alternatively the result of removing an edge from a spanning tree (from now on shortly called *spanning 2-forest*) and q^S is the external momentum flowing through the edges that would make a spanning tree into S when cut. The sums go over all spanning trees and spanning 2-forests respectively while the products are over all edges l that do not belong to T or S. Both polynomials are homogenous (i.e. all monomials have the same degree) and their degree is h and $h+1$ respectively. The second equality in (4.10) is not generally true for all graphs but in this work we are only concerned with so called *broken primitive divergent* graphs, which have exactly two external legs. Proper definitions and examples of the graph theoretical objects mentioned here can be found in appendix A.

The next step is to get rid of the exponential so that there are only polynomials in α_l in the numerator and denominator. Let λ be an arbitrary but non-empty set of internal edges l of G. Then we change the variables so that

$$\alpha_l = \beta_l t, \qquad t = \sum_{l \in \lambda} \alpha_l, \qquad \forall\, l, 1 \leq l \leq L. \tag{4.11}$$

Omitting the constant factor and setting $a := \sum a_l$, (4.8) is then

$$\int_0^\infty \prod_{l=1}^{L} \mathrm{d}(\beta_l t)\, \frac{(\beta_l t)^{a_l - 1} \exp(-\frac{\mathcal{V}_G t^{h+1}}{U_G t^h})}{(U_G t^h)^{\frac{D}{2}}} \tag{4.12}$$

$$\int_0^\infty \prod_{l=1}^{L} (\mathrm{d}\beta_l t + \beta_l \mathrm{d}t)\; t^{a-L-h\frac{D}{2}} \frac{(\beta_l)^{a_l-1} \exp(-\frac{\mathcal{V}_G}{U_G} t)}{U_G^{\frac{D}{2}}} \tag{4.13}$$

Multiplicating out $(\mathrm{d}\beta_l t + \beta_l \mathrm{d}t)$ we are left with three different types of terms:
1) One term without $\mathrm{d}t$:

$$\prod_{l=1}^{L} \mathrm{d}\beta_l t = t^L \prod_{l=1}^{L} \mathrm{d}\beta_l \tag{4.14}$$

This term vanishes because only $L-1$ of the $\mathrm{d}\beta_l$ are independent. Our boundary condition in (4.11) enables us to write one of the $\mathrm{d}\beta_l$ in terms of the $L-1$ others and as $\mathrm{d}\beta_i \mathrm{d}\beta_j = 0$ for $i = j$ the whole product is 0.
2) $L-1$ terms with more than one $\mathrm{d}t$. These vanish too, again because $\mathrm{d}t\mathrm{d}t = 0$.
3) L terms with exactly one $\mathrm{d}t$.

$$t\mathrm{d}\beta_1 ... \beta_i \mathrm{d}t ... t\mathrm{d}\beta_L, \quad i \in \{1, ..., L\} \tag{4.15}$$

We collect these terms by defining $\Omega_L := \sum_{i=1}^{L} (-1)^{i+1} \beta_i \mathrm{d}\beta_1 ... \widehat{\mathrm{d}\beta_i} ... \mathrm{d}\beta_L$ and find that

$$\int_0^\infty \mathrm{d}t \int_0^\infty \Omega_L\, t^{a-h\frac{D}{2}-1} \frac{\prod_{l=1}^{L} (\beta_l)^{a_l-1} \exp(-\frac{\mathcal{V}_G}{U_G} t)}{U_G^{\frac{D}{2}}} \tag{4.16}$$

has the form of a gamma function. We can now simply rename $\beta_l \to \alpha_l$ and call the hypersurface defined by the above boundary conditions $H_\lambda = \{\alpha_i : \sum_{l\in\lambda} \alpha_l = 1\}$. Taking all this into account we have the integral

$$\Gamma\left(a - h\frac{D}{2}\right) \int_{H_\lambda} \frac{\prod_{l=1}^{L} \alpha_l^{a_l - 1}}{U_G^{\frac{D}{2}}} \left(\frac{U_G}{\mathcal{V}_G}\right)^{a - h\frac{D}{2}} \Omega_L. \tag{4.17}$$

Remark 3. *The outcome of the integral is completely independent of the choice of edges in λ. This is due to the fact, that it is a projective integral. $(n-1)$-dimensional (real) projective space is the space of equivalence classes of points in $\mathbb{R}^n \setminus (0,...,0)$, so e.g. the points (a_1,a_2,a_3) and (ca_1,ca_2,ca_3), $c \in \mathbb{R}\setminus\{0\}$, in $\mathbb{R}^3$ are equal in $\mathbb{P}^2(\mathbb{R})$. Our boundary conditions confine the domain of integration $[0,\infty)^L$ in such a way that no two points $(\alpha_1,...,\alpha_L)$ and $(c\alpha_1,...,c\alpha_L)$ can be integrated over simultaneously. Formulated in another way, the boundary condition makes us choose exactly one point out of every equivalence class to integrate over, so the hypersurface H_λ is a subset of the projective space $\mathbb{P}^{L-1}(\mathbb{R})$. To see whence the freedom to choose λ arbitrarily, consider $[0,\infty)^2$ with points (x,y). One could choose $x=1$ as the condition and H_λ would contain e.g. $(1,\frac{1}{2})$, $(1,1)$, $(1,2)$. Choose $x+y=1$ instead and H_λ would contain e.g. $(\frac{2}{3},\frac{1}{3})$, $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{3},\frac{2}{3})$. Clearly, the projective space H_λ is the same, no matter how λ was chosen so we can always take the easiest way and choose one single edge for λ.*
A less mathematically elegant way to write (4.18) that physicists might be more comfortable with uses the delta function

$$\Gamma\left(a-h\frac{D}{2}\right)\int_{[0,\infty)^L}\frac{\prod_{l=1}^{L}\alpha_l^{a_l-1}}{U_G^{\frac{D}{2}}}\left(\frac{U_G}{\mathcal{V}_G}\right)^{a-h\frac{D}{2}}\delta\left(\sum_{l\in\lambda}\alpha_l-1\right) \tag{4.18}$$

Setting $D=4-2\epsilon$ it can be shown that (4.18) is proportional to the integral [2]

$$\int_{H_\lambda}\frac{\prod_{l=1}^{L+1}\alpha_l^{a_l-1}}{U_{\widetilde{G}}^{2-\epsilon}}\Omega_L \mathrm{d}\alpha_{L+1} \tag{4.19}$$

The decoration of the new edge has been set to $a_{L+1}=(h+1)\frac{D}{2}-a$. As mentioned before, if one closes the external legs of a broken primitive divergent graph G it becomes a primitive divergent graph $\widetilde{G}$ which has the graph polynomial

$$U_{\widetilde{G}} = V_G + \alpha_{L+1}U_G. \tag{4.20}$$

Finally we want to introduce a notation that will be convenient throughout the rest of this work.
From the definition of the graph polynomial (4.9) we can see that U_G is linear in every α_l for any graph G. Thus one can write it as

$$U_G = U_{G/\{i\}} + \alpha_i U_{G\setminus\{i\}} \tag{4.21}$$

from which (4.20) follows as a special case for $i=L+1$. $G/\{i\}$ is the graph G with edge i contracted and $G\setminus\{i\}$ is the graph G with edge i deleted. To make this clear we recall the definition of U_G.

$$U_G = \sum_T \prod_{l\notin T}\alpha_l$$

Adding an edge does not affect the vertices of the graph and obviously a spanning tree of $G\setminus\{i\}$ is also a spanning tree of G. That means that the new edge i is not an element of any of the spanning trees of $G\setminus\{i\}$ and we can just write $\alpha_i U_{G\setminus\{i\}}$. Of course by adding an edge we also created new spanning trees that we also need to sum over. Before adding an edge a spanning tree of G that contains the new edge i must have been a 2-forest with each of the two connected components containing

exactly one of the two vertices connected by the new edge. This is of course the same as to say that it is a spanning tree of $G/\{i\}$ so we write $U_{G/\{i\}}$. Similar arguments can be used to find the same result for V_G.
To shorten the tedious notation we will subsequently write for $X \in \{U, V\}$:

$$X^{(i)} = X_{G\setminus\{i\}} \qquad\qquad X_i = X_{G/\{i\}} \tag{4.22}$$

At last, note that for concrete calculations with these polynomials it is useful to write

$$X^{(i)} = \frac{\partial}{\partial \alpha_i} X \qquad\qquad X_i = X|_{\alpha_i=0} \tag{4.23}$$

and that multiple and mixed sub- and superscripts like $X^{(12)}$ or $X_1^{(2)}$ are possible.

4.2 Polylogarithms and integration

We have so far reduced our initial integrals (4.8) to (4.19) whose leading term in a Taylor expansion is nothing more than integrals over the square of a polynomial. After a few integration steps this will inevitably lead to special functions called *polylogarithms*. In this section we will work out some basic properties of these functions and introduce theoretical concepts that will be needed later.

Remark 4. *Although the functions that we will define in this section are not necessarily identical to the classical logarithms mentioned in the introduction (hence the notation L instead of* Li*), we will use the same words to denote them.*

Let $\Sigma = \{\sigma_0, \sigma_1, ..., \sigma_N\}$ be a set of distinct points $\sigma_i \in \mathbb{C}$ where σ_0 is always assumed to be 0. Furthermore, let $A = \{\mathsf{a}_0, \mathsf{a}_1, ..., \mathsf{a}_N\}$ be an alphabet of $N+1$ letters a_i, each associated with the corresponding σ_i and $A^\times$ the set of all words w over A, including the so-called *empty word e*. (Words, in the sense of formal languages, are (arbitrarily long) combinations of letters, e.g. $\mathsf{a}_0\mathsf{a}_2\mathsf{a}_1\mathsf{a}_4\mathsf{a}_0^3\mathsf{a}_3$.)
With each such word w we can associate a function

$$L_w(z)\colon \mathbb{C} \setminus \Sigma \to \mathbb{C}. \tag{4.24}$$

Definition 2. *Let* $\log(z)$ *be the logarithm function, i.e. a function satisfying* $\mathrm{d}\log(z) = z^{-1}\mathrm{d}z$. *To be more precise, as the logarithm is a multivalued function for $z \in \mathbb{C}$, let* $\log(z)$ *denote the principle branch of the logarithm function, i.e. with imaginary part $\theta \in (-\pi, \pi]$. Then the defining properties of the hyper-/polylogarithm $L_w(z)$ are:*

1. $L_e(z) = 1$
2. $L_{\mathsf{a}_0^n}(z) = \frac{1}{n!}\log^n(z) \quad \forall\ n \geq 1$
3. *For all $w \in A^\times$ and $0 \leq i \leq N$*

$$\frac{\partial}{\partial z} L_{\mathsf{a}_i w}(z) = \frac{1}{z - \sigma_i} L_w(z) \tag{4.25}$$

4. *For all non-empty $w \in A^\times$, $w \neq \mathsf{a}_0^n$*

$$\lim_{z\to 0} L_w(z) = 0 \tag{4.26}$$

Starting from 1 and 2, Def. 3 enables us to inductively construct $L_w(z)$ for arbitrary w. 4. is needed for uniqueness, as it serves as a condition on the constant of integration. The definitions imply

$$L_{\mathsf{a}_i}(z) = \log(z - \sigma_i) - \log(-\sigma_i) \quad \forall\, i \neq 0 \tag{4.27}$$

which will be used a lot in the sequel.

