HUMBOLDT-UNIVERSITÄT ZU BERLIN

FACULTY OF MATHEMATICS AND NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS

Cutkosky Cuts at Core Hopf Algebra

MASTER'S THESIS

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

Master of Science (M.Sc.)

in Mathematics

by
Anna Bley

Supervisors: Prof. Dr. Dirk Kreimer
Dr. Christian Bogner

Berlin, February 2016

Declaration of originality

Selbstständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig verfasst und noch nicht für andere Prüfungen eingereicht habe. Sämtliche Quellen, einschließlich Internetquellen, die unverändert oder abgewandelt wiedergegeben werden, insbesondere Quellen für Texte und Bilder, sind als solche kenntlich gemacht. Mir ist bekannt, dass bei Verstößen gegen diese Grundsätze ein Verfahren wegen Täuschungsversuchs bzw. Täuschung eingeleitet wird.

Berlin,

_____

Anna Bley

Contents

"Science may set limits to knowledge, but set no limits to imagination."
Bertrand Russel, 1872-1970

---

1. Introduction

Modern physics describes elementary particles and their interaction by quantum field theories. This union of classical field theory, special relativity theory and quantum mechanics has proven incredibly successful in predicting the high precision experiments performed in particle accelerators with enormous accuracy.
Nevertheless, the question of how to put quantum field theories in a solid mathematical framework is far from settled. Rather, the apparent discrepancy of physical calculations being extremely effective, while often turning out to be surprisingly hard to underlay with consistent mathematics, opens up an exciting as well as challenging playground for both mathematicians and physicists to cooperate.
One of the key advances in this direction was made by Kreimer showing that Feynman diagrams (or Feynman graphs) come with a rich algebraic structure, they form a Hopf algebra. This insight lead to explaining a problem which long troubled physicists: The process of renormalization, i.e. the handling of singularities occurring in the high energy limit of perturbative quantum field theories, long seeming like an ad hoc procedure, is now structurally understood in well defined algebraic terms of the so called renormalization Hopf algebra [6].
Beyond this, another promising direction for further research was mentioned in [1], namely the reconciliation of our understanding of renormalization with the understanding of the unitarity of the S-matrix. Intimately related to this endeavor are the concept of the core Hopf algebra, containing the above mentioned renormalization Hopf algebra as a quotient, on the one side, and (Cutkosky) cutting rules, illustrating propagator poles, as an implication of unitarity on the other.
The aim of this work is to contribute to this by introducing the algebraic structures leading to the core Hopf algebra and presenting the cutting rules with respect to application and unitarity. Finally, we show a connection between these two concepts by proving that the core coproduct is compatible with Cutkosky cutting as proposed in section 5 of [1]. The proof of a related proposition for more general cuttings, given in [2], is directly inferred. Both statements are useful for renormalization in the presence of discontinuities arising from propagator poles.
In the following chapter, we introduce the role of Feynman diagrams in quantum field theory and subsequently give a proper graph theoretic definition of them and certain associated graph operations, as far as relevant in the course of this work.
Chapter 3 is devoted to establishing the algebraic structures needed to understand the core Hopf algebra and its relation to the renormalization Hopf algebra. We begin with the general definition of Hopf algebra and then turn to its two physically meaningful realizations.
In chapter 4 we introduce unitarity as a key feature of a quantum field theory. Cutkosky cutting rules are derived and interpreted in relation to the optical theorem as an important consequence of unitarity. Moreover, the cutting rules are considered as a powerful tool for looking at the analytic structure of Feynman diagrams, allowing to calculate their imaginary parts, and via dispersion relations (also know as Hilbert transforms) the corresponding real part.
In chapter 5 we prove that the core coproduct in a certain way respects the cutting of a graph, precisely that there is a bijection between the several cut Feynman graphs Γ_{c_i} $(i = 1, .., n; n \in \mathbb{N})$ of a graph Γ as given by Cutkosky's prescription and the tensor products of subgraphs $\gamma' \subsetneq \Gamma$ as filtered by the coproduct Δ_c with the respective completely cut cographs γ''_{cc_j}: $\gamma' \otimes \gamma''_{cc_j}$, $(j = 1, ..., m; m \in \mathbb{N})$. The proof can almost identically be transferred to the more general statement for k-cuts, separating Γ into k connected components. Finally, implications of the two theorems are discussed.

2. Feynman diagrams

Feynman diagrams (Feynman graphs) are incredibly rich objects. They can bee looked at from a purely algebraic viewpoint just as one can study their combinatorial properties [8]. On the other hand, one can also consider the integral expression assigned to each Feynman diagram and study its analytic structure [12], or they can be treated as graph theoretical objects such that one can apply graph operations on them. In this chapter, we present the graph theoretical foundations, in the following we continue touching the other points.
The first section serves to classify the role of Feynman diagrams in quantum field theory after general textbook knowledge. Following [2, 3], we thereupon develop a graph theoretically precise definition of a Feynman diagram in section 2.2 as well as of its underlying forest structures in section 2.3 and finally, in section 2.4, introduce graph operations. We focus on subgraph shrinking and (Cutkosky) cutting as these are important concepts appearing in the following chapters 3 and 4 and eventually needed for the proofs of chapter 5.

2.1. Feynman diagrams in quantum field theory

High energy particle physics aims to understand the fundamental constituents of matter by performing particle collision experiments in accelerators. Quantum field theory as its theoretical foundation intends (and succeeds) to make precise predictions about these experiments as well as it tries to structurally understand the underlying "rules of the game".

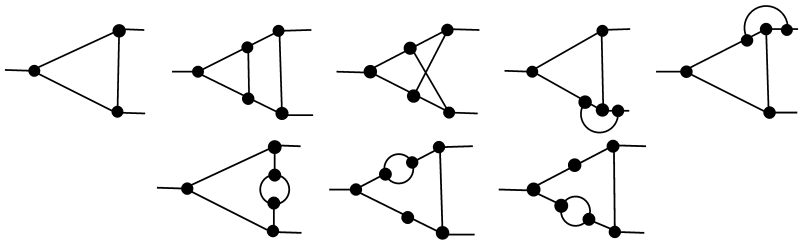


Figure 2.1: All graphs for $(1 \to 2)$-scattering up to second loop order as built from 3-valent vertices. Incoming particles on the left, outgoing on the right. Summation gives the corresponding amplitude A_2.

After quantum field theory was derived by a canonical quantization of the classical field theory at first, the so called path integral approach introduced by Feynman became very popular. This was in large part due to its intuitive diagrammatic representation of the path integral calculations to be performed.
For a free theory, i.e. a theory with no interactions between particles (and thereby finitely interesting), the path integrals can be calculated analytically, whereas for theories with interactions it turned out that only approximate calculations can be achieved. Perturbative expansions have proven to be a very successful method of approximation in this case.
Let us look at things in a little more detail. To calculate a physically measurable quantity which describes a scattering process the so called S-matrix

$$S = \mathbb{I} + iT \tag{2.1}$$

is introduced. The identity matrix $\mathbb{I}$ is identified with no interaction and the T-matrix represents the nontrivial part of the scattering. The latter is understood to describe a three-stage process of particles incoming, interacting, and afterwards outgoing. Formally, this is described by matrix elements $\langle f|T|i\rangle$ of the T-matrix, with an initial state $|i\rangle$ (contravariant vector) and a final state $\langle f|$ (covariant vector), both assumed to be described by a free theory when positioned infinitely far away from the interaction.
The matrix elements $\langle f|T|i\rangle$ can be related to so called (scattering) amplitudes A and calculated perturbatively as $T = \sum_{n\geq 0} g^n A_n$ with A_n denoting perturbative amplitudes. Finally,

these amplitudes A_n are sums of Feynman diagrams up to a certain loop order. The squared amplitudes $|A|^2$ then represent a scattering probability which can be measured.

Example 2.1.1. *Let us consider an example to get the idea in principle. The amplitude A_2 of a $(1 \to 2)$-scattering process is a sum of all Feynman graphs with 1 (fixed) incoming particle and 2 (fixed) outgoing particles that one can possibly draw given a certain set of rules (for edges and vertices). In- and outgoing particles are represented by "open" edges on the left and right side of the diagram respectively. Say, we are allowed to draw only vertices with 3 adjacent edges and one edge type. Given that our perturbative expansion is driven by the number of loops, our graphs contributing to A_2 can at most have two loops. The result is depicted in figure 2.1.*

2.2. Graph theoretic definition of Feynman diagrams

The physical term of a *Feynman graph* Γ is to be given a mathematically precise definition. For simplicity, the graph to be modeled is from a scalar theory[1] (ϕ^3- and ϕ^4-theory for illustrations) which suffices for our purposes. Details necessary to describe other theories should be reintroduced without difficulty (see figure 2.2 for examples). We start by recapitulating some basic graph theoretic terms.

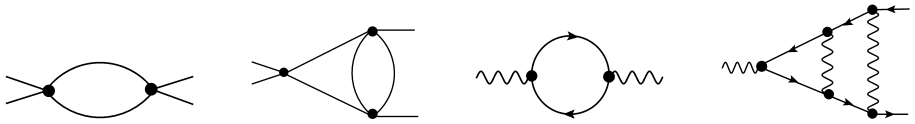


Figure 2.2: Examples for one- and two-loop Feynman diagrams from scalar ϕ^4-theory (left) and quantum electrodynamics (right).

Graph A *graph* G is a pair (V, E) of a vertex set V and an edge set E, with elements $e \in E$ formed by pairs of vertices $e = \{v, w\}$, s.t. $v, w \in V$. We consider finite graphs only, i.e. V and E are finite sets. Precisely, E is a multiset as its elements can occur more than once. Below, we define a labelling to address these so called multiple edges uniquely.

Orientation A graph is *unordered (or unoriented)* if $\{v, w\} = \{w, v\}$.

Multiple edge A graph has *multiple edges* if it has at least one pair of vertices $\{v, w\}$ connected by more than one edge. For instance, vertices b and c in figure 2.3 are connected by edge 3 and 4.

External half edge Apart from the usual (internal) edges, a Feynman graph has the specialty of *external half edges*[2], i.e. edges which are only connected to a vertex at one of its endpoints (see figure 2.3). This entails $E = E_{int} \sqcup E_{ext}$. From now on we assume the external half edges to be implicitly given by the valency of each vertex (defined below) and identify $E \equiv E_{int}$.

Graph labelling A *labelling* of edges $l : E \to \{1, ..., k \in \mathbb{N}\}, e \mapsto n$ maps to each edge a natural number. Analogously define a labelling of vertices, mapping to each vertex a letter.
While edges were first defined as pairs of vertices $\{v, w\}$, together with a labelling they are now given as triples $\{v, w, n\}$.

Remark 2.2.1. (Labelling w.r.t subgraphs and cographs)
In order to give this important note on labelling, we anticipate the terms of a subgraph and a cograph properly defined in chapter 2.4 below. Only with the extra information of the labelling

[1] Roughly speaking, this means there is only one edge type, orientation of edges does not matter and every vertex must be connected to the same fixed number of edges.

[2] These external edges correspond to incoming particles when drawn on the left and outgoing particles when drawn on the right.

we can address multiple edges uniquely and thereby uniquely identify subgraphs $\gamma \subsetneq \Gamma$ in a given graph Γ. As an example consider again the edges $\{b, c, 3\}$ and $\{b, c, 4\}$ in figure 2.3 giving rise to the different subgraphs $\gamma_1 = \{a, 1, b, 3, c, 2\}$ and $\gamma_2 = \{a, 1, b, 4, c, 2\}$.
Also the shrinking of a subgraph $\gamma \subsetneq \Gamma$ in Γ to obtain the cograph Γ/γ is a uniquely defined operation only in case that we have a labelling. Shrinking implies (besides deleting edges) the identification of the subgraph's vertices $V_\gamma \subsetneq V_\Gamma$, thereby accessing (possibly changing) the first two elements of triples $\{v, w, n\}$ in Γ. So relating the remaining edges of the cograph Γ/γ uniquely to edges in the original graph Γ needs the extra (untouched) information given by the third element in $\{v, w, n\}$. Consider as an example the lower line of graphs in figure 2.6, where for instance edge $\{v, w, 5\}$ of Γ/γ can be identified with edge $\{a, c, 5\}$ of Γ.