The set of all words $A^\times$ spans a vector space that we call $\mathbb{Q}\langle A\rangle$. In an abuse of notation we will also use w to denote the elements of this vector space:

$$w = \sum_{i=1}^{m} q_i w_i, \quad q_i \in \mathbb{Q},\; w_i \in A^\times \tag{4.28}$$

This enables us to extend the definition of $L_w(z)$:

$$L_w(z) = \sum_{i=1}^{m} q_i L_{w_i}(z) \tag{4.29}$$

The $L_w(z)$ also satisfy the *shuffle relation*

$$L_{w_1}(z) L_{w_2}(z) = L_{w_1 \sqcup\!\sqcup\, w_2}(z), \quad \forall\, w_1, w_2 \in \mathbb{Q}\langle A\rangle. \tag{4.30}$$

Definition 3. *The operation $\sqcup\!\sqcup$ is called shuffle product. It is a commutative multiplication on $\mathbb{Q}\langle A\rangle$ defined by:*

$$w \sqcup\!\sqcup\, e = e \sqcup\!\sqcup\, w = w, \qquad \forall\, w \in A^\times \tag{4.31}$$

$$\mathsf{a}_i w_1 \sqcup\!\sqcup\, \mathsf{a}_j w_2 = \mathsf{a}_i(w_1 \sqcup\!\sqcup\, \mathsf{a}_j w_2) + \mathsf{a}_j(\mathsf{a}_i w_1 \sqcup\!\sqcup\, w_2), \quad \forall\, w_1, w_2 \in A^\times, \mathsf{a}_i, \mathsf{a}_j \in A \tag{4.32}$$

4.2.1 Primitives

The functions $L_w(z)$ have an interesting algebraic structure. Let

$$\mathcal{O}_\Sigma := \mathbb{Q}\left[z, \frac{1}{z}, \frac{1}{z-\sigma_1}, \ldots, \frac{1}{z-\sigma_N}\right] \tag{4.33}$$

This is a ring of polynomials in $N+2$ variables z, $\frac{1}{z}$,..., $\frac{1}{z-\sigma_N}$ with rational coefficients. Just like vector spaces over fields we can define modules over rings. Hence, we can define $L(\Sigma)$ to be a $\mathcal{O}_\Sigma$-module. Elements of $L(\Sigma)$ are sums of terms of the form $o(z)L_w(z)$, $o(z) \in \mathcal{O}_\Sigma$, $w \in A^\times$.
Integration requires us to take primitives, i.e. finding functions F that satisfy

$$\frac{\partial F}{\partial z} = f, \quad f \in L(\Sigma) \tag{4.34}$$

where F is of by one higher weight than f. However, $L(\Sigma)$ is not always sufficient as can easily be seen from the example

$$\begin{aligned} f &= \frac{1}{(z-\sigma_i)(z-\sigma_j)} L_w(z) \\ &= \frac{1}{\sigma_i - \sigma_j}\left(\frac{1}{z-\sigma_i} - \frac{1}{z-\sigma_j}\right) L_w(z) \\ &= \frac{1}{\sigma_i - \sigma_j}\frac{\partial}{\partial z}\left(L_{\mathsf{a}_i w}(z) - L_{\mathsf{a}_j w}(z)\right) = \frac{\partial F}{\partial z} \end{aligned} \tag{4.35}$$

Therefore, we have to enlarge $\mathcal{O}_\Sigma$ to

$$\mathcal{O}_\Sigma^+ = \mathcal{O}_\Sigma \left[\sigma_i, \frac{1}{\sigma_i - \sigma_j}, 0 \leq i < j \leq N\right] \tag{4.36}$$

and $L^+(\Sigma)$ accordingly. As a general result we can state that for every $f \in L(\Sigma)$ we can find a primitive $F \in L^+(\Sigma)$. Furthermore we can state that by decomposing of the coefficient into partial fractions - as in the above example - the process of taking primitives is reduced to taking primitives of

$$(z - \sigma_i)^n L_w(z), \quad n \in \mathbb{Z} \tag{4.37}$$

where the case $n = -1$ is given by the definition and all other cases can be found via partial integration.

With the shuffle product introduced in the last section, $L(\Sigma)$ becomes a commutative algebra that allows us to formally integrate all functions by taking primitives and evaluating these in the integral's boundaries. Within our bounds any integral would be of the form

$$\int_0^\infty f(z)\mathrm{d}z = F(z)|_{z=\infty} - F(z)|_{z=0}\,. \tag{4.38}$$

Assigning meaningful values to both terms on the right hand side is our next task. Generally, both terms could be singular but the integral as a whole will be convergent. We will look at both terms separately, discarding the singular parts that would cancel each other anyway and keep the finite, *regularized* values for both. By parts 2 and 4 of our definition of L_w, the regularized value at 0 vanishes for all non-empty words.

$$\mathrm{Reg}_{z=0} L_w(z) = 0 \quad \forall\, w \neq e \tag{4.39}$$

4.2.2 Drinfeld's associator

In this section we introduce *Drinfeld's associator*, which will be needed in the following section. Instead of the polylogarithm functions, consider their generating series

$$\mathfrak{L}(z) = \sum_{w \in A^\times} w L_w(z) \tag{4.40}$$

and let

$$\mathbb{C}\langle\langle A \rangle\rangle = \left\{ \sum_{w \in A^\times} w S_w : S_w \in \mathbb{C} \right\} \tag{4.41}$$

be the set of formal power series in words $w \in A^\times$. $\mathfrak{L}(z)$ is a multivalued function on $\mathbb{C} \setminus \Sigma$ but instead of $\mathbb{C}$ it maps to $\mathbb{C}\langle\langle A \rangle\rangle$. Let $A = \{\mathsf{x}_0, \mathsf{x}_1\}$, $\Sigma = \{0, 1\}$. Equation (4.25) implies that $\mathfrak{L}(z)$ satisfies

$$\frac{\partial}{\partial z}\mathfrak{L}(z) = \left(\frac{\mathsf{x}_0}{z} + \frac{\mathsf{x}_1}{z-1}\right)\mathfrak{L}(z) \tag{4.42}$$

which is a special case of the *Knizhnik-Zamolodchikov equations*. Furthermore, the conditions $L_w(z) = 0$ for $z \to 0$, $w \neq \mathsf{x}_0^n$ and $L_{\mathsf{x}_0^n}(z) = \frac{1}{n!}\log^n(z)$ imply

$$\mathfrak{L}(z) \equiv \mathfrak{L}_0(z) = g(z)e^{\mathsf{x}_0 \log(z)} \tag{4.43}$$

for z near 0. $g(z)$ is $\mathbb{C}\langle\langle A\rangle\rangle$-valued, holomorphic in the neighbourhood of 0 and $g(0) = 1$. Similarly, it is possible to find another solution

$$\mathfrak{L}_1(z) = h(1-z)e^{\mathsf{x}_1 \log(1-z)} \tag{4.44}$$

for z near 1, $h(0) = 1$. $\mathfrak{L}_0$ and $\mathfrak{L}_1$ are well defined on the real interval $(0, 1)$. Since $\mathfrak{L}_0$ and $\mathfrak{L}_1$ are both solutions of (4.42) they must differ by some constant and invertible $\Phi(\mathsf{x}_0, \mathsf{x}_1) \in \mathbb{C}\langle\langle A\rangle\rangle$. This is Drinfeld's associator and we write

$$\Phi(\mathsf{x}_0, \mathsf{x}_1) = (\mathfrak{L}_1(z))^{-1}\,\mathfrak{L}_0(z). \tag{4.45}$$

Let $0 < \xi < 1$. We define

$$\mathfrak{L}_\xi(z) := \mathfrak{L}_0(z)\mathfrak{L}_0(\xi)^{-1} \tag{4.46}$$

such that $\mathfrak{L}_\xi(\xi) = 1$.
With (4.43 - 4.45) and by setting $\xi = 1 - z$ we get

$$\begin{aligned} e^{-\mathsf{x}_0 \log(z)}\mathfrak{L}_\xi(z)e^{\mathsf{x}_1 \log(z)} &= e^{-\mathsf{x}_0 \log(z)}\mathfrak{L}_0(z)\mathfrak{L}_0(\xi)^{-1}e^{\mathsf{x}_1 \log(z)} \\ &= e^{-\mathsf{x}_0 \log(z)}g(z)e^{\mathsf{x}_0 \log(z)}\Phi^{-1}(\mathsf{x}_0, \mathsf{x}_1)\mathfrak{L}_1(\xi)^{-1}e^{\mathsf{x}_1 \log(z)} \\ &= e^{-\mathsf{x}_0 \log(z)}g(z)e^{\mathsf{x}_0 \log(z)}\Phi^{-1}(\mathsf{x}_0, \mathsf{x}_1)e^{-\mathsf{x}_1 \log(z)}h^{-1}(z)e^{\mathsf{x}_1 \log(z)} \end{aligned} \tag{4.47}$$

When taking the limit $z \to 0^+$, all factors tend to 1 or cancel each other so on the right hand side only Φ^{-1} is left.

$$\Phi^{-1}(\mathsf{x}_0, \mathsf{x}_1) = \lim_{z\to 0^+} e^{-\mathsf{x}_0 \log(z)}\mathfrak{L}_0(z)\mathfrak{L}_0^{-1}(1-z)e^{\mathsf{x}_1 \log(z)} \tag{4.48}$$

By a similar calculation one could have found

$$\Phi(\mathsf{x}_0, \mathsf{x}_1) = \lim_{z\to 1^-} e^{-\mathsf{x}_1 \log(1-z)}\mathfrak{L}_0(z)\mathfrak{L}_0^{-1}(1-z)e^{\mathsf{x}_1 \log(1-z)} \tag{4.49}$$

The above formulae for Φ already suggest its nature as something like a regularized value in a singular point and indeed it is the generating series [1]

$$\Phi(\mathsf{x}_0, \mathsf{x}_1) = \sum_{w\in A^\times} w\mathrm{Reg}_{z=1}L_w(z) \tag{4.50}$$

and analogously for Φ^{-1} and $z = 0$.

Another way of writing Φ is in terms of iterated integrals

$$\Phi(\mathsf{x}_0, \mathsf{x}_1) = \lim_{z\to 1^-}\left[1 + \sum_{w\in A^\times}\left(\int_0^z \Omega(w)\right)\right] \tag{4.51}$$

where $\Omega(w)$ is a combination of the 1-forms

$$\Omega_0 = \frac{\mathrm{d}s}{s} \qquad \Omega_1 = \frac{\mathrm{d}s}{s-1} \tag{4.52}$$

associated to x_0 and x_1 in the word w. It can be shown that this evaluates to a series of multiple zeta values such that there is a unique function $\zeta : \mathbb{Q}\langle A\rangle \to \mathbb{R}$, satisfying [1],[7]

$$\Phi(\mathsf{x}_0, \mathsf{x}_1) = \sum_{w \in A^\times} w\zeta(w) \tag{4.53}$$

$$\zeta(\mathsf{x}_0) = \zeta(\mathsf{x}_1) = 0 \tag{4.54}$$

$$\zeta(\mathsf{x}_0^{s_l-1}\mathsf{x}_1 ... \mathsf{x}_0^{s_1-1}\mathsf{x}_1) = (-1)^l \zeta(s_1, ..., s_l) \tag{4.55}$$

$$\zeta(w)\zeta(w') = \zeta(w \sqcup\!\sqcup\, w') \tag{4.56}$$

where the shuffle product enables us to rewrite all words in the form required in (4.55). The first few terms of (4.53) are

$$\Phi(\mathsf{x}_0, \mathsf{x}_1) = 1 - \zeta(2)[\mathsf{x}_0, \mathsf{x}_1] + \zeta(3)\Big([[\mathsf{x}_0, \mathsf{x}_1], \mathsf{x}_1] - [\mathsf{x}_0, [\mathsf{x}_0, \mathsf{x}_1]]\Big) + ... \tag{4.57}$$

The inverse can be found similarly. The integral in (4.51) is replaced by [7]

$$\int_z^0 \Omega(w) = (-1)^{|w|} \int_0^z \Omega(\widetilde{w}) \tag{4.58}$$

where $\widetilde{w}$ is w reversed, i.e. read from back to front and its letters have been exchanged, i.e. $\mathsf{x}_0 \to \mathsf{x}_1$ and $\mathsf{x}_1 \to \mathsf{x}_0$. We refrain from proving this but give only the examples

$$\zeta(\mathsf{x}_0^2\mathsf{x}_1) = -\zeta(3) \tag{4.59}$$

$$\zeta(\mathsf{x}_0\mathsf{x}_1^2) = \zeta(1, 2) = \zeta(3) \tag{4.60}$$

Hence, the inverse has in every summand a factor $(-1)^{|w|}$ and the first terms are:

$$\Phi^{-1}(\mathsf{x}_0, \mathsf{x}_1) = 1 - \zeta(2)[\mathsf{x}_0, \mathsf{x}_1] - \zeta(3)\Big([[\mathsf{x}_0, \mathsf{x}_1], \mathsf{x}_1] - [\mathsf{x}_0, [\mathsf{x}_0, \mathsf{x}_1]]\Big) + ... \tag{4.61}$$

If we were interested in integrals on the interval $(0, 1)$ over functions that have singularities in the boundary of the interval we could just read off the regularized values from (4.61) and its inverse. Of course, our integrals are on $(0, \infty)$ (where they have to be holomorphic) and we will see that our functions have singularities in 0 and -1 only! Hence, we will need to modify the results of this section to serve our needs.

4.2.3 Logarithmic regularization at infinity

Consider again the Knizhnik-Zamolodchikov equations (4.42) and remember that in the last section the coefficients of the series $\mathfrak{L}(z)$ were functions $L_w(z)$ with singularities in 0 and 1. So far, we found the regularized values of the functions $L_w(z)$ in these points. We actually want the regularized value at infinity of a function with singularities in 0 and -1 or, equivalently, of the functions $L_w(z)$ from the last section in $-\infty$. A change of variable $z = \frac{y-1}{y}$ gives us

$$\begin{aligned}\frac{\partial}{\partial y}\mathfrak{L}\left(\frac{y-1}{y}\right) &= \left(\frac{-\mathsf{x}_0 - \mathsf{x}_1}{y} + \frac{\mathsf{x}_0}{y-1}\right)\mathfrak{L}\left(\frac{y-1}{y}\right) \\ \frac{\partial}{\partial y}\widetilde{\mathfrak{L}}(y) &= \left(\frac{\widetilde{\mathsf{x}}_0}{y} + \frac{\widetilde{\mathsf{x}}_1}{y-1}\right)\widetilde{\mathfrak{L}}(y).\end{aligned} \tag{4.62}$$

Remark 5. *This change of variables permuting the points* 0, ± 1 *and* ∞ *is called Möbius transformation. Generally they are defined as a map*

$$\phi: \ z \to \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, ad - bc \neq 0 \tag{4.63}$$

Another Möbius transformation occurred in (4.58) of the last section. $z \to 1 - z$ *exchanged* x_0 *and* x_1 *as well as the upper and lower limit.*

Clearly, (4.62) is another KZ equation that we can solve for $y \in (0,1)$. Checking the transformation shows that $y = 0$ corresponds to $z = -\infty$. Thus

$$\widetilde{\Phi}^{-1}(\widetilde{\mathsf{x}}_0, \widetilde{\mathsf{x}}_1) = \begin{cases}\sum_{w\in A^\times} w \mathrm{Reg}_{z=-\infty} L_w(z) & \text{if } \Sigma = \{0,1\} \\ \sum_{w\in A^\times} w \mathrm{Reg}_{z=+\infty} L_w(z) & \text{if } \Sigma = \{0,-1\}\end{cases} \tag{4.64}$$

We have, up to words of weight 3:

$$\begin{aligned}&\widetilde{\Phi}^{-1}(\widetilde{\mathsf{x}}_0, \widetilde{\mathsf{x}}_1) = \Phi^{-1}(-\mathsf{x}_0 - \mathsf{x}_1, \mathsf{x}_0) = \\ &\qquad 1 + \zeta(2)(\mathsf{x}_1\mathsf{x}_0 - \mathsf{x}_0\mathsf{x}_1) + \zeta(3)(\mathsf{x}_0\mathsf{x}_1^2 + \mathsf{x}_1^2\mathsf{x}_0 - 2\mathsf{x}_1\mathsf{x}_0\mathsf{x}_1) + \ldots\end{aligned} \tag{4.65}$$

We can use this to conveniently read off all regularized values at infinity of functions in $L(\Sigma = \{0,-1\})$. A generalization to singularities σ_i is only needed for words of weight one, as we will see in section 4.4.4. It is, again immediately following from the definition of L_w:

$$\mathrm{Reg}_{z=\infty} L_{\mathsf{a}_0} = 0 \quad \mathrm{Reg}_{z=\infty} L_{\mathsf{a}_i} = -\log(-\sigma_i) \tag{4.66}$$

Remark 6. *Due to the enlargement of* $\mathcal{O}_\Sigma$ *to* $\mathcal{O}_\Sigma^+$ *in (4.36) there may be more singular loci in* $\sigma_i = \sigma_j$*. These singularities would usually cancel out because they occur in both parts of a partial fraction decomposition of the form*

$$\frac{1}{\sigma_i - \sigma_j}\left(\frac{1}{z-\sigma_i} - \frac{1}{z-\sigma_j}\right) \tag{4.67}$$

but as we want to look at the summands separately we have to deal with them by introducing a restricted regularization that maps superfluous singularities to 0*. This will not be needed in our calculations, so we will not elaborate further.*

4.3 Reduction algorithm for polynomials

Here we will outline a reduction algorithm that can be used to check whether the coefficients of the Taylor expansion of a given integral of the form (4.19) evaluate to rational linear combinations of multiple zeta values. [2]
The idea is to keep track of the singularities of the various polynomials occurring throughout the integrations.

4.3.1 The simple reduction algorithm

Let $S = \{P_1, ..., P_N\}$ be a set of polynomials in variables $\alpha_1, ..., \alpha_q$ with rational coefficients. If every polynomial is linear with respect to a variable α_i then we can use the notation of (4.23) to write $S = \{(P_1)_i + \alpha_i(P_1)^{(i)}, ..., (P_N)_i + \alpha_i(P_N)^{(i)}\}$.
Now, let

$$\widetilde{S}_{(i)} = \{P_1)_i, ..., (P_N)_i, (P_1)^{(i)}, ..., (P_N)^{(i)}, [(P_n)_i(P_m)^{(i)} - (P_n)^{(i)}(P_m)_i]_{1 \leq n < m \leq N}\} \tag{4.68}$$

be another set of polynomials in the same variables except α_i and $S_{(i)}$ the set of irreducible factors of the polynomials in $\widetilde{S}_{(i)}$. This means that if it is possible to write a polynomial in $\widetilde{S}_{(i)}$ as a product of two or more polynomials of a lower degree (e.g. $x^2 - 1 = (x-1)(x+1)$), $S_{(i)}$ contains these factors instead of the original reducible polynomial but if all polynomials in $\widetilde{S}_{(i)}$ are already irreducible then the two sets are equal. Furthermore we can drop all elements that are constants or monomials, as these are irrelevant. (Ultimately, this algorithm serves to keep track of singularities, so multiplicative constants and monomials in a denominator do not carry useful information for us.)
This process can be repeated if all polynomials in $S_{(i)}$ are linear in at least one variable. The notation is then $S_{(i_1,i_2)}$, $S_{(i_1,i_2,i_3)}$ and so on.
If there is a sequence $(i_1, i_2, ..., i_q)$ such that the polynomials can be reduced with respect to all q variables, then we call the set S *simply reducible.*

4.3.2 The Fubini reduction algorithm

Due to Fubini's theorem the outcome of the integration should be independent of the order of the variables. The condition for simple reducibility only required at least one sequence to exist such that the algorithm can be repeated through all variables. This means that it is very well possible for the algorithm to give one result in one sequence but a different result in another, which should not be the case. Thus, we need to amend the algorithm in such a way that its result is independent of the chosen sequence of reduction.

The basic idea behind the Fubini reduction algorithm is to apply the simple reduction algorithm in all possible sequences and take the intersections of the resulting sets. Let all polynomials in S be linear with respect to variables α_{i_1} and α_{i_2}. Then we define recursively:

$$
\begin{aligned}
S_{[i_1,i_2]} &:= S_{(i_1,i_2)} \cap S_{(i_2,i_1)} \\
S_{[i_1,i_2,i_3]} &:= S_{[i_1,i_2](i_3)} \cap S_{[i_1,i_3](i_2)} \cap S_{[i_2,i_3](i_1)} \\
&\ldots
\end{aligned} \tag{4.69}
$$

Should at any step occur a polynomial not linear in a variable then its reduction with respect to that variable is not defined and thus omitted from the intersection. If this is the case for all sets in the intersection then the resulting set is not defined. The set S is called *Fubini reducible* if at every step all polynomials in $S_{[i_1,\ldots]}$ are linear in at least one of the remaining variables and if there is a sequence of q variables so that it is possible to reduce the polynomials with respect to all variables. Whether the algorithm terminates at all may still depend on the order of variables but the end result for all sequences that do let the algorithm go through will now be the same.