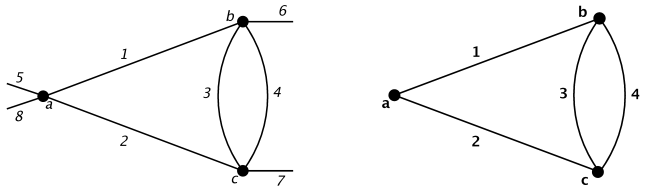
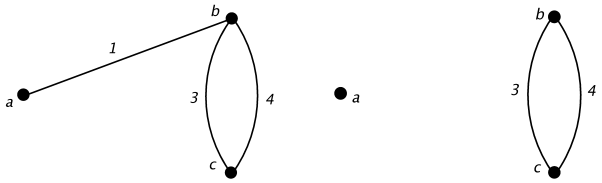


Figure 2.3: A Feynman graph Γ with labelling. From now on, external half edges are implicitly assumed given by the fixed valency $Val(v)$ of each vertex and therefore omitted.

Connectivity A graph is *connected* if every two of its vertices can be connected through a finite sequence of edges and vertices $\{v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, ..., e_{j_k}, v_{i_k}\}$ with $i, j, k = 1, ...n \in \mathbb{N}$. Also a single vertex is considered connected.
A graph is *k-connected* if removing $k-1$ of its internal edges leaves it connected, i.e. if Γ is connected, so is $\Gamma - \{e_1, ..., e_{k-1}\}$ for any edge $e \in E$. Important for us is the case of *2-connected or 1PI* graphs, which stay connected after removal of one internal edge (see figure 2.4). These are the graphs we build our Hopf algebras on in chapter 3.



$\Gamma - \{e_2\}$ $\Gamma - \{e_1, e_2\}$

Figure 2.4: Removing in the graph Γ of figure 2.3 the single edge e_2 leaves it connected. Removing edges e_1 and e_2 separates the graph in two components.

Loops A *loop* of a graph is a sequence of its edges and vertices, such that each edge and vertex are passed only once and initial and final vertex agree. For instance, in the graph of figure 2.3 the sequence $\{a, 1, b, 3, c, 2, a\}$ is a loop.

Self-loops A *self loop* is an edge connected at both ends to the same vertex, i.e. $e = \{v, v\}$.

Simple graphs A graph is *simple* if it has no multiple edges or self loops.

Vertex valency The *valency* of a vertex is the number of its adjacent half edges, i.e. be E_v the set of half edges connected to a vertex v then $Val(v) = |E_v|$. For instance, the two graphs on the l.h.s of figure 2.2 have $Val(v) = 4$ for each vertex, whereas the two graphs on the right have $Val(v) = 3$.
A *k-regular* graph is a graph where all vertices are of fixed valency k, $k \in \mathbb{N}$.

The below definition of a Feynman graph is used in the following. We will sometimes refer to Feynman graphs simply as graphs from now on.

Definition 2.2.1. (Feynman graph)
A Feynman graph Γ *is an unordered, connected and two-connected (1PI) graph, with multiple edges between vertices and self-loops allowed. For each vertex*[1] *be* $Val(v) \geq 3$. *Additionally it contains external half edges which if not explicitly drawn can be inferred from the valency of each vertex. All vertices and (half) edges are uniquely labelled by finite sets of letters and natural numbers respectively.*

Furthermore, let V_Γ be the vertex set of Γ, $|V_\Gamma| = v_\Gamma$ the number of vertices. E_Γ and $|E_\Gamma| = e_\Gamma$ be the set and number of edges respectively. $|\Gamma| := |H_1(\Gamma)|$ (first Betti number) be the loop number of a the graph Γ.

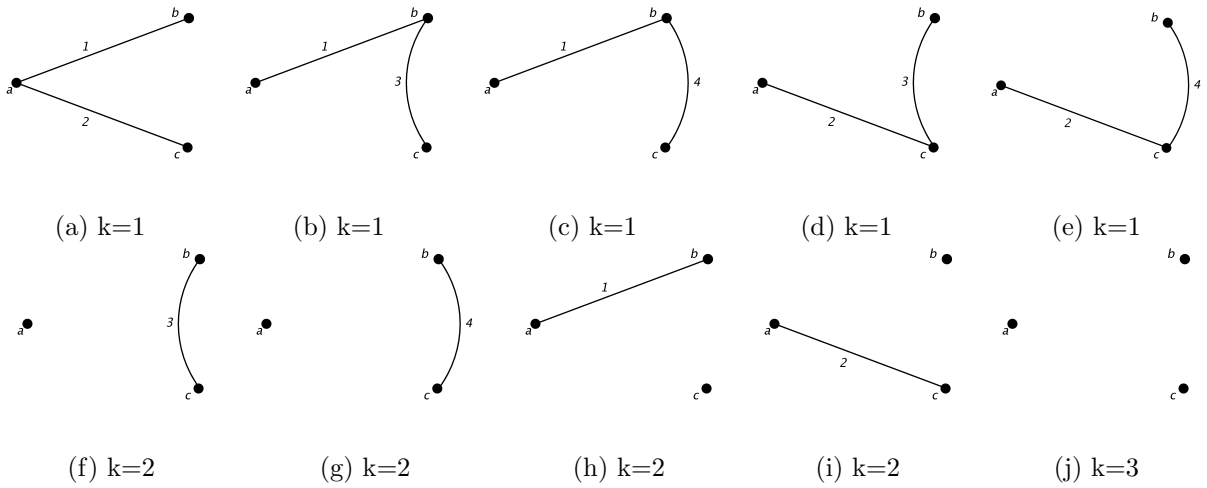
2.3. Tree and forest structures

With this definition in place, we can define spanning trees T and spanning forests F of a graph Γ. These concepts add to the graph some more structure and prove useful for the proper definition of graph cutting later on.

Definition 2.3.1. (Spanning tree)
A spanning tree *of* Γ *is a proper subgraph* $T \subsetneq \Gamma$, *such that* $V_T = V_\Gamma$, *and* T *is simply connected.*

Definition 2.3.2. (Spanning k-forest)
A spanning k-forest F_k, $k = 1, ..., v_\Gamma$, *is a disjoint union* $\cup_{i=1}^{k} T_i$ *of trees,* $T_i \subsetneq \Gamma$, *such that* $\cup_i V_{T_i} = V_\Gamma$.

Yes, a k-forest with $k = 1$ is a tree.



(a) k=1 (b) k=1 (c) k=1 (d) k=1 (e) k=1

(f) k=2 (g) k=2 (h) k=2 (i) k=2 (j) k=3

Figure 2.5: All possible k-forests of the graph Γ in figure 2.3.

It turns out that spanning 2-forests are of special interest as they induce Cutkosky cuts, which separate a graph Γ into exactly two connected components $\Gamma = \Gamma_1 \sqcup \Gamma_2$ under some additional physical constraints.
Generally, spanning k-forests separate a graph into several $k = 1, ..., v_\Gamma \in \mathbb{N}$ connected components via their induced vertex partitioning. However, forest (f) and (g) of figure 2.5, for

[1] We consider mainly graphs from ϕ^4-theory (4-regular graphs) and ϕ^3-theory (3-regular graphs) for illustration. Exceptionally, two-valent vertices may occur in cographs at a the place where a propagator correction subgraph is shrunken.

instance, both yield the same vertex partition $\{a\} \sqcup \{b, c\}$. There are more spanning k-forests than allowed for uniquely partitioning vertices, thus it makes sense to introduce the following equivalence relation.

Definition 2.3.3. (Equivalence) *Two spanning k-forests F_k^1 and F_k^2, $k = 1, 2, ..., v_\Gamma \in \mathbb{N}$, are said to be* equivalent *if every two trees T_{i_r} ($i = 1, 2$, $r = 1, ..., k$) from different forests share the same vertex set $V_{T_{i_r}}$.*

From now on, we only consider k-forests up to equivalence. The set of k-forests of a graph Γ is denoted by $\mathcal{F}_k$.

2.4. Graph operations

Among the various kinds of operations on graphs, shrinking of subgraphs and removing edge sets are of special interest here as these are the operations performed later when applying coproducts and cutting edges respectively.

2.4.1. Shrinking and inserting subgraphs

Generally, a subgraph $\gamma \subseteq \Gamma$ is a pair (V_γ, E_γ) with $V_\gamma \subseteq V_\Gamma$ and $E_\gamma \subseteq E_\Gamma$. Here, we always consider as subgraphs disjoint unions of $1PI$ (two-connected) graphs $\gamma = \sqcup_{i=1}^k \gamma_i$, with $\gamma = \Gamma$ allowed. For instance, subgraphs are given by the middle graphs in figure 2.6.

Definition 2.4.1. (Cograph)
The cograph Γ/γ*, for $\gamma \subseteq \Gamma$ a disjoint union of 1PI subgraphs $\gamma = \sqcup_{i=1}^k \gamma_i$, denotes the graph obtained from Γ by shrinking all internal edges of γ to zero length while all vertices $v_l \in V_\gamma$ are merged within each of the k subgraph components γ_i and identified with k new vertices w_i, labelled by the disjoint union of the letters of the v_l.*

As the name suggests, cographs appear in the coproducts Δ and Δ_c introduced in chapter 3. Note that it is possible to regain from the cograph Γ/γ the original graph Γ by the knowledge of γ when given a suitable labelling as described in remark 2.2.1.

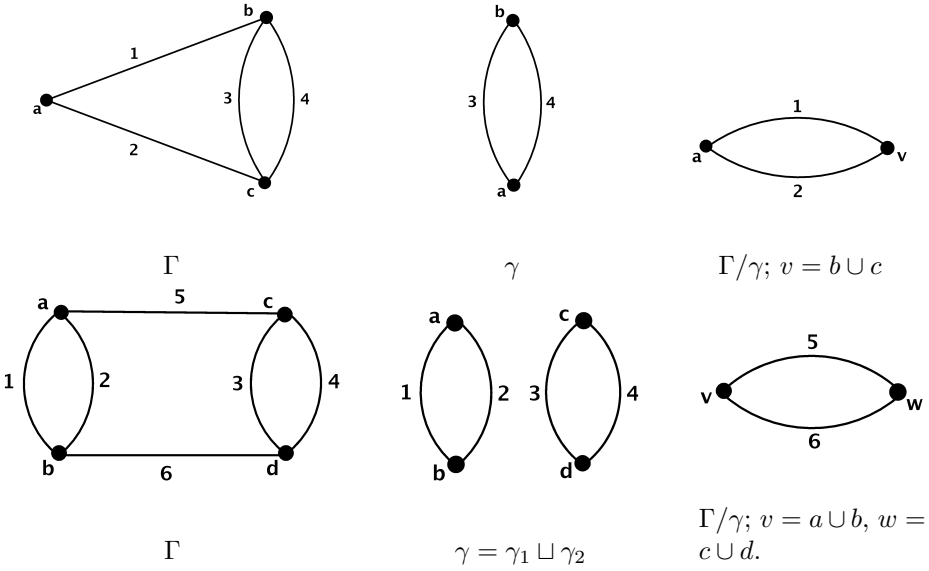


Figure 2.6: Shrinking the subgraph γ (possibly a disjoint union) in the graph Γ gives the cograph Γ/γ.

Example 2.4.2. (Constructing cographs)
In the upper line of figure 2.6 a one component subgraph $\gamma \subsetneq \Gamma$ is shrunken within Γ by contracting edges 3 and 4 and identifying the union of the vertices b and c with a new vertex $w = b \cup c$.

In the lower line of figure 2.6 a two component subgraph $\gamma = \gamma_1 \sqcup \gamma_2 \subsetneq \Gamma$ *is shrunken within* Γ *by contracting edges* 1 *and* 2 *and identifying the union of the vertices* a *and* b *with a new vertex* $v = a \cup b$*, while at the same time contracting edges* 3 *and* 4 *and identifying the union of the vertices* b *and* c *with a new vertex* $w = c \cup d$*.*

Let us look explicitly at the special case of shrinking a propagator correction subgraph γ, i.e. a subgraph with only two (implicitly given) external edges.