4.3.3 Ramification

As we have seen in section 4.2 , integrals over polynomials with singularities in 0, -1 and ∞ will result in multiple polylogarithms, or hyperlogarithms at the penultimate stage which in turn become multiple zeta values after the final integration. Hence, for the sequence of sets

$$
S_{(i_1)}, S_{[i_1,i_2]}, \ldots, S_{[i_1,\ldots,i_q]} \tag{4.70}
$$

we need to check the polynomials $P_1, \ldots, P_{M_k}$ in every set $S_{[i_1,\ldots,i_k]}$. At the k-th stage we have polynomials $P_n = a_n \alpha_{n_{k+1}} + b_n,\ 1 \leq n \leq M_k, a_n \neq 0$ and call the set of singularities

$$
\Sigma_{\alpha_k} = \left\{ -\frac{b_n}{a_n}, n \in \{1, \ldots,\ M_k\} \right\}. \tag{4.71}
$$

The set Σ_{α_k} is called *unramified* if

$$
\lim_{\alpha_q \to 0} \left(\lim_{\alpha_{q-1} \to 0} \left(\ldots \left(\lim_{\alpha_{k+2} \to 0} \Sigma_{\alpha_k} \right) \ldots \right) \right) \subseteq \{0, -1, \infty\} \tag{4.72}
$$

The sequence of sets (4.70) is called unramified if Σ_{α_k} is unramified for all $1 \leq k \leq q-1$ and the set S is unramified if it is Fubini reducible and there is a sequence $(i_1, \ldots,\ i_q)$ such that the corresponding sequence of sets is unramified.

Finally, it follows that for any broken primitive divergent graph G the coefficients of the Taylor expansion of the integral $I(G)$ (4.19) are multiple zeta values if the set $S_G = \{U_{\widetilde{G}}\}$ is Fubini reducible and unramified.

4.4 The Wheel with Three Spokes

Having prepared all necessary theoretical tools, we can now calculate again the period of the wheel with three spokes. We shall start from (4.19)

$$
\int_{H_\lambda} \frac{\prod_{l=1}^{L+1} \alpha_l^{a_l-1}}{U_{\widetilde{G}}^{2-\epsilon}} \Omega_L \mathrm{d}\alpha_{L+1}
$$

The first term of the Taylor expansion around $\epsilon = 0$ is

$$T_G = \int_{H_\lambda} \frac{\Omega_L \mathrm{d}\alpha_{L+1}}{U_{\widetilde{G}}^2}. \tag{4.73}$$

To find $U_{\widetilde{G}}$ we use (4.20)

$$U_{\widetilde{G}} = V_G + \alpha_{L+1} U_G,$$

assuming that the edges labeled 1 to 5 belong to the master two loop diagram (left in Fig. 1.2) and the 6th edge closes the external legs to build the wheel with three spokes.

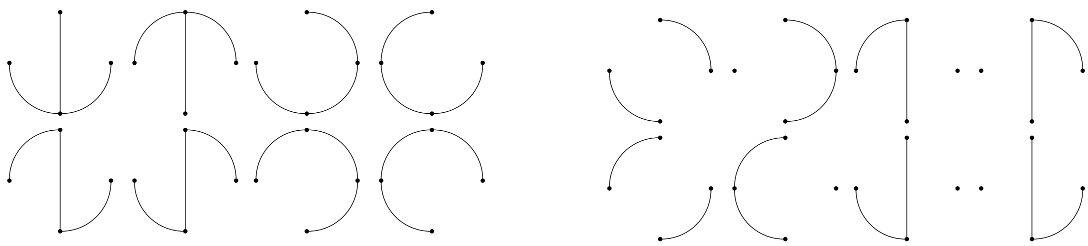


Figure 4.1: The spanning trees and 2-forests of the two-loop graph.

As the definitions of the polynomials involved the spanning trees and 2-forests respectively, we have to find these at first. For our two-loop diagram there are exactly 8 of each kind, depicted in Fig. 4.1. Let the edges be labeled as in Fig. 4.2. The graph polynomials are then

$$\begin{aligned} U_G &= \alpha_1\alpha_2 + \alpha_4\alpha_5 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_2\alpha_5 + \alpha_1\alpha_4 + \alpha_3\alpha_5 + \alpha_3\alpha_4 \\ &= (\alpha_1 + \alpha_5)(\alpha_2 + \alpha_4) + \alpha_3(\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5) \end{aligned} \tag{4.74}$$

$$\begin{aligned} V_G &= \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_5 + \alpha_2\alpha_4\alpha_5 + \alpha_1\alpha_4\alpha_5 + \alpha_2\alpha_3\alpha_5 + \alpha_2\alpha_3\alpha_4 \\ &\qquad + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_2\alpha_5 \qquad (4.75) \\ &= \alpha_3(\alpha_1 + \alpha_2)(\alpha_4 + \alpha_5) + (\alpha_2\alpha_4\alpha_5 + \alpha_1\alpha_4\alpha_5 + \alpha_1\alpha_2\alpha_5 + \alpha_1\alpha_2\alpha_4) \end{aligned}$$

Note that they are both linear in all variables.

4.4.1 Checking reducibility and ramification

The set of polynomials to be reduced is $S = \{U_{\widetilde{G}}\}$. Clearly, we have $S_{(6)} = \{U_G, V_G\}$. Reducing again with respect to α_1 gives

$$\widetilde{S}_{(6,1)} = \{U_1, U^{(1)}, V_1, V^{(1)}, U_1V^{(1)} - U^{(1)}V_1\} \tag{4.76}$$

The last polynomial is quadratic in every variable but it factorizes as the square of a linear polynomial (see 4.91), so the set of *irreducible* polynomials is

$$S_{(6,1)} = \{U_1, U^{(1)}, V_1, V^{(1)}, D\} \tag{4.77}$$

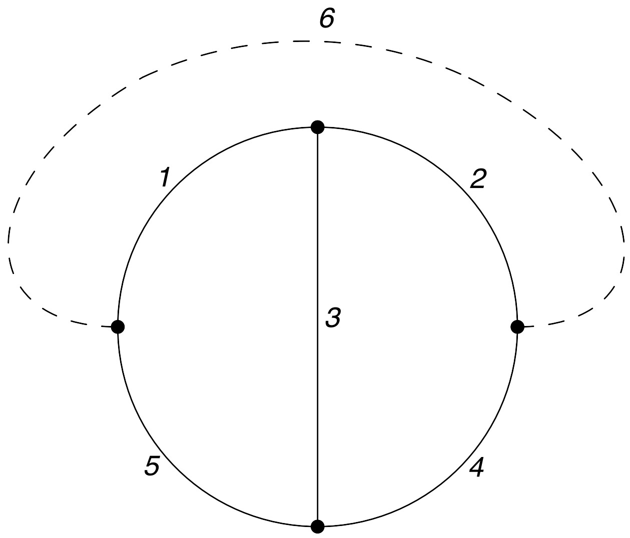


Figure 4.2: A labeled two-loop graph and the 6th edge

Setting one of the edge parameters to 1 and going on with the reduction would lead to a polynomial with singularities in 0, 1 and -1. This is not enough, so we apply the Fubini reduction algorithm. After setting $\alpha_5 = 1$ and $S' = S|_{\alpha_5=1}$ the sets are

$$\begin{aligned}
S'_{[6]} &= \{\alpha_1\alpha_2 + \alpha_1\alpha_4 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_2 + \alpha_3 + \alpha_4, \\
&\quad \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_4 + \alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_1\alpha_4 + \alpha_1\alpha_2\} \\
S'_{[6,1]} &= \{\alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_3\alpha_4 + \alpha_3 + \alpha_4, \\
&\quad \alpha_2\alpha_4 + \alpha_3\alpha_4 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_3\alpha_4 + \alpha_2 + \alpha_3 + \alpha_4\} \\
S'_{[6,1,2]} &= \{\alpha_4 + 1, \alpha_3 + 1, \alpha_3\alpha_4 + \alpha_3 + \alpha_4, \alpha_3 + \alpha_4\} \\
S'_{[6,1,2,3]} &= \{\alpha_4 + 1\}
\end{aligned} \tag{4.78}$$

We see that the set S' is Fubini reducible for the sequence $[6, 1, 2, 3, 4]$.

To check for ramification we first obtain the sets of singularities from the above sets:

$$\begin{aligned}
\Sigma_1 &= \left\{0, -\frac{\alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_2 + \alpha_3 + \alpha_4}{\alpha_2 + \alpha_3 + \alpha_4}, -\frac{\alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_2\alpha_3\alpha_4}{\alpha_2 + \alpha_3 + \alpha_4 + \alpha_2\alpha_4 + \alpha_3\alpha_4}\right\} \\
\Sigma_2 &= \left\{0, -\frac{\alpha_3\alpha_4 + \alpha_3 + \alpha_4}{\alpha_3 + 1}, -\alpha_3 - \alpha_4, -\frac{\alpha_3\alpha_4 + \alpha_3 + \alpha_4}{\alpha_4 + 1}, -\alpha_3\alpha_4 - \alpha_3 - \alpha_4\right\} \\
\Sigma_3 &= \left\{0, -1, -\frac{\alpha_4}{\alpha_4 + 1}, -\alpha_4\right\} \qquad \Sigma_4 = \{0, -1\}
\end{aligned} \tag{4.79}$$

It is easy to check that taking the limits $\alpha_2 \to 0$, $\alpha_3 \to 0$, $\alpha_4 \to 0$, in that order, according to (4.72), indeed yields subsets of $\{0, -1, \infty\}$ only. Therefore, S' is unramified and all coefficients of the corresponding integral's Taylor expansion, particularly the first term (4.73), will be MZVs.

4.4.2 Common integration in linear variables

We choose for H_λ the hyperplane $\alpha_5 = 1$. The integration over α_6 can now easily be executed, as U_G and V_G depend only on α_1 to α_5.

$$T_G = \int_{\alpha_5=1} \int_0^\infty \frac{\mathrm{d}\alpha_6}{(V_G + \alpha_6 U_G)^2} \Omega_5 = \int_{\alpha_5=1} \frac{1}{U_G V_G} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \tag{4.80}$$

Choosing α_1 as the next variable for integration we can write

$$T_G = \int_{\alpha_5=1} \frac{1}{(U_1 + \alpha_1 U^{(1)})(V_1 + \alpha_1 V^{(1)})} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \tag{4.81}$$

which can be decomposed into partial fractions.

$$T_G = \int_{\alpha_5=1} \frac{1}{U^{(1)} V_1 - U_1 V^{(1)}} \left(\frac{U^{(1)}}{U_1 + \alpha_1 U^{(1)}} - \frac{V^{(1)}}{V_1 + \alpha_1 V^{(1)}} \right) \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \tag{4.82}$$

This is again easily integrable with respect to α_1 and delivers

$$\begin{aligned} T_G &= \int_{\alpha_5=1} \frac{\left[\log\left(U_1 + \alpha_1 U^{(1)}\right) - \log\left(V_1 + \alpha_1 V^{(1)}\right)\right]_0^\infty}{U^{(1)} V_1 - U_1 V^{(1)}} \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \\ &= \int_{\alpha_5=1} \frac{\log\left(\frac{U_1 + \alpha_1 U^{(1)}}{V_1 + \alpha_1 V^{(1)}}\right)\Big|_{\alpha_1 \to \infty} - \log\left(\frac{U_1}{V_1}\right)}{U^{(1)} V_1 - U_1 V^{(1)}} \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \\ &= \int_{\alpha_5=1} \frac{\log U^{(1)} - \log V^{(1)} - \log U_1 + \log V_1}{U^{(1)} V_1 - U_1 V^{(1)}} \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \end{aligned} \tag{4.83}$$