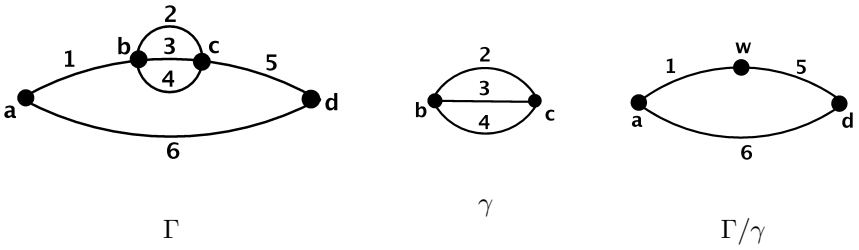


Figure 2.7: Shrinking in the graph Γ the propagator correction subgraph γ yields the cograph Γ/γ with the new vertex $w = b \cup c$.

Example 2.4.3. (Cographs from propagator subgraphs)
In figure 2.7 the propagator correction subgraph $\gamma \subsetneq \Gamma$ *(with two implicitly given external edges) is shrunken to a 2-valent vertex* $w = b \cup c$ *in the cograph* Γ/γ*. This is worth noting, for in the later performed cutting of cographs, both of its two adjacent edges* 1 *and* 5 *facilitate a cut.*

2.4.2. Cutkosky cuts

Now we are prepared to define the Cutkosky cutting of Feynman diagrams in purely graph theoretic terms. For its physical origin and interpretation we refer to chapter 4. Generally, a Cutkosky cut graph Γ_c is a graph Γ with additional "cut information" of how to separate it into two connected components under certain (physical) constraints. The definition we give here is based on forests F which are suitable to induce on the graph Γ this looked for additional structure, i.e. we consider pairs[1] (Γ, F). More generally, this notion allows to define cuts of Γ into k connected components.

Let us start from the general case and consider a pair (Γ, F), with F a spanning k-forest for the graph Γ. The k connected components of F separate the vertices in k partitions. Thereby, F defines a set $\mathcal{G}^F$ of k mutually disjoint graphs $\Gamma_i \subsetneq \Gamma$, $1 \leq i \leq k$, with every edge $e \in \Gamma_i$ having both of its boundary vertices in Γ_i. Note that a spanning 2-forest yields exactly two disjoint graphs $\mathcal{G}^F = \{\Gamma_1, \Gamma_2\}$.
At the same time, by defining $\Gamma_F := \Gamma/(\cup_{i=1}^k \Gamma_i)$, F induces a unique set of edges E_{Γ_F} which connect vertices of different Γ_i. Γ_F is based on edges $e \in E_{\Gamma_F}$ and k vertices only. In particular, for a two-forest we obtain two vertices connected by a multiple edge as shown in column three of figure 2.8.
Now let us restrict our considerations to 2-forests and impose on them the constraint that every component of the forest (equivalently the induced graphs of $\mathcal{G}^{\mathcal{F}}$) must be connected to at least one external half edge. Then the induced edge set E_{Γ_F} can be identified with a Cutkosky cut when we interpret cutting as a removal[2] of all edges $e \in E_{\Gamma_F}$ in Γ. The resulting cut graph $\Gamma_c = \Gamma - E_{\Gamma_f} (\equiv \sqcup_{i=1}^k \Gamma_i)$ can contain isolated vertices and as well as intact subgraphs. Figure 2.8 together with example 2.4.6 shows a concrete construction.

[1] Cutkosky cuts can be defined directly via removal or marking of a certain edge set $E \subseteq E_\Gamma$ without an underlying forest structure but we intend to connect to the framework of [2], leading beyond.

[2] Equivalently, one can of course mark the edges.

Definition 2.4.4. (Cut graph)
A (Cutkosky) cut graph *is a pair* (Γ, F) *of a graph* Γ *and one of its* 2*-forests* F *together with the condition that each element* $\Gamma_i \in \mathcal{G}^F$*,* $i = 1, 2$*, of the induced set of graphs contains at least one external half edge.*

The following important definition is needed later on as well.

Definition 2.4.5. (Completely cut graph) *A* completely cut graph *is a pair* (Γ, F) *of a graph* Γ *and one of its* 2*-forests* F *together with the two conditions that each element* $\Gamma_i \in \mathcal{G}^F$*,* $i = 1, 2$*, of the induced set of graphs contains at least one external half edge, while none of them contains intact subgraphs.*

In chapter 5 we denote cut graphs as Γ_c and completely cut graphs as Γ_{cc}. This is convenient for there the focus is not anymore on how cuts are constructed in detail but rather on the combinatorial operations to be performed with them. To get acquainted, this notation also appears in the following two examples.

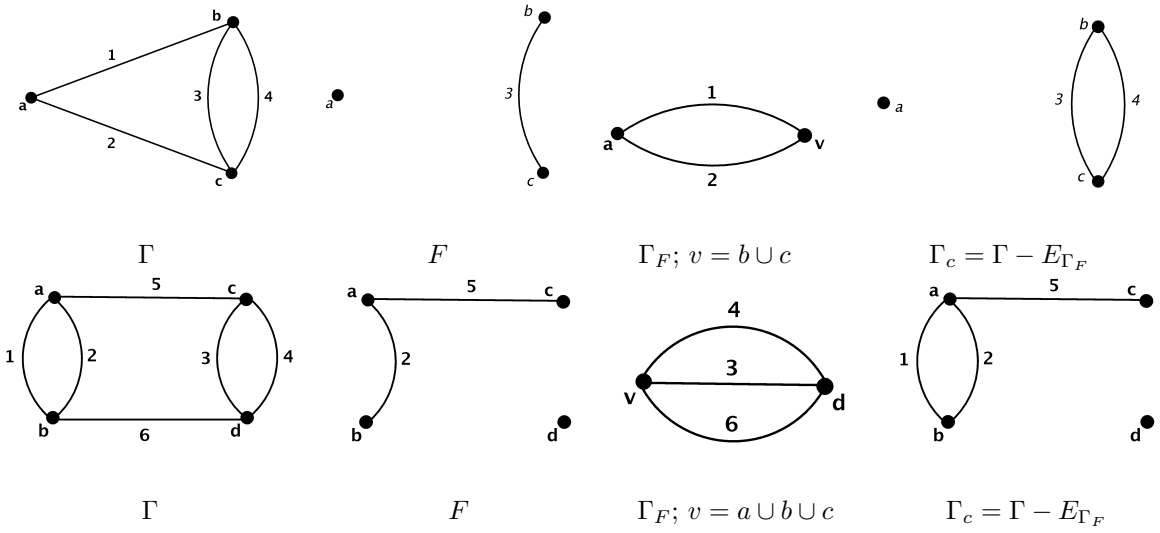


Figure 2.8: A cut graph Γ_c of the graph Γ as constructed from a forest F.

Example 2.4.6. (Cutting and completely cutting graphs)
The 2-loop graph Γ *in the upper line of figure 2.8 has (among others) the 2-forest* F *which separates the vertices of the graph in two sets. This partition of vertices induces a set* $\mathcal{G}^F$ *of two graphs* $\Gamma_1 = \{1\}$ *and* $\Gamma_2 = \{c, 3, b, 4\}$ *which contain edges with both end vertices in their respective component only. Now consider* $\Gamma_F = \Gamma/(\Gamma_1 \sqcup \Gamma_2)$*, made solely of edges with one vertex in component* Γ_1 *and the other in* Γ_2*, namely* $E_{\Gamma_F} = \{1, 2\}$*. Removing these edges gives the cut graph* Γ_c *which has the subgraph* $\gamma = \{c, 3, b, 4\}$ *intact. For the 3-loop graph in the line below the cutting works identically.*
For an example of a completely cut graph, consider another 2-forest F *in the upper line of figure 2.9 which induces a different edge set* $E_{\Gamma_f} = \{2, 3, 4\}$*. This yields a cut graph* Γ_c *with no intact (1PI) subgraphs left and thereby is identical to* F*. Again, for the 3-loop graph below the cutting works identically.*

The two above definitions are essential for later proving theorem 5.2.1.

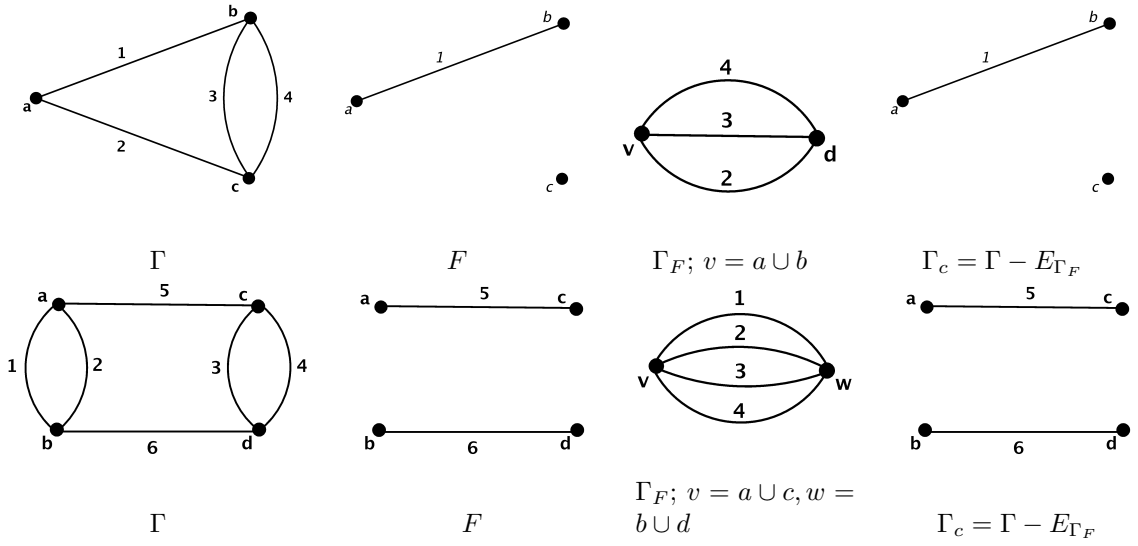


Figure 2.9: A completely cut graph Γ_c of the graph Γ as constructed from a forest F.

Also note, that there are graphs Γ which allow for no cuts for they can not be separated in at least two components. Here, we only state that these graphs are mapped to zero by a cutting and come back to a more detailed consideration of this fact in remark 5.1.

2.4.3. K-cuts

It is possible to more generally define cuts of graphs by using k-forests (instead of 2-forests) and dropping the condition of connectedness to external half edges.

Definition 2.4.7. (K-cut graph)
A k-cut graph *is a pair* (Γ, F) *of a graph* Γ *and one of its* k*-forests* F *which induces elements* $\Gamma_i \in \mathcal{G}^F$, $1 \leq i \leq k$.

Definition 2.4.8. (Completely k-cut graph)
A completely k-cut graph *is a pair* (Γ, F) *of a graph* Γ *and one of its* k*-forests* F *such that the elements* $\Gamma_i \in \mathcal{G}^F$, $1 \leq i \leq k$, *in the corresponding induced set of graphs contain no intact subgraphs.*

These definitions become important for the generalization of theorem 5.2.1 to arbitrary k-cuts, theorem 5.3.1.

3. Hopf algebraic structures

The discovery of the Hopf algebra structure on the set of Feynman graphs enabled a systematic understanding of renormalization[1] from an algebraic viewpoint [6]. But there are further promising Hopf algebra structures to be explored, namely the core Hopf algebra $\mathcal{H}_c$, first introduced in [1], which holds a connection to unitarity and analytic properties of Feynman diagrams via Cutkosky cutting rules. The core Hopf algebra $\mathcal{H}_c$ can be constructed from dropping the requirement for divergence generally entering the definition of the coproduct such that it filters for all $1PI$ subgraphs. This way, the core Hopf algebra can be regarded as a Hopf algebra for a field theory in infinite dimension as well as a generalization of the renormalization Hopf algebra, including the latter as a quotient.
This chapter serves to introduce the general concept of a Hopf algebra (section 3.1, following [5]) and the two physically important cases of the renormalization Hopf algebra (section 3.2, after [6, 2]) and the core Hopf algebra (section 3.3, after [1, 4, 2, 7]). As our focus is to understand the core Hopf algebra, we keep the other two chapters short and restrict to presenting concepts of direct importance for this goal. For details and omitted proofs we refer to the cited literature.

3.1. The Hopf algebra

Let $\mathbb{K}$ be a field. All vector spaces and tensor products are to be understood over this field. Also we canonically identify the vector space V with $V \cong \mathbb{K} \otimes V \cong V \otimes \mathbb{K}$.
Further, let $\tau_{V,W} : V \otimes W \to W \otimes V, v \otimes w \mapsto w \otimes v$ denote the twist map.

Definition 3.1.1. (Associative unital algebra)
An associative unital $\mathbb{K}$-algebra $(A, m, \mathbb{I})$ *is a* $\mathbb{K}$*-vector space* A *together with two linear maps,* $m : A \otimes_{\mathbb{K}} A \to A$ *(multiplication) and* $\mathbb{I} : \mathbb{K} \to A$ *(unit) such that:*

$$m \circ (id \otimes m) = m \circ (m \otimes id) \tag{3.1}$$

$$m \circ (\mathbb{I} \otimes id) = m \circ (id \otimes \mathbb{I}) \tag{3.2}$$

The multiplication is commutative if $m = m \circ \tau$. Equation (3.1) and (3.2) are equivalent to the commutativity of the following diagrams.

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes id} & A \otimes A \\ {\scriptstyle id \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow[m]{} & A \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{\mathbb{I} \otimes id} & A \otimes A & \xleftarrow{id \otimes \mathbb{I}} & A \otimes \mathbb{K} \\ & \searrow^{\simeq} & \downarrow{\scriptstyle m} & \swarrow_{\simeq} & \\ & & A & & \end{array} \tag{3.3}$$

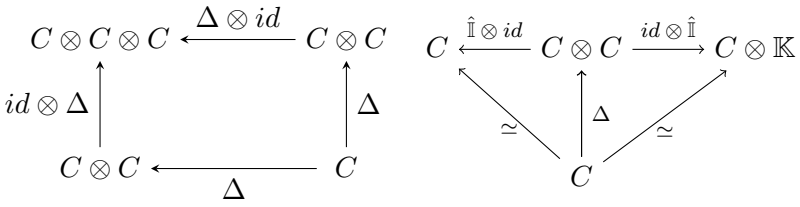
Definition 3.1.2. (Coassociative counital coalgebra)
A coassociative counital $\mathbb{K}$-coalgebra $(C, \Delta, \hat{\mathbb{I}})$ *is a* $\mathbb{K}$*-vector space* C *together with two linear maps* $\Delta : C \to C \otimes C$ *(compultiplication) and* $\hat{\mathbb{I}} : C \to \mathbb{K}$ *(counit) such that:*

$$(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta \tag{3.4}$$

$$(id \otimes \hat{\mathbb{I}}) \circ \Delta = \tau_{C,\mathbb{K}}(\hat{\mathbb{I}} \otimes id) \circ \Delta \tag{3.5}$$

The coproduct is cocommutative if $\tau \circ \Delta = \Delta$. As the coalgebra is the object dual to the algebra, the commutative diagrams corresponding to equations (3.4) and (3.5) can be obtained by simply reversing the arrows in equation (3.3).

[1] The treatment of divergencies in Feynman integrals occurring in the high energy limit of the momenta, i.e. $p^\mu \to \infty$, such as to obtain finite expressions from them.



(3.6)

A common way to denote the coproduct is *Sweedler's notation*, $\Delta(x) = \sum_x x' \otimes x''$, which is frequently used in the following chapters within the context of the physical Hopf algebras.

Definition 3.1.3. (Algebra morphism)
For two algebras $(A, m_A, \mathbb{I}_A)$ *and* $(B, m_B, \mathbb{I}_B)$ *a linear map* $\phi : A \to B$ *is an* algebra morphism *if*

$$\phi \circ \mathbb{I}_{\mathbb{A}} = \mathbb{I}_B \tag{3.7}$$

$$\phi \circ m_A = m_B \circ (\phi \otimes \phi) \tag{3.8}$$

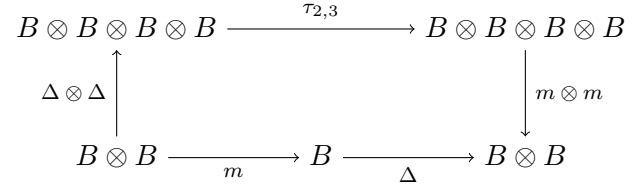
For two coalgebras $(C, \Delta_C, \hat{\mathbb{I}}_C)$ *and* $(D, \Delta_D, \hat{\mathbb{I}}_D)$ *a linear map* $\psi : C \to D$ *is an* algebra morphism *if*

$$\hat{\mathbb{I}}_D \circ \psi = \hat{\mathbb{I}}_C \tag{3.9}$$

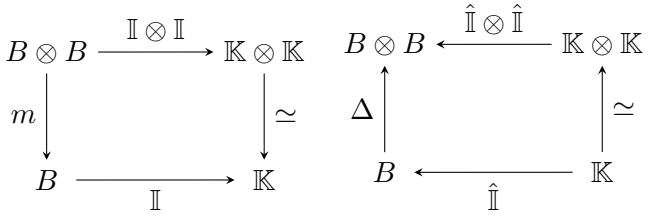
$$\Delta_D \circ \psi = (\psi \otimes \psi) \circ \Delta_C \tag{3.10}$$

Definition 3.1.4. (Bialgebra)
A $\mathbb{K}$-bialgebra $(B, m, \mathbb{I}, \Delta, \hat{\mathbb{I}})$ *is a* $\mathbb{K}$*-vector space* B *which has a* $\mathbb{K}$*-algebra structure* $(m, \mathbb{I})$ *and a* $\mathbb{K}$*-coalgebra structure* $(\Delta, \hat{\mathbb{I}})$*, such that* $(m, \mathbb{I})$ *are coalgebra morphisms and* $(\Delta, \hat{\mathbb{I}})$ *are algebra morphisms.*

We note that m is a coalgebra morphism if and only if Δ is an algebra morphism. So the bialgebra can equivalently be defined via the commutativity of the following diagrams.



(3.11)

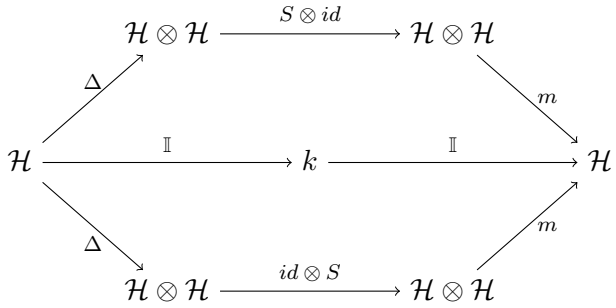


(3.12)

Definition 3.1.5. (Hopf algebra)
A Hopf algebra $(\mathcal{H}, m, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S)$ *is a bialgebra* $(\mathcal{H}, m, \mathbb{I}, \Delta, \hat{\mathbb{I}})$ *endowed with an antipode, i.e. a map* $S \in Hom(\mathcal{H}, \mathcal{H})$ *that satisfies*

$$m \circ (S \otimes id)\Delta = m \circ (id \otimes S)\Delta = \mathbb{I} \circ \hat{\mathbb{I}} \tag{3.13}$$

Equivalently to (3.13) the diagram below commutes. The antipode of a Hopf algebra is unique and closely related to the convolution product introduced below.



(3.14)

Convolution product

Take $f, g \in Hom(\mathcal{H}, \mathcal{H})$. Then the so called convolution product[1]

$$f * g := m \circ (f \otimes g) \circ \Delta \tag{3.15}$$

defines another linear map on $\mathcal{H}$. $(Hom(\mathcal{H}, \mathcal{H}), *, e)$ can be shown to form an algebra with neutral element $e := \mathbb{I} \otimes \hat{\mathbb{I}}$. The $*$-inverse is given by $f^{-1} = f \circ S$ with S the above given antipode. Now, equation (3.13) allows to interpret the antipode as the $*$-inverse of the identity map id on $\mathcal{H}$.

Filtration and connectedness

The Hopf algebra $\mathcal{H}$ comes with a filtered structure which can be useful for performing inductive proofs. Especially for connected Hopf algebras the start of induction is trivial.

Definition 3.1.6. (Graduation) *A Hopf algebra $\mathcal{H}$ is* graded *if*

$$\mathcal{H} = \oplus_{i=0}^{\infty} \mathcal{H}_n, \tag{3.16}$$

$$m(\mathcal{H}_n \otimes \mathcal{H}_m) \subseteq \mathcal{H}_{n+m}, \tag{3.17}$$

$$\Delta(\mathcal{H}_n) \subseteq \oplus_{j+k=n} \mathcal{H}_j \otimes \mathcal{H}_k = \oplus_{i=0}^{n} \mathcal{H}_i \otimes \mathcal{H}_{n-i}, \tag{3.18}$$

$$S(\mathcal{H}_n) \subseteq \mathcal{H}_n. \tag{3.19}$$

Definition 3.1.7. (Connectedness)
A graded Hopf algebra is connected *if*

$$\mathcal{H}_0 \simeq \mathbb{K}\mathbb{I}. \tag{3.20}$$

Definition 3.1.8. (Filtration)
A Hopf algebra $\mathcal{H}$ is filtered *if there is a growing sequence of subspaces $\mathcal{H}^n \subseteq \mathcal{H}^{n+1}$, $n \in \mathbb{N}_0$, fulfilling the conditions*

$$\mathcal{H} = \sum_{i=0}^{\infty} \mathcal{H}^n, \tag{3.21}$$

$$m(\mathcal{H}^n \otimes \mathcal{H}^m) \subseteq \mathcal{H}^{n+m}, \tag{3.22}$$

$$\Delta(\mathcal{H}^n) \subseteq \sum_{i=0}^{n} \mathcal{H}^i \otimes \mathcal{H}^{n-i}, \tag{3.23}$$

$$S(\mathcal{H}^n) \subseteq \mathcal{H}^n. \tag{3.24}$$

Every graduation induces a filtration by $\mathcal{H}^n = \oplus_{k=0}^{n} \mathcal{H}_k$.

[1] Generally, it suffices to take a $\mathbb{K}$-algebra A and $\mathbb{K}$-coalgebra C to define the convolution product $f * g := m_A \circ (f \otimes g) \circ \Delta_c$.

3.2. The renormalization Hopf algebra

Let us now look at the first physically important realization of a Hopf algebra, namely the renormalization Hopf algebra $\mathcal{H}$. To begin with, we enlist its constituents explicitly, then we give a brief overview over the renormalization procedure known from quantum field theory and relate it to its new representation in purely algebraic terms.

3.2.1. Constituents and properties

In order to introduce the Hopf algebra $\mathcal{H}$ as a framework for renormalization, the notion of a (superficially) divergent graph is essential. It is defined via the concept of a weight[1] of a graph Γ originating from powercounting in the integral expression assigned to a Feynman diagram.
To all edges $e \in E_\Gamma$ and vertices $v \in V_\Gamma$, we assign weights $w(e), w(v) \in \mathbb{Z}$, and define the *weight of a graph* to be the integer

$$w(\Gamma) = D|\Gamma| - \sum_{v \in V_\Gamma} w(v) - \sum_{e \in E_\Gamma} w(e), \tag{3.25}$$

where D denotes the dimension of space time. For in this thesis we restrict our considerations to scalar graphs, we always set $w(v) = 0$ and $w(e) = 2$ in the following.
A graph is called *superficially divergent* if $w(\Gamma) \geq 0$. For instance, in dimension $D = 4$ the graph in figure 2.3 has weight $w(\Gamma) = 4 \cdot 2 - 0 - 8 = 0$, i.e. it is superficially divergent.