At this step, the next integration is not as openly visible as before. We have to make use of the *Dodgson identity.*It basically says that we can rewrite the denominator of (4.83) as the square of a polynomial that is again linear in all remaining variables.

$$D^2 = U_1 V^{(1)} - U^{(1)} V_1 \tag{4.84}$$

With this new polynomial we can express (4.83) as

$$\begin{aligned} T_G = \int_{\alpha_5=1} \frac{-1}{(D_2 + \alpha_2 D^{(2)})^2} &\Big[\log(U_2^{(1)} + \alpha_2 U^{(12)}) - \log(V_2^{(1)} + \alpha_2 V^{(12)}) \\ &- \log(U_{12} + \alpha_2 U_1^{(2)}) + \log(V_{12} + \alpha_2 V_1^{(2)})\Big] \mathrm{d}\alpha_2 \mathrm{d}\alpha_3 \mathrm{d}\alpha_4 \end{aligned} \tag{4.85}$$

Our integral is now a sum of four integrals of the form

$$\int_0^\infty \frac{\log(q + px)}{(s + rx)^2} \mathrm{d}x \tag{4.86}$$

which can via partial integration and partial fraction decomposition be found to evaluate to

$$\frac{\log q}{rs} + \frac{p(\log p - \log q - \log r + \log s)}{r(ps - qr)} \tag{4.87}$$

So far we have made no assumptions about our polynomials U and V other than linearity in all variables. Making use of the notation $\{p, q|r, s\}$ for (4.87) we can hence state as a general result that (4.73) is equal to

$$\begin{aligned}\int_{\alpha_L=1} \Big(&-\{U^{(ij)}, U_j^{(i)}|D^{(j)}, D_j\} + \{V^{(ij)}, V_j^{(i)}|D^{(j)}, D_j\} \\ &+\{U_i^{(j)}, U_{ij}|D^{(j)}, D_j\} - \{V_i^{(j)}, V_{ij}|D^{(j)}, D_j\}\Big)\ \mathrm{d}\alpha_1 ... \widehat{\mathrm{d}\alpha_i} ... \widehat{\mathrm{d}\alpha_j} ... \mathrm{d}\alpha_{L-1}\end{aligned} \tag{4.88}$$

It is possible to continue with this formula as long as all polynomials are linear but we will see that this is not the case for our specific integral.

4.4.3 Primitives in the polylogarithm algebra

To proceed, we need to calculate $U^{(12)}$, $U_2^{(1)}$ etc. using the formulae (4.23):

$$\begin{aligned} U &= \alpha_1\alpha_2 + \alpha_4 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_2 + \alpha_1\alpha_4 + \alpha_3 + \alpha_3\alpha_4 \\ U_1 &= \alpha_4 + \alpha_2\alpha_3 + \alpha_2 + \alpha_3 + \alpha_3\alpha_4 \\ U^{(1)} &= \alpha_2 + \alpha_3 + \alpha_4 \\ U_{12} &= \alpha_3 + \alpha_4 + \alpha_3\alpha_4 \qquad U_2^{(1)} = \alpha_3 + \alpha_4 \\ U_1^{(2)} &= 1 + \alpha_3 \qquad U^{(12)} = 1 \end{aligned} \tag{4.89}$$

$$\begin{aligned} V &= \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_3 + \alpha_2\alpha_4 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_2 \\ V_1 &= \alpha_2\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 \\ V^{(1)} &= \alpha_3\alpha_4 + \alpha_3 + \alpha_4 + \alpha_2\alpha_4 + \alpha_2 \\ V_{12} &= 0 \qquad V_2^{(1)} = \alpha_3\alpha_4 + \alpha_3 + \alpha_4 \\ V_1^{(2)} &= \alpha_4 + \alpha_3 + \alpha_3\alpha_4 \qquad V^{(12)} = \alpha_4 + 1 \end{aligned} \tag{4.90}$$

$$\begin{aligned} D^2 &= U_1V^{(1)} - U^{(1)}V_1 \\ &= (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_2\alpha_3 + \alpha_3\alpha_4)(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_2\alpha_4 + \alpha_3\alpha_4) \\ &\qquad - (\alpha_2 + \alpha_3 + \alpha_4)(\alpha_2\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4) \\ &= (\alpha_2^2 + \alpha_3^2 + \alpha_4^2 + 2\alpha_3\alpha_4^2 + 2\alpha_3^2\alpha_4 + 2\alpha_3\alpha_4 + 2\alpha_2\alpha_3 + 2\alpha_2\alpha_4 + 2\alpha_2\alpha_3\alpha_4 + \alpha_3^2\alpha_4^2) \\ &= (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_3\alpha_4)^2 \\ D_2 &= \alpha_3 + \alpha_4 + \alpha_3\alpha_4 \qquad D^{(2)} = 1 \end{aligned} \tag{4.91}$$

Inserting these results into (4.85) and collecting all terms results in

$$\begin{aligned}T_G = \int_0^\infty \int_0^\infty \frac{1}{\alpha_3\alpha_4}&(\log(\alpha_3+1)+\log(\alpha_3+\alpha_4)-\log(\alpha_3+\alpha_4+\alpha_3\alpha_4))\\ &+\frac{(\alpha_4+1)\log(\alpha_4+1)-\log(\alpha_3+1)-\alpha_4\log(\alpha_3+\alpha_4)}{\alpha_4(\alpha_3+\alpha_4+\alpha_3\alpha_4)}\mathrm{d}\alpha_3\mathrm{d}\alpha_4.\end{aligned} \tag{4.92}$$

As mentioned before, we have now reached an integral that we cannot simply integrate but will need us to make use of polylogarithms. Let us rename variables to simplify notation, say $y=\alpha_3$, $x=\alpha_4$. We can then introduce the sets with corresponding alphabets

$$\begin{aligned}\Sigma_x &= \{0,-1\} & X &= [\mathsf{x}_0,\mathsf{x}_1]\\ \Sigma_y &= \left\{0,-1,-x,-\frac{x}{x+1}\right\} & Y &= [\mathsf{y}_0,\mathsf{y}_1,\mathsf{y}_2,\mathsf{y}_3]\end{aligned} \tag{4.93}$$

Using these alphabets and (4.27) it is possible to write T_G in terms of polylogarithms.

$$\begin{aligned}T_G &= \int_0^\infty\int_0^\infty \frac{1}{xy}\left(L_{\mathsf{y}_1}(y)+L_{\mathsf{y}_2}(y)-L_{\mathsf{y}_3}(y)\right)\\ &\qquad +\frac{(x+1)L_{\mathsf{x}_1}(x)-L_{\mathsf{y}_1}(y)-x(L_{\mathsf{y}_2}(y)+L_{\mathsf{x}_0}(x))}{x(x+y+xy)}\mathrm{d}y\mathrm{d}x\\ &= \int_0^\infty\int_0^\infty \frac{1}{xy}\left(L_{\mathsf{y}_1}(y)+L_{\mathsf{y}_2}(y)-L_{\mathsf{y}_3}(y)\right)+\frac{1}{x}\frac{1}{y+\frac{x}{x+1}}L_{\mathsf{x}_1}(x)\\ &\qquad -\frac{1}{x(x+1)}\frac{1}{y+\frac{x}{x+1}}\left(L_{\mathsf{y}_1}(y)+xL_{\mathsf{y}_2}(y)+xL_{\mathsf{x}_0}(x)\right)\mathrm{d}y\mathrm{d}x\end{aligned} \tag{4.94}$$

In the second step we have written the prefactors in a way that allows us to take primitives according to section 4.2.1.

$$\begin{aligned}T_G = \int_0^\infty \frac{1}{x}&\mathrm{Reg}_{y=\infty}\Big(L_{\mathsf{y}_0\mathsf{y}_1}(y)+L_{\mathsf{y}_0\mathsf{y}_2}(y)-L_{\mathsf{y}_0\mathsf{y}_3}(y)+L_{\mathsf{x}_1}(x)L_{\mathsf{y}_3}(y)\Big)\\ &-\frac{1}{x(x+1)}\mathrm{Reg}_{y=\infty}\Big(L_{\mathsf{y}_3\mathsf{y}_1}(y)+xL_{\mathsf{y}_3\mathsf{y}_2}(y)+xL_{\mathsf{x}_0}(x)L_{\mathsf{y}_3}(y)\Big)\,\mathrm{d}x\end{aligned} \tag{4.95}$$

4.4.4 Regularized values in two variables

For polylogarithms we can calculate the regularized values according to section 4.2.3. Specifically we find the values for words of weight one from (4.66):

$$\begin{aligned}&\mathrm{Reg}_{x=\infty}L_{\mathsf{x}_0}(x)=\mathrm{Reg}_{y=\infty}L_{\mathsf{y}_0}(y)=0\\ &\mathrm{Reg}_{x=\infty}L_{\mathsf{x}_1}(x)=\mathrm{Reg}_{y=\infty}L_{\mathsf{y}_1}(y)=0\\ &\mathrm{Reg}_{y=\infty}L_{\mathsf{y}_2}(y)=-\log(x)=-L_{\mathsf{x}_0}(x)\\ &\mathrm{Reg}_{y=\infty}L_{\mathsf{y}_3}(y)=-\log(x)+\log(x+1)=L_{\mathsf{x}_1}(x)-L_{\mathsf{x}_0}(x)\end{aligned} \tag{4.96}$$

Higher weight regularized values in one variable, i.e. of hyperlogarithms in the words y_0 and y_1, whose corresponding singularities are the constants 0 and -1 can be read off (4.65). For now, we only need

$$\mathrm{Reg}_{y=\infty} L_{\mathsf{y}_0\mathsf{y}_1}(y) = -\zeta(2) \tag{4.97}$$

For multiple polylogarithms we need to use a trick. Say, we want to find the regularized value of $L_w(y)$ with $|w| = 2$ and at least one letter in w is not y_0 or y_1. We can differentiate with respect to x, thereby lowering the weight by one. For the regularized value of this derivative the formulae (4.96) can be used. To get the desired regularized value of the original function one now simply has to take the primitive with respect to x.

$$\mathrm{Reg}_{y=\infty} L_w(y) = \int \mathrm{Reg}_{y=\infty} \left(\frac{\partial}{\partial x} L_w(y) \right) \mathrm{d}x \tag{4.98}$$

The constant of integration is $\mathrm{Reg}_{y=\infty}\mathrm{Reg}_{x=0} L_w(y)$. The x-dependent singularities should reduce to 0 or -1. Thus, the regularized value at infinity can be read of (4.65) just like before and is also a multiple zeta value.