With this in place, the ingredients of the renormalization Hopf algebra take the following explicit form. Consider the free commutative $\mathbb{Q}$-algebra

$$\mathcal{H} = \oplus_{i \geq 0} \mathcal{H}^{(i)}, \mathcal{H}^{(0)} \sim \mathbb{Q}\mathbb{I}, \tag{3.26}$$

generated by all (infinitely many) 1PI graphs as free generators. The full renormalization Hopf algebra $\mathcal{H}(m, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S)$ is then given by a multiplication m defined as the disjoint union of graphs

$$m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}, \gamma_i \otimes \gamma_j \mapsto \gamma_i \sqcup \gamma_j, \tag{3.27}$$

and an algebra unit

$$\mathbb{I} : \mathbb{Q} \to \mathcal{H}, q \mapsto q\mathbb{I}. \tag{3.28}$$

Further, we have a coproduct Δ which assigns to a graph Γ the sum over all tensor products of its superficially divergent subgraphs γ with their respective cographs Γ/γ as defined in section 2.4:

$$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma, \gamma = \sqcup_i \gamma_i, w(\gamma_i) \geq 0} \gamma \otimes \Gamma/\gamma, \tag{3.29}$$

while the counit

$$\hat{\mathbb{I}} : \mathcal{H} \to \mathbb{Q}, q\mathbb{I} \mapsto q, \tag{3.30}$$

maps the augmentation ideal $Aug = \oplus_{i \geq 1} \mathcal{H}^{(i)}$ (i.e. all nontrivial Hopf algebra elements, precisely, all graphs except for tree-level graphs and the empty graph) to zero.
Finally, the antipode is given by

$$S : \mathcal{H} \to \mathcal{H}, S(\Gamma) = -\Gamma - \sum_{\gamma \subsetneq \Gamma, \gamma = \sqcup_i \gamma_i, w(\gamma_i) \geq 0} S(\Gamma)\Gamma/\gamma, \tag{3.31}$$

and plays a leading role for renormalization (see section 3.2.2).

[1] Feynman rules allow to transfer the power counting in the integrand, a rational function of the external momenta q_i^μ, to a counting of graph components. See chapter 4.2 for an explicit integral expression.

A most useful *grading* on $\mathcal{H}$ is given by the superficial degree of divergence of the graphs.

It is helpful to define the *reduced coproduct* (for an example see figure 3.2a),

$$\tilde{\Delta}(\Gamma) := \Delta(\Gamma) - (\Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma) = \sum_{\gamma \subsetneq \Gamma, \gamma = \sqcup_i \gamma_i, w(\gamma_i) \geq 0} \gamma \otimes \Gamma/\gamma, \tag{3.32}$$

in order to define an important class of Feynman graphs called primitive elements.
We call a Feynman graph *primitive* if $\tilde{\Delta}(\Gamma) = 0$. Thus, for the renormalization Hopf algebra the primitive elements are those graphs Γ which contain no superficially divergent subgraphs $\gamma \subsetneq \Gamma$.

3.2.2. The renormalization procedure

Let us give a brief conceptual overview of renormalization as known from quantum field theory to see how it relates to the above Hopf algebraic notions.

In quantum field theory, renormalization is employed as mathematically consistent method to receive finite expressions from divergent Feynman integrals. Essentially, the renormalization procedure is based on the systematic subtraction of integrals. Let us give a brief motivation. Consider the logarithmically divergent integral

$$I = \int_0^\infty \frac{1}{x+1} dx \to \infty. \tag{3.33}$$

The first step to handling a divergent integral is to introduce a regulator. For instance, introducing a complex parameter α, with $Re(\alpha) > 0$, as a power in the denominator, ensures the convergence of the integral

$$I(\alpha) = \int_0^\infty \frac{1}{(x+1)^\alpha} dx. \tag{3.34}$$

Note, that for $\alpha \to 1$ we recover the original integrand. Even though we can calculate the integral now, the correct physical result is obtained in the limit $\alpha \to 1$, which is still infinite. This is where renormalization comes into play. Introducing the regulator allows to view the original integral as a function $I(\alpha)$, which has a Laurent series in α. In the series, we are interested mostly in the pole terms at $\alpha = 1$. The key idea of renormalization is that two expressions with the same pole part in α in their Laurent series can be subtracted, such that poles cancel mutually and we get a meaningful finite result when finally taking limit $\alpha \to 1$.

Now, let us look at the full renormalization procedure. We benefit from the fact that Feynman rules allow to capture renormalization structurally in terms of graphs Γ rather than (regularized) integrals.
Further, we are interested in the renormalization procedure itself, i.e. the canceling of divergent parts of the Laurent series. Therefore we forget about possible finite terms and consider graphs equivalent, $\Gamma_i \sim \Gamma_j$, if they give Laurent series with identical pole parts. Obviously, for a finite expression this means $\Gamma \sim 0$.
Generally, a graph Γ can not only give rise to one overall divergence, but rather contain subdivergences from its subgraphs $\gamma \subsetneq \Gamma$. The same can be true again for each γ. Therefore, one must pay special attention to properly cancel every single divergence to finally obtain a finite result. If (sub-) divergencies are present, can be identified by calculating the weight for Γ and all subgraphs γ after equation (3.25).
Renormalization constructs for a divergent graph Γ a new expression $\hat{\Gamma}$, which has all its subdivergences (from subgraphs) renormalized. From this, it calculates the so called counterterm $Z(\hat{\Gamma})$ with the same pole part as $\hat{\Gamma}$ but opposite sign. Then the quantity $\hat{\Gamma} + Z(\hat{\Gamma})$ is finite.

Zimmermann's forest[1] *formula* organizes this canceling in a beautiful way by recursively constructing counterterms

$$Z(\Gamma) = -R(\Gamma) - \sum_{\gamma \in \Gamma} R(Z(\gamma)\Gamma/\gamma), \tag{3.35}$$

where we wrote $Z(\Gamma)$ for convenience, but still mean that Z only constructs overall counterterms, i.e. counterterms for graphs $\hat{\Gamma}$ with all subdivergences renormalized. For instance, a subgraph $\gamma \in \Gamma$ with no overall divergence has $Z(\gamma) = 0$. In the case of subgraphs γ containing several disconnected components, e.g. $\gamma = \gamma_i \sqcup \gamma_j$, we have $Z(\gamma) = Z(\gamma_i)Z(\gamma_j)$. $R(\Gamma)$ denotes the renormalization map.

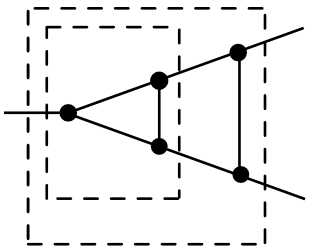


Figure 3.1: An overall divergent graph Γ (outer box) with one divergent subgraph $\gamma \subsetneq \Gamma$ (inner box).

Example 3.2.1. (Renormalization)
Zimmerman's forest formula is best illustrated in the simple example of figure 3.1. Note, that for this graph $\Gamma/\gamma = \gamma$. Generally, we start calculating counterterms for subgraphs and work our way up to the final overall divergence of Γ.

$$Z(\gamma) = -R(\gamma) \tag{3.36}$$

$$\hat{\Gamma} = \Gamma - Z(\gamma)\gamma \tag{3.37}$$

$$Z(\Gamma) = R(\hat{\Gamma}) = -R(\Gamma) + R(Z(\gamma)\gamma) \tag{3.38}$$

$$\hat{\Gamma} + Z(\Gamma) = \Gamma - Z(\gamma)\gamma - R(\Gamma) + R(Z(\gamma)\gamma) \tag{3.39}$$

In the last line (3.39), everything is expressed in terms of graphs without subdivergences (primitives). It corresponds to the renormalized (finite) graph $\hat{\Gamma} + Z(\Gamma) \sim 0$.

The renormalization Hopf algebra $(\mathcal{H}, m, \mathbb{I}, \Delta, \hat{\mathbb{I}}, S)$ models the above process for, as indicated in section 3.2.1, its ingredients are constructed to take the following tasks:

- The counit $\hat{\mathbb{I}}$ annihilates graphs.
- The coproduct Δ generates all terms necessary to compensate the subdivergences.
- The renormalization schemes correspond to maps $R : \mathcal{P} \to \mathcal{P}$ fulfilling $R(\Gamma) \sim \Gamma$, where $\mathcal{P}$ denotes the set of all primitive graphs.
- The antipode $S(R(\Gamma))$ equals the counterterm $Z(\Gamma)$ for a given graph Γ.
- Finally, the defining equation (3.13) for the antipode $m \circ (S \otimes id)\Delta(\Gamma) = \mathbb{I} \circ \hat{\mathbb{I}}(\Gamma)$ gives the finite term ~ 0, corresponding to the renormalized graph. For our example graph in figure 3.1 this result is equal to equation (3.39).

We refer to [6] for examples, further or more detailed information.

[1] Here, forests mean families of divergent $1PI$ subgraphs which are non-overlapping, i.e. pairwise disjoint w.r.t. edges and vertices or included one in the other w.r.t their edges. In fact, this is equivalent to our notion of subgraphs γ introduced in chapter 2.4.1.

3.3. The core Hopf algebra

Beyond the Hopf algebra for renormalization, one can rightly suspect further algebra structures to underly perturbative quantum field theory. The core Hopf algebra $\mathcal{H}_c$ can be constructed from the renormalization Hopf algebra by dropping the condition of superficial divergence entering the definition of the coproduct Δ and the antipode S.
In chapter 5 we will see that $\mathcal{H}_c$ combines well with the notion of Cutkosky cutting graphs, which again is closely related to the unitarity of the S-matrix and analytic properties of Feynman graphs (see chapter 4).

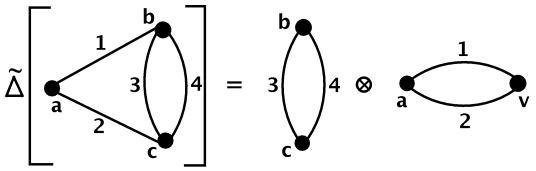
Accordingly, $\mathcal{H}(m, \mathbb{I}, \Delta_c, \hat{\mathbb{I}}, S_c)$ is given with multiplication m, algebra unit $\mathbb{I}$, and counit $\hat{\mathbb{I}}$ as defined in section 3.2 but with a new core coproduct Δ_c and antipode S_c defined as

$$\Delta_c : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \Delta_c(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma, \gamma = \sqcup_i \gamma_i} \gamma \otimes \Gamma/\gamma, \tag{3.40}$$

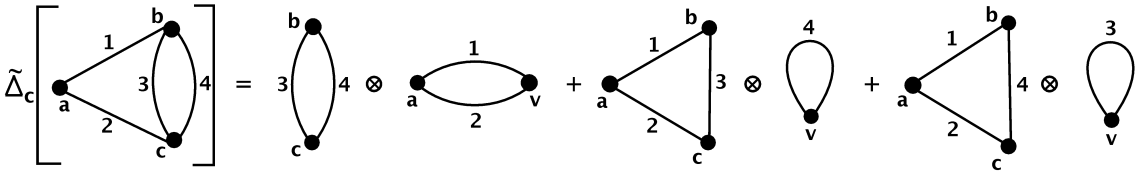
and

$$S_c : \mathcal{H} \to \mathcal{H}, S_c(\Gamma) = -\Gamma - \sum_{\gamma \subsetneq \Gamma, \gamma = \sqcup_i \gamma_i} S(\Gamma)\Gamma/\gamma. \tag{3.41}$$

A natural *grading* on $\mathcal{H}_c$ is given by the loop number due to $|\Gamma| = |\gamma| + |\Gamma/\gamma|$ as filtered by the core coproduct Δ_c.



(a) The reduced renormalization coproduct $\tilde{\Delta}$ in dimension $D = 4$.



(b) The reduced core coproduct $\tilde{\Delta}_c$.

Figure 3.2: Coproduct application to a 4-regular 2-loop graph.

Analogously to before, we define the *reduced core coproduct* $\tilde{\Delta}_c := \Delta_c - (\Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma)$.