To be able to compute the derivative with respect to x in the first place, one has to employ the same trick with respect to y first, i.e.:

$$\frac{\partial}{\partial x} L_w(y) = \int \left(\frac{\partial}{\partial x} \frac{\partial}{\partial y} L_w(y) \right) \mathrm{d}y \tag{4.99}$$

Initially it is unclear how the derivative with respect to x of a hyperlogarithm in y - but with implicit x-dependence - can be computed. Differentiation with respect to y unveils the x-dependence in $\sigma_i = \sigma_i(x)$. This should become clearer in the first example:

$$\begin{aligned}
\frac{\partial}{\partial x} L_{\mathsf{y}_0\mathsf{y}_2}(y) &= \int \left(\frac{\partial}{\partial x} \frac{\partial}{\partial y} L_{\mathsf{y}_0\mathsf{y}_2}(y) \right) \mathrm{d}y \\
&= \int \left(\frac{\partial}{\partial x} \frac{1}{y} L_{\mathsf{y}_2}(y) \right) \mathrm{d}y \\
&= \int \left(\frac{\partial}{\partial x} \frac{1}{y} (\log(y+x) - \log(x)) \right) \mathrm{d}y \\
&= \int \left(\frac{1}{y} \left(\frac{1}{x+y} - \frac{1}{x} \right) \right) \mathrm{d}y \\
&= \int \frac{-1}{x(x+y)} \mathrm{d}y \\
&= -\frac{1}{x} L_{\mathsf{y}_2}(y)
\end{aligned} \tag{4.100}$$

With that result the regularized value is

$$
\begin{aligned}
\mathrm{Reg}_{y=\infty} L_{\mathsf{y}_0\mathsf{y}_2}(y) &= \int \mathrm{Reg}_{y=\infty}\left(\frac{\partial}{\partial x} L_{\mathsf{y}_0\mathsf{y}_2}(y)\right) \mathrm{d}x \\
&= \int -\frac{1}{x} \mathrm{Reg}_{y=\infty} L_{\mathsf{y}_2}(y) \mathrm{d}x \\
&= \int \frac{\log(x)}{x} \mathrm{d}x \\
&= L_{\mathsf{x}_0^2}(x) = \frac{1}{2}\log^2(x).
\end{aligned}
\tag{4.101}
$$

The last step and the vanishing constant of integration follow directly from the definition. Similarly, we can compute the remaining regularized values:

$$
\begin{aligned}
\mathrm{Reg}_{y=\infty} L_{\mathsf{y}_0\mathsf{y}_3}(y) &= L_{\mathsf{x}_0^2}(x) - L_{\mathsf{x}_1\mathsf{x}_0}(x) + L_{\mathsf{x}_1^2}(x) - L_{\mathsf{x}_0\mathsf{x}_1}(x) \\
\mathrm{Reg}_{y=\infty} L_{\mathsf{y}_3\mathsf{y}_2}(y) &= L_{\mathsf{x}_0\mathsf{x}_1}(x) - L_{\mathsf{x}_1^2}(x) + L_{\mathsf{x}_0^2}(x) \\
\mathrm{Reg}_{y=\infty} L_{\mathsf{y}_3\mathsf{y}_1}(y) &= L_{\mathsf{x}_1\mathsf{x}_0}(x) - L_{\mathsf{x}_1^2}(x) - \zeta(2)
\end{aligned}
\tag{4.102}
$$

For the two products of hyperlogarithms in x and y we remember the shuffle relation (4.30):

$$
\begin{aligned}
\mathrm{Reg}_{y=\infty} L_{\mathsf{x}_1}(x) L_{\mathsf{y}_3}(y) &= L_{\mathsf{x}_1}(x)(L_{\mathsf{x}_1}(x) - L_{\mathsf{x}_0}(x)) \\
&= L_{\mathsf{x}_1 ш \mathsf{x}_1}(x) - L_{\mathsf{x}_1 ш \mathsf{x}_0}(x) \\
&= 2L_{\mathsf{x}_1^2}(x) - L_{\mathsf{x}_1\mathsf{x}_0}(x) - L_{\mathsf{x}_0\mathsf{x}_1}(x)
\end{aligned}
\tag{4.103}
$$

$$
\mathrm{Reg}_{y=\infty} L_{\mathsf{x}_0}(x) L_{\mathsf{y}_3}(y) = L_{\mathsf{x}_1\mathsf{x}_0}(x) + L_{\mathsf{x}_0\mathsf{x}_1}(x) - 2L_{\mathsf{x}_1^2}(x)
$$

We now have everything we need to finally evaluate T_G. Inserting all the results yields

$$
\begin{aligned}
T_G = \int_0^\infty & \frac{1}{x}\Big(-\zeta(2) + L_{\mathsf{x}_0^2}(x) - L_{\mathsf{x}_0^2}(x) + L_{\mathsf{x}_1\mathsf{x}_0}(x) - L_{\mathsf{x}_1^2}(x) + L_{\mathsf{x}_0\mathsf{x}_1}(x) \\
& \quad + 2L_{\mathsf{x}_1^2}(x) - L_{\mathsf{x}_1\mathsf{x}_0}(x) - L_{\mathsf{x}_0\mathsf{x}_1}(x)\Big) \\
& - \frac{1}{x(x+1)}\Big(L_{\mathsf{x}_1\mathsf{x}_0}(x) - L_{\mathsf{x}_1^2}(x) - \zeta(2) + x(L_{\mathsf{x}_0\mathsf{x}_1}(x) - L_{\mathsf{x}_1^2}(x) + L_{\mathsf{x}_0^2}(x)) \\
& \quad + x(L_{\mathsf{x}_1\mathsf{x}_0}(x) + L_{\mathsf{x}_0\mathsf{x}_1}(x) - 2L_{\mathsf{x}_1^2}(x))\Big)\, \mathrm{d}x.
\end{aligned}
\tag{4.104}
$$

Expanding the second summand and collecting all terms results in

$$T_G = \int_0^\infty \frac{1}{x}\Big(L_{\mathsf{x}_1^2}(x) - \zeta(2)\Big) + \left(\frac{1}{x+1} - \frac{1}{x}\right)\Big(L_{\mathsf{x}_1\mathsf{x}_0}(x) - L_{\mathsf{x}_1^2}(x) - \zeta(2)\Big)$$

$$- \frac{1}{x+1}\Big(2L_{\mathsf{x}_0\mathsf{x}_1}(x) - 3L_{\mathsf{x}_1^2}(x) + L_{\mathsf{x}_0^2}(x) + L_{\mathsf{x}_1\mathsf{x}_0}(x)\Big)\,\mathrm{d}x$$

$$= \int_0^\infty \frac{1}{x}\Big(2L_{\mathsf{x}_1^2}(x) - L_{\mathsf{x}_1\mathsf{x}_0}(x)\Big) + \frac{1}{x+1}\Big(2L_{\mathsf{x}_1^2}(x) - \zeta(2) - 2L_{\mathsf{x}_0\mathsf{x}_1}(x) - L_{\mathsf{x}_0^2}(x)\Big) \tag{4.105}$$

and taking primitives one last time leaves us with regularized values that we can again find in(4.65):

$$T_G = \mathrm{Reg}_{x=\infty}\Big(2L_{\mathsf{x}_0\mathsf{x}_1^2}(x) - L_{\mathsf{x}_0\mathsf{x}_1\mathsf{x}_0}(x) + 2L_{\mathsf{x}_1^3}(x) - \zeta(2)L_{\mathsf{x}_1}(x) - 2L_{\mathsf{x}_1\mathsf{x}_0\mathsf{x}_1}(x) - L_{\mathsf{x}_1\mathsf{x}_0^2}(x)\Big)$$

$$= 2\zeta(3) - 0 + 0 - 0 - 2(-2)\zeta(3) - 0 = 6\zeta(3). \tag{4.106}$$

Remember that our integral had a prefactor consisting of four gamma functions depending on a, h and D. Inserting our values yields again

$$I = \frac{6\zeta(3)}{\epsilon} + \mathcal{O}(\epsilon^0). \tag{4.107}$$

Chapter 5

Conclusion

We have shown in three different ways that the period of the wheel with three spokes is $6\zeta(3)$. Integration by parts only required basic mathematics but especially for more complicated graphs the calculations can become quite tedious. More fatally, there is no general rule to find a function F that allows us to split the integral into easier integrable parts and for many graphs such a suitable function might not even exist. The Gegenbauer polynomial x-space technique, on the other hand, follows more or less always the same steps: Translate the propagators using (3.1), integrate via (3.2) and evaluate the resulting series. Furthermore, generalizations to the formulae used in this work exist, e.g. for products of polynomials of the form $C_n^\lambda(\hat{x}_1 \cdot \hat{q})C_m^\lambda(\hat{x}_2 \cdot \hat{q})C_l^\lambda(\hat{x}_3 \cdot \hat{q})$. The series obtained in the last step can be problematic though. They can be 3-fold, 4-fold or more and often contain dozens of rather complicated terms such that they can only be evaluated numerically. The method of chapter 4 allows for a wide variety of graphs to be computed. Moreover, analyzing it yields an algorithm to check beforehand whether the result will be an MZV. The technique reliably delivers the periods of planar graphs up to 5 loops and a generalization of the ramification condition (4.72) allows for non-planar graphs with crossing number 1.[2] It fails though, for five loop graphs with crossing number 2 like the graphs obtained from $K_{3,4}$ (Fig. A.3) by cutting one of its edges. The algorithm offers a handful of starting points for possible extensions. First of all, when factorizing the polynomials, the outcome may depend on the chosen class of numbers and larger classes than the rationals e.g. rationals with imaginary factors or even the algebraic numbers $\overline{\mathbb{Q}}$ might allow for polynomials that are not reducible over $\mathbb{Q}$ like $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$. Another possibility is to split the integral into parts with different sets of singularities, all of which have the necessary properties, while their union does not. Although we showed the outcome to be independent of the chosen hyperplane H_λ, designating one edge over the others could be seen as breaking the graphs symmetry, so there might be cases where it is worthwhile to run the algorithm with a set λ containing more than one edge and see if more graphs become computable. Lastly, the relevant algebraic-geometric property of the objects defined by the polynomials (their genus) can under circumstances be the same for linear and quadratic polynomials. Thus, conditionally permitting quadratic terms might extend the class of computable graphs.

Independent of the limitations of the algorithm at hand, the framework of polylogarithm algebras has in recent years shown to be fruitful in various applications to the theory of Feynman graphs.

Danksagung

Ich danke Professor Kreimer für die ausgezeichnete Betreuung sowie Erik Panzer und Oliver Schnetz für die vielen beantworteten Fragen und gegebenen Hinweise. Weiterhin möchte ich meinen Eltern und Xiaorui Wang für ihre Unterstützung in allen Lebenslagen, sowie Claudia Zielke für ihr gewissenhaftes Korrekturlesen danken.

Bibliography

[1] Francis Brown. Iterated integrals in quantum field theory. `http://www.math.jussieu.fr/~brown/ColombiaNotes7.pdf`, 2009. Lecture Notes.

[2] Francis Brown. The massless higher-loop two-point function. *Communications in Mathematical Physics*, 287:925–958, 2009. `arXiv:0804.1660` [math.AG].