Note that now the *primitive elements* ($\tilde{\Delta}_c = 0$) are graphs with no 1PI subgraphs (i.e. one-loop graphs).

Example 3.3.1. *As an example for the application of the two coproducts Δ and Δ_c, see figure 3.2. Compared with the renormalization coproduct Δ in (3.2a), the core coproduct Δ_c in (3.2b) yields two more graphs. They are filtered out by Δ as they are not superficially divergent after equation (3.25), assumed dimension*[1] *$D = 4$.*

[1]Dimension $D = 4$ is a convenient choice as then ϕ^4-theory is renormalizable.

Relations to the renormalization Hopf algebra

Another pretty viewpoint for looking at the core Hopf algebra is by starting from the power counting formula (3.25). There we observe that if the space time dimension D is high enough, any graph becomes divergent. Thus, we can regard the core Hopf algebra $\mathcal{H}_c$, as the renormalization Hopf algebra of a field theory in infinite dimension.

More generally, we can build a tower of Hopf algebras by incrementally increasing in the power counting formula (3.25) the dimension[1] $D = 4, 6, ..., 2n, .., \infty$. Doing so, allows for more and more graphs to be regarded divergent. Now, consider the coproduct Δ as the defining map for our Hopf algebras. It filters subgraphs only if they are divergent for dimension D, such that it gives a Hopf algebra for each growing dimension:

$$\mathcal{H}_4 \subset \mathcal{H}_6 \subset ... \subset \mathcal{H}_{2n} \subset ... \subset \mathcal{H}_\infty \equiv \mathcal{H}_c, \tag{3.42}$$

with $\mathcal{H}_c := \cup_{D=0}^{\infty} \mathcal{H}_{\mathcal{D}}$. $\mathcal{H}_D$ is a subalgebra of $\mathcal{H}_{D+n}$, but not a Hopf subalgebra as their respective coproducts differ.

We can also show that the renormalization Hopf algebra $\mathcal{H}$ is contained in the core Hopf algebra $\mathcal{H}_c$ as a quotient. Again, let us start from general considerations. In $\mathcal{H}_{D+n}$ we can consider the following $\mathbb{Q}$-ideal

$$\mathcal{I} := \langle \Gamma \in \mathcal{G} | w_D(\Gamma) > 0 \text{ and } w_{D+n}(\Gamma) \leq 0 \rangle, \tag{3.43}$$

with $\mathcal{G}$ the set of all Feynman graphs. The ideal filters out all graphs that are not divergent in D dimensions and thereby excluded from $\mathcal{H}_D$ but divergent in $D + n$ and included in $\mathcal{H}_{D+n}$. With respect to this ideal we can build the quotient

$$\mathcal{H}_D \simeq \mathcal{H}_{D+n}/\mathcal{I}, \tag{3.44}$$

and regard $\mathcal{H}_D$ as a quotient Hopf algebra of $\mathcal{H}_{D+n}$. The renormalization Hopf algebra is then just a special case of $\mathcal{H}_D$ while we can identify $\mathcal{H}_{D+n}$ with $\mathcal{H}_c$.

We remark that there is a strong connection between the core Hopf algebra $\mathcal{H}_c$ and quantum gravity (see for instance [4, 9]). Especially, for theories with a power counting like quantum gravity the core Hopf algebra becomes the renormalization Hopf algebra.

[1] We consider even dimensions only, for the analytic structures of field theories in even dimensions bear resemblances among each other but not w.r.t. odd dimensions. The same holds true vice versa.

4. Cutkosky cutting rules

Cutkosky cutting rules serve as a powerful tool for they allow to calculate imaginary parts[1] of Feynman diagrams. These imaginary parts originate from poles or branch cuts in the integrand of a Feynman integral, arising when intermediate (virtual) particles can go on shell[2] at the same time. Together with dispersion relations, more familiar to mathematicians under the name of Hilbert transforms, one can also calculate the corresponding real part out of them [11]. Above this, the cutting rules are intimately related to two fundamental features of a quantum field theory, namely causality[3] and conservation of probability, the latter provided by the unitarity of the S-matrix [10].
This chapter introduces Cutkosky cutting rules following established physics literature. We begin in section 4.1 with a heuristic reflection on how Feynman diagrams give rise to imaginary parts after [12, 14, 15]. In section 4.2 we make a connection to the corresponding real part via dispersion relations following [16, 13] and in section 4.3 we present a derivation of the Cutkosky cutting rules after [10, 11]. Finally, in section 4.4 we link the cutting rules to unitarity and the above mentioned imaginary parts after [10, 11, 15].

4.1. Imaginary parts from Feynman diagrams

After plenty of algebra in the previous chapter, now, in order to get an idea of how imaginary parts of Feynman diagrams arise, we intend to open the toolbox of complex analysis. We explicitly view the integrand of a Feynman integral as a complex function of (complex) momentum p. While only real momenta p are physically interesting, this change of perspective has proven to allow for powerful calculations, which can finally be shown to lead back to the real case. The latter is shown in the following section.

Consider a single Feynman integral $G(\{q_i\})$, $i = 1, ..., n$, as given by a Feynman diagram in its most general form

$$G(\{q_i\}) = \lim_{\epsilon \to 0} \prod_r \int d^d k_r \frac{N(k_r, q_i)}{D(k_r, q_i)}, \tag{4.1}$$

with denominator

$$D(k_r, q_i) = \prod_{j=1}^{n} (l_j^2(k_r, q_i) - m_j^2 + i\epsilon), \tag{4.2}$$

such that $l_j(k_r, q_i)$ denotes the momenta of the internal edges as a linear function of loop momenta $\{k_r\}$ and external momenta $\{q_i\}$, and m_j denotes the particle mass.
In practice, evaluating such integrals is nontrivial. As seen in chapter 3.2.2, the integrands do not necessarily fall off fast enough at infinity for the corresponding integrals to converge. For now, assume the divergencies from momenta $k_i \to \infty$ to be treated by renormalization such that we can neglect them.

Apart from this, the integrands can diverge at propagator (denominator) poles. So, as our focus is on D now, equal for particles with spin and scalar particles, we set $N = 1$ and restrict to scalar integrals

$$G(\{q_i\}) = \lim_{\epsilon \to 0} \int \frac{d^d k_1 \cdot ... \cdot d^d k_r}{\prod_{j=1}^{n} (p_j^2 - m_j^2 + i\epsilon)}, \tag{4.3}$$

[1] Equally referred to as absorbative parts for the imaginary part of a complex-valued refractive index describes the attenuation of light entering material.
[2] Often abbreviated as loop momenta reaching values for particle production.
[3] For a treatment of causality we refer to [10].

where we directly write out $p_j = l_j(k_r, q_i)$ for the momenta of the n internal lines.
For illustration, let us look at the simplest case of $G(\{q_i\})$ which is the tree-level process contributing to the 1-particle decay, given by the single propagator without any loop integrations (see figure 4.1 on the l.h.s). In this case we have only one momentum $q_1 = p$ flowing in from the left such that

$$G(\{p\}) = \frac{1}{p^2 - m^2 + i\epsilon}. \tag{4.4}$$

G has a pole at the on shell value for the momentum $p^2 = m^2$ when $\epsilon \to 0$. At the same time, its imaginary part is by complex extension

$$Im(\frac{1}{p^2 - m^2 + i\epsilon}) = -\frac{\epsilon}{(p^2 - m^2)^2 + \epsilon^2}. \tag{4.5}$$

Given an arbitrary finite value $const = (p^2 - m^2)^2$, this expression vanishes properly for $\epsilon \to 0$ as the constant factor saves the denominator from becoming zero. At $p^2 = m^2$ though, the expression equals $1/\epsilon$ which obviously diverges as $\epsilon \to 0$. If we integrate over this imaginary part of the propagator, like we would in loop calculations, we get[1]

$$\int_{-\infty}^{\infty} \frac{\epsilon}{(p^2 - m^2)^2 + \epsilon^2} dp^2 = -\pi, \tag{4.6}$$

suggesting that

$$Im(\frac{i}{p^2 - m^2 + i\epsilon}) = -\pi\delta(p^2 - m^2), \tag{4.7}$$

which strongly resembles the Cutkosky rules presented at the end of chapter 4.4.
These results already give a heuristic idea of how propagator poles arise at on shell values of the momenta and contribute imaginary parts to Feynman diagrams, which in turn can be calculated from Cutkosky cutting rules.

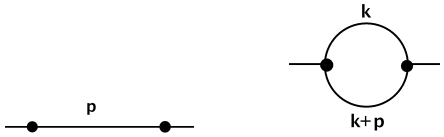


Figure 4.1: Tree-level (left) and one-loop (right) contribution to the 1-particle decay.

Now, consider the first loop-order contribution to the 1-particle decay, depicted in 4.1 on the right. From standard textbook knowledge, the corresponding integral expression, after (Pauli-Villars) regularization and Feynman parametrization, gives

$$G(\{p\}) = g^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{[(k+p)^2 - m^2 + i\epsilon]} \frac{1}{[k^2 - m^2 + i\epsilon]} \tag{4.8}$$

$$= \frac{g^2}{16\pi^2} \int_0^1 dx \; ln\left(\frac{m^2 - p^2 x(1-x)}{\Lambda}\right), \tag{4.9}$$

with Λ denoting the regulator.
Let us analyze the pole structure. From $0 \leq x \leq 1$ it follows that $x(1-x) \leq 1/4$, which means that for $p < 2m$ the logarithm is real. For $p \geq 2m$ the argument in the logarithm becomes negative and creates a branch cut. We are not interested in calculating any results[2], but note

[1] See appendix 4.6.
[2] See for instance [11, 15].

that this branch cut yields a contribution to the imaginary part by using the principal arm of the complex logarithm to find the logarithm of negative arguments $ln(-x) = ln|x| + i\pi$.
From the beforehand tree-level considerations we saw that propagators give rise to imaginary parts when they are put on-shell. Looking one more time at equation (4.8), it becomes obvious that a momentum $p \geq 2m$ is large enough for both propagators to be put on shell at the same time.
The general version of this condition for arbitrary integrals $G(\{q_i\})$ in (4.3) is given by Cutkosky rules, which "cut" propagators in all possible ways such that they can be put on shell simultaneously.

4.2. Dispersion relations

In the previous chapter, we saw that Feynman integrals are generally real valued, while giving rise to imaginary parts at poles or branch cuts. This threatens to cloud our mood for we are persistently interested in the real part along the branch cut. Luckily, dispersion relations allow to circumvent this problem by employing the knowledge of the imaginary part along the branch cut to recover the real part. In chapter 4.4, it is shown how this imaginary part is calculated from Cutkosky rules, for now we assume it to be given.

For in the following we would like to even more employ the toolbox of complex analysis, it makes sense to restart more conceptually and view the integrand (or an amplitude) of a (possibly transformed) complex momentum p^2 as a general complex function $f(s)$ of complex variable s.

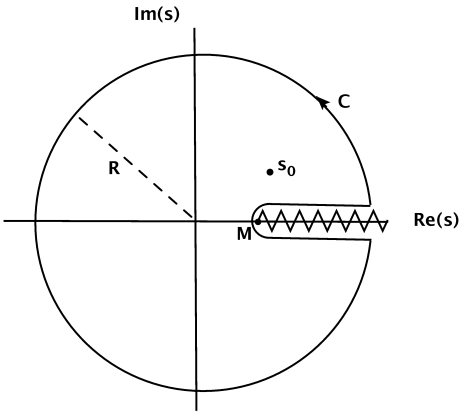


Figure 4.2: Contour of the integration path C in the complex s-plane.