[3] K.G. Chetyrkin, A.L. Kataev, and F.V. Tkachov. New approach to evaluation of multiloop feynman integrals: The gegenbauer polynomial x-space technique. *Nuclear Physics*, B174:345–377, 1980.

[4] John C. Collins. *Renormalization*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1986.

[5] Arthur Erdelyi, editor. *Higher Transcendental Functions*, volume 1. McGraw-Hill, New York, 1953.

[6] Claude Itzykson and Jean Bernard Zuber. *Quantum Field Theory*. Dover Publications Inc., Mineola, NY, Dover edition, 2005.

[7] Christian Kassel. *Quantum Groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[8] Maxim Kontsevich and Don Zagier. Periods. In Björn Engquist and Wilfried Schmid, editors, *Mathematics Unlimited - 2001 and Beyond*. Springer-Verlag, Berlin, 2001.

[9] Dirk Kreimer. *Knots and Feynman Diagrams*, volume 13 of *Cambridge Lecture Notes in Physics*. Cambridge University Press, Cambridge, 2000.

[10] Oliver Schnetz. Quantum periods: A census of ϕ^4-transcendentals. *Communications in Number Theory and Physics*, 4:1–48, 2010. `arXiv:0801.2856v2` [hep-th].

[11] Vladimir A. Smirnov. *Renormalization and Asymptotic Expansions*, volume 14 of *Progress in Physics*. Birkhäuser Verlag, Basel, 1991.

[12] Vladimir A. Smirnov. *Evaluating Feynman Integrals*, volume 211 of *Springer Tracts in Modern Physics*. Springer-Verlag, Berlin, 2004.

[13] Richard P. Stanley. Topics in algebraic combinatorics. `http://math.mit.edu/~rstan/algcomb/algcomb.pdf`. Version of 1 February 2013.

Appendix A

Basic Graph Theoretical Definitions

This appendix shall serve as a collection of definitions and illustrating examples for some graph theoretical objects that are widely used thoughout this work and in the literature. For a more comprehensive account the reader is referred to textbooks on the topic.

A *graph* G is an ordered pair (V, E) of sets V, called *vertices* and E, called *edges* such that $E \subseteq [V]^2$ and $V \cap E = \emptyset$. A vertex v is said to be *incident* with an edge e if $v \in e$. Two vertices x,y are called *adjacent* if there is an edge $e = \{x, y\}$. A graph $G' = (V' \subseteq V, E' \subseteq E) \subseteq G$ is called a *subgraph* of G. If $V' = V$, then G' is called a *spanning subgraph* of G. A *path* of length k is a non-empty graph

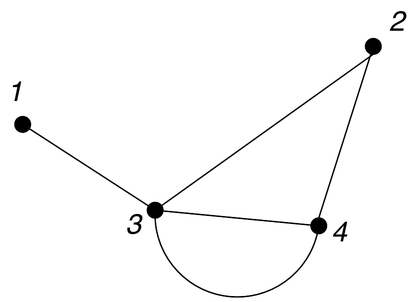


Figure A.1: A graph with edges $E = \{\{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{3, 4\}\}$ and vertices $V = \{1, 2, 3, 4\}$

$P = (\{x_0, x_1, ..., x_k\}, \{\{x_0, x_1\}, \{x_1, x_2\}, ..., \{x_{k-1}, x_k\}\})$. If $x_k = x_0$ then P is called a *cycle* of length k or *k-cycle.* (Note that in physics literature the term *loop* is often used instead of any cycle whereas in graph theory loop is used only for *1-cycles.*)
A graph G is called *connected* if for any two of its vertices there is a path that contains both and is a subgraph of G. A graph that is connected and contains no cycles is called *tree* and a disjoint union of trees is a *forest*
A graph is called *bipartite* if the set of vertices V can be divided into two disjoint subsets V_1 and V_2 such that all edges of G connect a vertex from V_1 and a vertex from V_2, i.e. all edges are of the form $e = \{v_1, v_2\}, v_1 \in V_1, v_2 \in V_2$. A bipartite graph is called *complete* if every vertex in one of the subsets is adjacent to all vertices in the other subset. We write $K_{\alpha,\beta}$ for the bipartite graph with $|V_1| = \alpha$ and $|V_2| = \beta$.
If it is possible to draw a graph in a plane without intersecting edges then it is called *planar.* For non-planar graphs the *crossing number* is the minimal number of

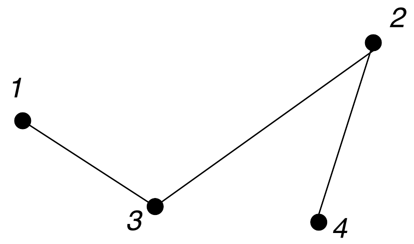


Figure A.2: A spanning subgraph of the graph from Fig. A.1 that is also a path and a tree.

intersections in a plane drawing of the graph.
To treat graphs algebraically one associates with them two matrices.

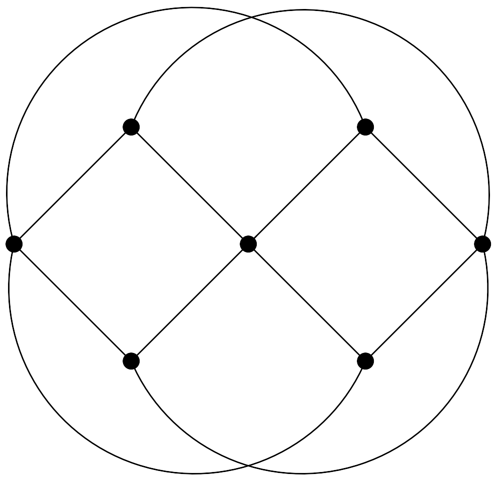


Figure A.3: The complete bipartite graph $K_{3,4}$ with crossing number 2.

The *incidence matrix* of a graph $G = (\{v_1, ..., v_n\}, \{e_1, ..., e_m\})$ is defined as an $n \times m$-matrix where the element $m_{ij} \in \{0, 1, 2\}$ is the number of times that vertex v_i and edge e_j are incident. This definition can be enhanced by introducing a direction, as in the matrix (ϵ_{vl}) in appendix B.
The *adjacency matrix* is the $n \times n$-matrix where every element a_{ij} is the number of edges connecting the vertices v_i and v_j.

	e_1	e_2	e_3	e_4	e_5
v_1	1	0	0	0	0
v_2	0	1	1	0	0
v_3	1	1	0	1	1
v_4	0	1	1	1	1

	v_1	v_2	v_3	v_4
v_1	0	0	1	0
v_2	0	0	1	1
v_3	1	1	0	2
v_4	0	1	2	0

Table A.1: Incidence and adjacency matrix for the graph from Fig. A.1

Appendix B

Derivation of equation (4.8)

In this appendix we derive a general parametric representation of a diagram G in a scalar theory, in large parts following the approach of Itzykson and Zuber [6].

We start by introducing an orientation for all internal lines. Therefore the incidence matrix of G (see appendix A) has the elements

$$\epsilon_{vl} = \begin{cases} 1 & \text{if vertex } v \text{ is the starting point of directed edge } l \\ -1 & \text{if vertex } v \text{ is the ending point of directed edge } l \\ 0 & \text{if } l \text{ is not incident on } v \end{cases} \tag{B.1}$$

This orientation could be interpreted as the direction of the momentum associated with an edge. Let P_v be the external momentum entering a vertex v. Then a dimensionally regularized Euclidean Feynman integral representing that graph would look like

$$I(G) = \int \prod_{l=1}^{L} \mathrm{d}^D k_l \, \frac{1}{(k_l^2 - m^2)^{a_l}} \prod_{v=1}^{V} \delta^D \left(P_v - \sum_{l=1}^{L} \epsilon_{vl} k_l \right). \tag{B.2}$$

Now we use the gamma function to generalize (4.2)

$$\Gamma(z) = \int_0^\infty \mathrm{d}t \; t^{z-1} e^{-t}. \tag{B.3}$$

For $t = \alpha_l(k^2 - m^2)$ and $z = a_l$ this becomes

$$\Gamma(a_l) = \int_0^\infty \mathrm{d}\alpha_l \; (k^2 - m^2)(\alpha_l(k^2 - m^2))^{a_l - 1} e^{-\alpha_l(k^2 - m^2)} \tag{B.4}$$

from which a representation of the propagator immediately follows:

$$\frac{1}{(k^2 - m^2)^{a_l}} = \int_0^\infty \mathrm{d}\alpha_l \, \frac{\alpha_l^{a_l - 1}}{\Gamma(a_l)} e^{-\alpha_l(k^2 - m^2)} \tag{B.5}$$

Furthermore we need the Fourier transform of the delta function.

$$\delta^D \left(P_v - \sum_l^L \epsilon_{vl} k_l \right) = \frac{1}{(2\pi)^D} \int \mathrm{d}^D y_v \exp \left[-i y_v \cdot \left(P_v - \sum_l^L \epsilon_{vl} k_l \right) \right] \tag{B.6}$$

Our integral has become

$$I(G) = \int \prod_{l=1}^{L} \mathrm{d}^D k_l \int_0^\infty \mathrm{d}\alpha_l \, \frac{\alpha_l^{a_l-1}}{\Gamma(a_l)} e^{-\alpha_l(k^2-m^2)} \prod_{v=1}^{V} \int \frac{\mathrm{d}^D y_v}{(2\pi)^D} \exp\left[-i y_v \cdot \left(P_v - \sum_{l=1}^{L} \epsilon_{vl} k_l\right)\right] \tag{B.7}$$

where the regularization allows us to change the order of integration and rearrange it to [6]

$$\begin{aligned} I(G) = \int_0^\infty \left(\prod_{l=1}^{L} \mathrm{d}\alpha_l \, \frac{\alpha_l^{a_l-1}}{\Gamma(a_l)}\right) \int \left(\prod_{v=1}^{V} \frac{\mathrm{d}^D y_v}{(2\pi)^D}\right) \int \left(\prod_{l=1}^{L} \mathrm{d}^D k_l \exp\left[\alpha_l m^2 - i \sum_{v=1}^{V} y_v P_v\right]\right. \\ \left. \times \exp\left[-\alpha_l \left\{k_l^2 + \frac{i}{\alpha_l} \sum_{v=1}^{V} y_v \epsilon_{vl} k_l\right\}\right]\right). \end{aligned} \tag{B.8}$$

Note the new summation sign in the first and the change of the summing variable in the second exponential. In (B.6) and (B.7) the summation over v in the exponent was implicit because of the multiplication over v. Now, this is the case for l but not v so the summation sign has to be written explicitly.
The second exponential in this integral can by quadratic extension be brought into the form of a gaussian integrand:

$$\begin{aligned} &\exp\left[-\alpha_l \left(k_l^2 + \frac{i}{\alpha_l} \sum_{v=1}^{V} y_v \epsilon_{vl} k_l\right)\right] \\ &\quad = \exp\left[-\alpha_l \left(k_l + \frac{i}{2\alpha_l} \sum_{v=1}^{V} y_v \epsilon_{vl}\right)^2\right] \exp\left[\frac{1}{4\alpha_l} \left(\sum_{v=1}^{V} y_v \epsilon_{vl}\right)^2\right] \end{aligned} \tag{B.9}$$