In order to model the above described situation, let $f(s)$ be real for real $s < M$ and let $f(s)$ have a branch cut for real $s \geq M$. Away from the branch cut, let $f(s)$ be analytic[1] for complex s on the upper half plane. We fix the sign of the imaginary part of f above the branch cut by

$$f(s + i\epsilon) = Ref(s) + iImf(s), \tag{4.10}$$

with $\epsilon > 0$ is infinitesimal. Further, assuming $f(s^*) = f^*(s)$ (for our setup is symmetric w.r.t the real axis) we have all requirements fulfilled to reflect[2] f across the real axis and thereby analytically continue it to entire complex plane. Applying the above property we get

$$f(s + i\epsilon) - f(s - i\epsilon) = 2iImf(s). \tag{4.11}$$

Within the contour $\mathcal{C}$, as depicted in figure 4.2, f is analytic at each point s. So we can apply Cauchy's integral formula together with equation (4.11) to give the value of f at an arbitrary point s_0 inside $\mathcal{C}$.

[1] Convergence is guaranteed, when working with renormalized expressions.
[2] By Schwartz' reflection principle.

$$f(s_0) = \frac{1}{2\pi i}\oint_C ds \frac{f(s)}{s-s_0} \tag{4.12}$$

$$= \frac{1}{2\pi i}\left(\int_M^R ds \frac{f(s+i\epsilon)-f(s-i\epsilon)}{s-s_0} + \oint_{|s|=R} ds \frac{f(s)}{s-s_0}\right) \tag{4.13}$$

$$= \frac{1}{\pi}\int_{M^2}^R ds \frac{Imf(s)}{s-s_0-i\epsilon} + \frac{1}{2\pi i}\oint_{|s|=R} ds \frac{f(s)}{s-s_0} \tag{4.14}$$

Take a look at equation (4.14). As we only know $Im(f)$ along the branch cut (from Cutkosky rules) we run into a problem as f also appears in the integral along the circle with radius R. If the right integral vanishes as $R \to \infty$, we obtain the dispersion relation

$$f(s_0) = \frac{1}{\pi}\int_M^\infty ds \frac{Imf(s)}{s-s_0-i\epsilon}, \tag{4.15}$$

which allows to reconstruct f at any point s_0 in $\mathcal{C}$ from $Imf(s)$, and especially the real part via a suitable limiting procedure

$$Ref(s_0) = \lim_{R-\infty}\frac{1}{\pi}\int_M^R ds \frac{Imf(s)}{s-s_0}. \tag{4.16}$$

Generally though, the right integral in equation (4.14) is nonzero. However, to obtain a dispersion relation one can, for instance, subtract from equation (4.12) its value at some real point $s_1 < M$,

$$f(s_0)-f(s_1) = \frac{s_0-s_1}{\pi}\int_M^R ds \frac{1}{s-s_1}\frac{Imf(s)}{s-s_0-i\epsilon} + \frac{s_0-s_1}{2\pi i}\oint_{|s|=R} ds \frac{1}{s-s_1}\frac{f(s)}{s-s_0}. \tag{4.17}$$

Now, the right integrand has a better convergence behavior than the right integral in equation (4.14), precisely $O(1/s^2)$ compared to $O(1/s)$ before, such that the integral vanishes for $R \to \infty$. We arrive at the subtracted dispersion relation

$$f(s_0) = f(s_1) + \frac{s_0-s_1}{\pi}\int_M^\infty ds \frac{1}{s-s_1}\frac{Imf(s)}{s-s_0-i\epsilon}. \tag{4.18}$$

This procedure may be repeated if necessary.

4.3. Derivation of the Cutkosky cutting rules

This section reveals the physical origin of our purely graph theoretic definition of Cutkosky cut graphs in chapter 2.4.2. The cutting rules can be derived by analyzing the energy flow trough a Feynman diagram. For simplicity, the derivation is again restricted to a scalar field theory (ϕ^3-theory for illustrations) and integrals are assumed to be regulated, so no divergencies occur.

Let $x_i \equiv x_i^\mu$, $\mu = 0,1,2,3$, be the four vector with time component x_i^0 and let $i = 1,...,n$ denote different points in space time. The starting point for deriving the cutting rules is to consider the scalar Feynman propagator in coordinate space

$$\Delta_{ij} \equiv \Delta_F(x_i - x_j) = \langle 0|T(\phi(x_i)\phi(x_j))|0\rangle, \tag{4.19}$$

with the time ordered product $T(\phi(x_i),\phi(x_j))$ of the fields

$$T(\phi(x_i),\phi(x_j)) = \begin{cases} \phi(x_i)\phi(x_j), & x_i^0 > x_j^0 \\ \phi(x_j)\phi(x_i), & x_i^0 < x_j^0. \end{cases} \tag{4.20}$$

We write down the explicit Fourier representation of the propagator

$$\Delta_{ij} = \int \frac{d^d k}{(2\pi)^d} e^{-ikx} \frac{i}{k^2 - m^2 + i\epsilon}, \tag{4.21}$$

where we abbreviate $x = x_i - x_j$ inside the integral.

We now want to decompose the propagator $\Delta_F(x)$ into positive and negative energy parts. Using the heavyside function

$$\theta(x) = \begin{cases} 1, & x^0 > 0 \\ 0, & x^0 < 0, \end{cases} \tag{4.22}$$

we can express the time ordered product T of fields by a product of normal fields to receive the following expression

$$\Delta_{ij} = \theta(x_i - x_j)\Delta^+_{ij} + \theta(x_j - x_i)\Delta^-_{ij}, \tag{4.23}$$

with

$$\Delta^+_{ij} = \langle 0|\phi(x_i)\phi(x_j)|0\rangle = \int \frac{d^d k}{(2\pi)^d} e^{-ikx}\theta(+k)2\pi\delta(k^2 - m^2), \tag{4.24}$$

and

$$\Delta^-_{ij} = \langle 0|\phi(x_j)\phi(x_i)|0\rangle = \int \frac{d^d k}{(2\pi)^d} e^{-ikx}\theta(-k)2\pi\delta(k^2 - m^2). \tag{4.25}$$

We note that Δ^+ and Δ^- fulfill

$$\Delta^\pm_{ij} = \Delta^\mp_{ji} \quad \text{and} \quad \Delta^\pm_{ij} = (\Delta^\mp_{ij})^*, \tag{4.26}$$

such that we obtain the complex conjugate of the propagator

$$\Delta^*_{ij} = \theta(x_i - x_j)\Delta^-_{ij} + \theta(x_j - x_i)\Delta^+_{ij}. \tag{4.27}$$

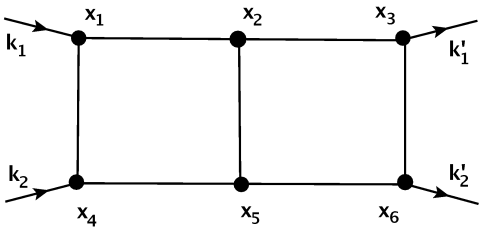


Figure 4.3: A (Φ^3-theory) Feynman graph in coordinate space.

Now, consider a graph with n vertices in coordinate space (an example with $n = 6$ depicted in figure 4.3). By convention incoming particles with momentum k_i, $i = 1, 2$, are drawn on the l.h.s., outgoing particles with momentum k'_i, $i = 1, 2$, on the r.h.s. as indicated by the arrows.

We can calculate the amplitude of this diagram by applying the usual coordinate space Feynman rules for a scalar theory. Assign to:

1. Each vertex x_i a factor $-ig$, with g the coupling constant.
2. Each internal edge from x_i to x_j a propagator Δ_{ij}. In scalar theories we have $\Delta_{ij} = \Delta_{ji}$, i.e. the orientation of the edges is irrelevant.

3. Each incoming particle with impulse k_i, represented by an external edge attached to a vertex x_j, a factor $e^{-k_i x_j}$.

4. Each outgoing particle with impulse k_i, represented by an external edge with attached to a vertex x_j, a factor $e^{k_i' x_j}$.

5. Integrate over all vertices x_i.

Above, we gave the full set of rules again in order to compare them to the new rules to be constructed below. For now, we consider only the part of the amplitude $F(x_1, ..., x_n)$ before multiplying with external particles and before integration (i.e. disregarding points 4 and 5). For instance the graph in figure 4.3 gives $F(x_1, x_2, x_3, x_4, x_5, x_6) = (-ig)^6 \Delta_{12}\Delta_{23}\Delta_{36}\Delta_{56}\Delta_{45}\Delta_{14}\Delta_{25}$.

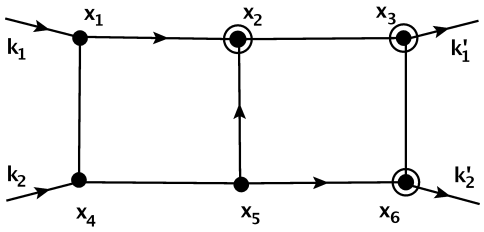


Figure 4.4: A (Φ^3-theory) Feynman graph in coordinate space with circled vertices.

Next, consider the same graph with some of its vertices circled as shown in figure 4.4. For clarity, arrows are drawn on edges from uncircled vertices to circled vertices. Lines with end vertices of the same type remain as before. Later, we will see that these arrows, together with those from the incoming and outgoing particles, indicate the energy flow through the diagram.

For a diagram with an arbitrary number of vertices circled, we calculate a similar function as above, namely, $F(x_1, ..., \underline{x}_i, ..., x_j, ..., \underline{x}_m, ..., x_n)$, with the $\underline{x}_l$ representing circled vertices. In order to calculate this F, we extend the above given Feynman rules to adapt to the newly generated cases in the diagram. Assign:

6. $-ig \to ig$ for each circled vertex $\underline{x}_i$, i.e. the complex conjugate of an uncircled vertex.

7. $\Delta_{ij} \to \Delta_{i\underline{j}} = \Delta^-_{ij}$ for each edge from a circled vertex $\underline{x}_j$ to an uncircled vertex x_i.

8. $\Delta_{ij} \to \Delta_{\underline{i}j} = \Delta^+_{ij}$ for each edge from an uncircled vertex x_j a circled vertex $\underline{x}_i$.

9. $\Delta_{ij} \to \Delta_{\underline{i}\underline{j}} = \Delta^*_{ij}$ for each edge connecting two circled vertices $\underline{x}_i$ and $\underline{x}_j$, i.e. the complex conjugate propagator.

As before, Δ_{ij} and Δ^*_{ij} are invariant under exchange of vertices but $\Delta^+_{ij} = \Delta^-_{ji}$ by the l.h.s. property of equation (4.26). In the case where all vertices are uncircled we regain $F(x_1, ..., x_n)$. As an example the diagram in figure 4.4 gives (again disregarding rules 4 and 5): $F(x_1, \underline{x}_2, \underline{x}_3, x_4, x_5, \underline{x}_6) = (ig)^3(-ig)^3\Delta^-_{12}\Delta^*_{23}\Delta^*_{36}\Delta^-_{56}\Delta_{45}\Delta_{14}\Delta^+_{25}$.

The largest time equation

With the previous rules set up, we can now write down a first important statement.

Theorem 4.3.1. (Largest time equation) *In a graph with an arbitrary number of circled vertices, be x_m the vertex of largest time, i.e. $x^0_m > x^0_i$ for all $i \neq m$. Then*

$$F(x_1, ..., \underline{x}_i, ..., x_m, ..., x_n) + F(x_1, ..., \underline{x}_i, ..., \underline{x}_m, ..., x_n) = 0, \tag{4.28}$$

i.e. the two sides differ only in that the vertex x_m is circled in one graph and uncircled in the other.

The two terms are identical except for the minus sign in the right graph containing $\underline{x}_m$, originating from the replacement $-ig \to ig$ for a circled vertex by rule 6.
Now let us examine the energy flow by looking at the explicit propagators $\Delta_{ij}^{\pm}$ and Δ_{ij}. The factor $e^{-ik(x_i - x_j)}$ in the integrals corresponds to a momentum k entering the vertex x_i (as familiar from the multiplicative factors of the incoming particles). So, for all edges between vertices of different types we have $\Delta^{\pm}$ such that the energy flow is always directed towards the circled vertex, whereas for all edges between vertices of the same type (Δ_{ij} or Δ_{ij}^{*}) energy can flow in both directions.

From theorem 4.3.1 follows an important statement:

Corollary 4.3.1.

$$\sum F(x_1, ..., x_n) = 0, \tag{4.29}$$

where the sum runs over all possibilities to circle vertices in the graph.