The internal momentum k_l is isolated in this part of the integrand and the integration can be executed using

$$\int_{-\infty}^{+\infty} e^{-a(x+b)^2} \mathrm{d}x = \sqrt{\frac{\pi}{a}} \tag{B.10}$$

$$\begin{aligned} \Longrightarrow \; I(G) = \int_0^\infty \left(\prod_{l=1}^{L} \mathrm{d}\alpha_l \, \frac{\pi^{\frac{D}{2}} \alpha_l^{a_l-1}}{\alpha_l^{\frac{D}{2}} \Gamma(a_l)}\right) \int \left(\prod_{v=1}^{V} \frac{\mathrm{d}^D y_v}{(2\pi)^D}\right) \exp\left[-i \sum_{v=1}^{V} y_v P_v\right] \\ \times \exp\left[\alpha_l m^2 + \frac{1}{4\alpha_l} \left(\sum_{v=1}^{V} y_v \epsilon_{vl}\right)^2\right] \end{aligned} \tag{B.11}$$

At this stage we change the variables y_v in such a way that we can execute one of the V integrations easily. Let

$$\begin{aligned} y_v &=: z_v + z_V \quad \forall\, 1 \leq v < V \\ y_V &=: z_V. \end{aligned} \tag{B.12}$$

The general form of the integral stays the same:

$$I(G) = \int_0^\infty \left(\prod_{l=1}^{L} \mathrm{d}\alpha_l \, \frac{\pi^{\frac{D}{2}} \alpha_l^{a_l - 1}}{\alpha_l^{\frac{D}{2}} \Gamma(a_l)} \right) \int \left(\prod_{v=1}^{V} \frac{\mathrm{d}^D z_v}{(2\pi)^D} \right) \exp \left[-i \sum_{v=1}^{V-1} (z_v + z_V) P_v - i z_V P_V \right]$$
$$\times \exp \left[\alpha_l m^2 + \frac{1}{4\alpha_l} \left(\sum_{v=1}^{V-1} (z_v + z_V) \epsilon_{vl} + z_V \epsilon_{Vl} \right)^2 \right] \tag{B.13}$$

From the definition of the incidence matrix it immediately follows that $\sum_{v=1}^{V} \epsilon_{vl} = 0$ for any l. Thus, the term containing z_V vanishes in the second exponent and the integration over z_V is the delta function of momentum conservation.

$$I(G) = \int_0^\infty \left(\prod_{l=1}^{L} \mathrm{d}\alpha_l \, \frac{\pi^{\frac{D}{2}} \alpha_l^{a_l - 1}}{\alpha_l^{\frac{D}{2}} \Gamma(a_l)} \right) \int \left(\prod_{v=1}^{V} \frac{\mathrm{d}^D z_v}{(2\pi)^D} \right) \exp \left[-i z_V \sum_{v=1}^{V} P_v \right]$$
$$\times \exp \left[-i \sum_{v=1}^{V-1} z_v P_v \right] \exp \left[\alpha_l m^2 + \frac{1}{4\alpha_l} \left(\sum_{v=1}^{V-1} z_v \epsilon_{vl} \right)^2 \right] \tag{B.14}$$

To go on we first define a matrix d_G:

$$[d_G]_{v_1 v_2} := \sum_l \epsilon_{v_1 l} \epsilon_{v_2 l} \alpha_l^{-1}, \quad \forall \, v_1, v_2 \in \{1, ..., V-1\} \tag{B.15}$$

If the indices v_i ranged from 1 to V and there was no factor α_l^{-1}, we would call this a *Laplacian matrix*. Our d_G is basically a laplacian matrix with its last row and column deleted and additional factors for the summands of every element. For usual laplacian matrices $L(G)$ the *matrix tree theorem* yields the result [13]

$$\det L_0(G) = \kappa(G) \tag{B.16}$$

where L_0 is the laplacian matrix with one column and one row removed and $\kappa(G)$ is the number of spanning trees of the graph G. We can enhance the proof of (B.16) to construct the determinant of d_G.
Moreover, we need the *Binet-Cauchy theorem*. Let A be an $m \times n$-matrix and B an $n \times m$-matrix. If $m \leq n$ then

$$\det(AB) = \sum_Q \det(A[Q]) \det(B[Q]) \tag{B.17}$$

where $Q \subseteq \{1, ..., n\}$, $|Q| = m$ and $X[Q]$ is a quadratic matrix consisting of the columns of X (or rows, respectively) whose index is in Q.
Since α_l is just a number we can write $\widetilde{\epsilon}_{v_2 l} := \alpha_l^{-1} \epsilon_{v_2 l}$ and from (B.17) we have

$$\det d_G = \sum_Q \det((\epsilon_{v_1 l})[Q]) \det((\widetilde{\epsilon}_{v_2 l})^T [Q]). \tag{B.18}$$

Here, Q is a $(V-1)$-element subset of the set of all edges $\{1, ..., L\}$.
To calculate the determinants we need to further examine the incidence matrix. Say, the edges in Q do not form a spanning tree of G. This means that the edges of some subset of Q form a cycle. The columns of an incidence matrix belonging to a cycle

are linearly dependent and thus the determinant of said matrix is 0. The linear dependence is clear as every vertex in a cycle has exactly two edges attached to it, so in every row exactly two of the k columns representing the cycle are ± 1 while the others are 0.
If the edges in Q do indeed form a spanning tree T and an edge $e \in T$ is adjacent with the vertex that is indexed by the last row of the incidence matrix of the graph (the row that is deleted to get $(\epsilon_{v_1 l})$), then the column representing e in $(\epsilon_{v_1 l})[Q]$ has exactly one non-zero entry ± 1, as the other one would have been in the deleted row. Deleting the column and row containing this entry yields a smaller matrix whose determinant is the same as the determinant of $(\epsilon_{v_1 l})[Q]$ up to a factor of ± 1. Now let T' be the tree obtained from T by contracting e. Delete again the row and column with the only non-zero entry and repeat until there is only a 1×1-matrix with an entry ± 1 left. Thus, $\det(\epsilon_{v_1 l})[Q] = \det(\epsilon_{v_1 l})^T[Q] = \pm 1$.

For $(\widetilde{\epsilon}_{v_2 l}[Q])$ we can proceed analogously. The only difference is the one non-zero entry in column e, which is now $\pm \alpha_e^{-1}$. Consequently, $\det(\widetilde{\epsilon}_{v_2 l}[Q]) = \pm \prod_{l \in T} \alpha_l^{-1}$ and

$$\det d_G = \sum_T \prod_{l \in T} \alpha_l^{-1} \neq 0. \tag{B.19}$$

With this result we make use of another general gaussian integral to integrate z_v:

$$\begin{aligned} I(G) = \int_0^\infty \left(\prod_{l=1}^L \mathrm{d}\alpha_l \, \frac{\pi^{\frac{D}{2}} \alpha_l^{a_l - 1}}{\alpha_l^{\frac{D}{2}} \Gamma(a_l)} \exp\left(\alpha_l m^2\right) \right) \frac{1}{(2\pi)^{D(V-1)}} \left(\frac{(2\pi)^{V-1}}{\det d_G} \right)^{\frac{D}{2}} \\ \times \exp\left[- \sum_{v_1, v_2} [d_G^{-1}]_{v_1, v_2} P_{v_1} P_{v_2} \right]. \end{aligned} \tag{B.20}$$

A further look at the functions of α_l allows us to conveniently rename them. In the denominator we have:

$$\left(\prod_{l=1}^L \alpha_l \right) \cdot \det d_G = \left(\prod_{l=1}^L \alpha_l \right) \sum_T \prod_{l \in T} \alpha_l^{-1} = \sum_T \prod_{l \notin T} \alpha_l =: U_G. \tag{B.21}$$

And in the exponent there is

$$\sum_{v_1, v_2} [d_G^{-1}]_{v_1, v_2} P_{v_1} P_{v_2} \tag{B.22}$$

One possibility to write the inverse of a matrix is

$$d_G^{-1} = \frac{1}{\det d_G} \left(C_{ij} \right). \tag{B.23}$$

Where (C_{ij}) is the matrix of cofactors of d_G. Let $d_G^{(i,j)}$ be the matrix d_G with row i and column j removed. A cofactor is then defined as $C_{ij} = (-1)^{i+j} \det(d_G^{(i,j)})$. The determinant of $d_G^{(i,j)}$ can again be calculated with the formula

$$\det d_G^{(i,j)} = \sum_Q \det((\epsilon_{v \setminus i, l})[Q]) \det((\widetilde{\epsilon}_{v \setminus j, l})^T[Q]) \tag{B.24}$$

We find again that $\det((\epsilon_{v \setminus i, l})[Q])$ is ± 1 or 0 but we cannot generally say that the determinant only vanishes if some of the edges form a cycle. If $i = j$ both determinants on the right side have the same sign. As exchanging two rows or columns changes the sign of the determinant the signs are also the same if the difference of i and j is even but different if odd. Hence, the sign of $\det d_G^{(i,j)}$ can be given as $(-1)^{i+j}$. Instead of all spanning trees, as in (B.18), the sum is over a subset of subgraphs with one less edge than the spanning trees. These subgraphs could also be interpreted as spanning 2-forests $S = T_1 \cup T_2$. Therefore, writing $S_{(i,j)}$ for the spanning 2-forests that occur when deleting row i and column j,

$$\begin{aligned}[][d_G^{-1}]_{i,j} &= \frac{\alpha_1 \ldots \alpha_L}{U_G} (-1)^{2(i+j)} \sum_{S_{(i,j)}} \prod_{l \in S_{(i,j)}} \alpha_l^{-1} \\ &= \frac{1}{U_G} \sum_{S_{(i,j)}} \prod_{l \notin S_{(i,j)}} \alpha_l . \end{aligned} \tag{B.25}$$

Joining this result with the external momenta yields

$$\begin{aligned} \sum_{v_1,v_2} [d_G^{-1}]_{v_1,v_2} P_{v_1} P_{v_2} &= \sum_{v_1,v_2} P_{v_1} P_{v_2} \frac{1}{U_G} \sum_{S_{(v_1,v_2)}} \prod_{l \notin S_{(v_1,v_2)}} \alpha_l \\ &= \frac{1}{U_G} \sum_S \prod_{l \notin S} \alpha_l (q(S))^2 =: \frac{\mathcal{V}_G}{U_G} \end{aligned} \tag{B.26}$$

where $q(S)$ is the momentum flowing through the removed edge e and the sum is now over all spanning 2-forests. Finally, our integral now has the desired form

$$I(G) = \frac{\pi^{h\frac{D}{2}}}{\prod_{l=1}^{L} \Gamma(a_l)} \int_0^\infty \prod_{l=1}^{L} \mathrm{d}\alpha_l \ \exp\left(\alpha_l m^2\right) \frac{\alpha_l^{a_l - 1} \exp(-\frac{\mathcal{V}_G}{U_G})}{U_G^{\frac{D}{2}}} . \tag{B.27}$$

Selbständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Berlin, den