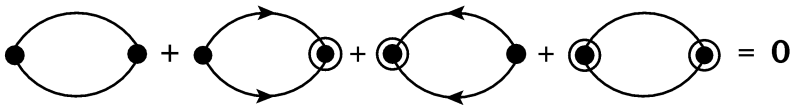


Figure 4.5: An example for corollary 4.3.1.

This can be seen as follows: Be x_m the vertex of largest time. Then the above sum can be decomposed into pairs so that the two terms in the pair are identical except for one space time point x_i being circled in one term but not in the other. These terms cancel by 4.3.1 such that the entire sum adds up to zero. For an example see figure 4.5.

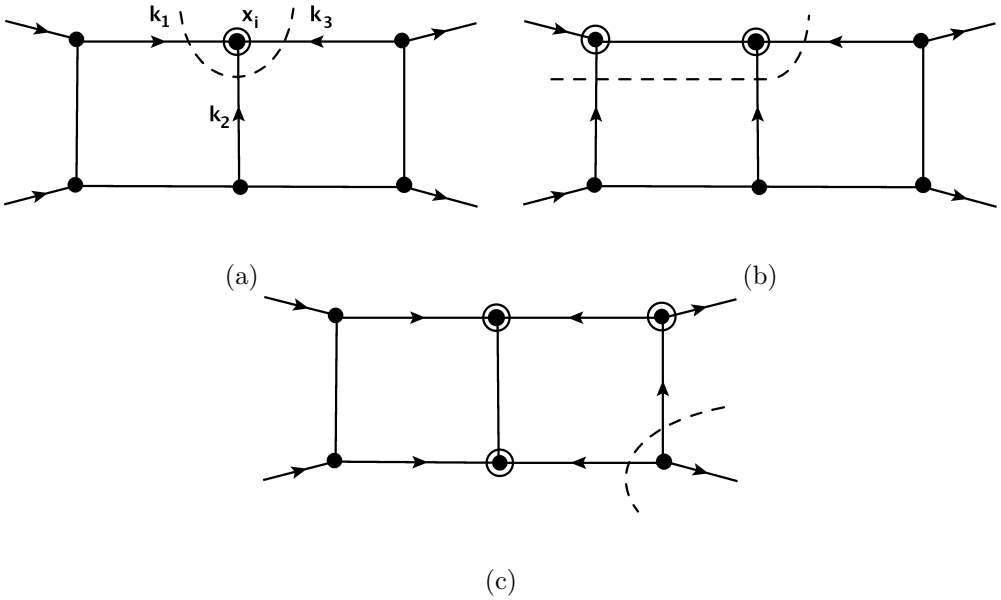


Figure 4.6: Examples for graphs with zero contribution to equation (4.29). Circled and uncircled regions marked by a line.

When looking at all graphs contributing to equation 4.29, one finds that some diagrams are subject to mutually conflicting requirements and therefore must give a zero contribution. For instance, graph (a) in figure 4.6 vanishes because (as described above) the $\Delta^{\pm}$ force the energy

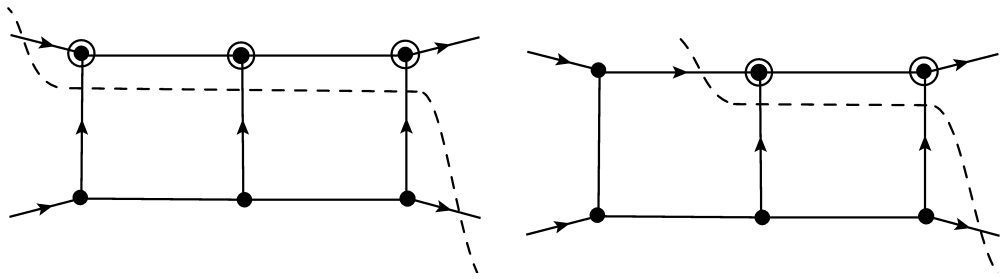
to flow into the vertex $\underline{x}_i$. But in this case, no energy can leave $\underline{x}_i$ to exit the diagram which conflicts with energy conservation.
Similarly, graphs (b) and (c) in figure 4.6 break this requirement. In graph (b) only positive energy enters the circled region (indicated by the arrows), whereas graph (c) gives a zero contribution for only positive energy leaves the uncircled region.

Cutkosky rules

Now we are close to making the final step towards Cutkosky rules. In summary of what we just found, non-vanishing diagrams must fulfill two conditions:

- The circled vertices must constitute a (by internal edges) connected region, linked to one or more outgoing edges through which the inflowing energy can exit.
- Uncircled vertices must constitute a (by internal edges) connected region, linked to one ore more incoming edges through which the energy can flow into the diagram.

In diagrams fulfilling these two conditions, the circled and uncircled vertices can be separated by a so called "cut" which separates the diagram in a "sunny side" linked to the ingoing vertices and a "shady side" linked to the outgoing vertices. Cutting of external half edges has no physical meaning.



(a) (b)

Figure 4.7: Examples allowed cuts of graphs.

From the above preliminary Feynman rules (ordinary Feynman rules 1.-5. plus extension 6.-9.) we can now write down the proper momentum space Feynman rules for graphs with cuts.

- To each "sunny-side" propagator with impulse k assign $\frac{i}{(2\pi)^d}\frac{1}{k^2-m^2+i\epsilon}$.
- To each "shady-side" propagator with impulse k assign $-\frac{i}{(2\pi)^d}\frac{1}{k^2-m^2+i\epsilon}$.
- To each cut propagator through which the impulse k flows from the "sunny-side" to the "shady-side" assign $2\pi\delta(k^2-m^2)\theta(k)$.
- To each vertex on the "sunny-side" assign $(-ig)(2\pi)^d\delta^d(\sum k_i)$.
- To each vertex on the "shady-side" assign $(+ig)(2\pi)^d\delta^d(\sum k_i)$.

Now in corollary 4.3.1 consider the two extreme cases of a totally uncircled amplitude $F(x_1,...,x_n)$ and a totally circled amplitude $\overline{F} = F(\underline{x}_1,...,\underline{x}_n)$. Together with the just introduced notion of a "cut", corollary 4.3.1 can be rewritten as the equation giving the *Cutkosky rules*.

Corollary 4.3.2. (Cutkosky rules)

$$F(x_1,...,x_n) + \overline{F}(x_1,...,x_n) = -\sum_{cuts} F_c(x_1,...,x_n), \tag{4.30}$$

where F is evaluated using the "sunny-side" rules only, $\overline{F}$ using the "shady-side" rules only and F_c on the r.h.s according to the Feynman rules for cut graphs given above. The sum runs over all possible cuts with energy flowing from sun to shade.

Certain cuts can give zero contribution to the r.h.s. sum as shown in the previous section. Note that equation (4.30) is still valid after multiplying with external vertices and integration to calculate the amplitude.
From the above analysis of the energy flow through a Feynman diagram it is clear where our purely graph theoretic definition of Cutkosky cut graphs in chapter 2.4.2 has its roots.

4.4. Unitarity

Conservation of probability is a crucial feature for a quantum field theory and represented by the property of the S-matrix to be unitary ($S^\dagger = S^{-1}$). This unitarity can be seen as a general requirement posed on the theory from which one can deduce further results. One of the most important implications of unitarity is the *optical theorem*.

Recall the definition of the S-matrix of a scattering process in chapter 2.1, namely $S = \mathbb{I} + iT$. Requiring unitarity implies

$$S^\dagger S = \mathbb{I} \Leftrightarrow \quad i(T - T^\dagger) = -T^\dagger T. \tag{4.31}$$

Now, rewrite this in terms of T-matrix elements $\langle f|T|i\rangle \equiv T_{fi}$ and squeeze in a complete set[1] of intermediate states $\mathbb{I} = \sum |a\rangle\langle a|$. It follows that

$$i(\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle) = -\sum_a \langle f|T^\dagger|a\rangle\langle a|T|i\rangle. \tag{4.32}$$

This equation is known as the general *optical theorem*. Remember from chapter 2.1 that the T-matrix elements are directly related to perturbative amplitudes A_n, such that the theorem must work order by order in perturbation theory. Note also that in equation (4.32) on the left we have matrix elements, while on the right there are matrix elements squared. For instance, in a calculation up to order $\alpha = g^2$, for each loop calculation on the left there must be corresponding a tree-level calculation on the right.

Assuming time reversal invariance of the T-matrix[2] ($T_{if} = T_{fi}$) and writing out the matrix elements as complex numbers $T_{if} = Re(T_{if}) + iIm(T_{if})$ on the left side of equation (4.32) it is obvious that

$$2iIm(T_{if}) = \sum_a T^\dagger_{fa} T_{ai}. \tag{4.33}$$

Again, for instance, up to order α this means that imaginary parts of loop amplitudes are completely determined by tree-level amplitudes.

Unitarity from Cutkosky cuts

The formal analogy of the optical theorem (4.32) and equation (4.30) giving the Cutkosky rules is striking. Nevertheless, equation (4.30) works for any theory described by a Lagrangian $\mathcal{L}$, it does not need to be unitary.
In contrast to working top down, deriving the optical theorem from the general unitarity requirement as we just did, we can also work from the bottom up and infer the optical theorem

[1] Integration over the appropriate phase space element for the intermediate states $|a\rangle$ is implicitly assumed.
[2] Electromagnetic, gravitational, strong and weak interactions are invariant [17].

and thereby unitarity from the cutting rules. For this statement to hold true, the Lagrangian $\mathcal{L}$ from which the S-matrix is derived must be hermitian, i.e. $\mathcal{L} = \mathcal{L}^*$.
This can be seen as follows: In order for equation (4.32) to agree with (4.30) the diagrams occurring in $S^\dagger$ must be equal to the diagrams where all vertices are on the "shady side". Matrix elements of $S^\dagger$ are usually defined as

$$\langle a|S^\dagger|b\rangle = \langle b|S|a\rangle^*, \tag{4.34}$$

but they can also be obtained another way. Define the matrix $\overline{S}$ calculated as usual from $\mathcal{L}^*$ together with the replacement $i \to -i$ in the propagators and the vertices. This is equivalent to using our previously defined "shady-side" Feynman rules. Then by [10] one can show that

$$\langle a|S^\dagger(\mathcal{L}, i)|b\rangle = \langle a|\overline{S}(\mathcal{L}^*, -i)|b\rangle. \tag{4.35}$$

Given $\mathcal{L} = \mathcal{L}^*$ we can rewrite the optical theorem (4.32) to

$$i(T_{if} + \overline{T}_{fi}) = -\sum_a \overline{T}_{fa} T_{ai}. \tag{4.36}$$

This is precisely equation (4.30) with $T = -iF$. So it is allowed to say that the cutting rules fulfill the optical theorem (4.32) even diagram wise when we identify the intermediate states $|a\rangle$ with cuts of the graph. This way we can now calculate imaginary parts directly from Cutkosky cuts.

Finally, we have made the link between Cutkosky cuts and imaginary parts via the optical theorem. Practically, the prescription to calculate imaginary parts from Cutkosky cuts is:

- Cut a diagrams in all possible ways which allow to put the cut propagators on shell simultaneously.
- Replace for each cut propagator $\frac{1}{p^2 - m^2 + i\epsilon} \to 2\pi i \delta(p^2 - m^2)$.
- Sum over all cuts.

With this slender recipe we conclude this chapter and move on to the final part of the thesis, namely the linking between Cutkosky cuts and the core Hopf algebra.

5. A connection: Cutkosky cutting and the core Hopf algebra

Having collected the necessary background knowledge from the previous chapters in order to understand the core Hopf algebra and (Cutkosky) cutting rules, the following part of the thesis is devoted to the linking between these two concepts.
Why would this be an interesting thing to do? Most generally, a relation between the core Hopf algebra and the cutting of graphs would mean, that the beautiful theory of Hopf algebras, which did us a great service in conceptually understanding renormalization, receives a link to the unitarity of the S-matrix, another most fundamental feature of a quantum field theory.
Tied to this on the more practical level is a also a connection to the analytic structure of Feynman graphs: The coproduct of the core Hopf algebra filters in graphs for certain subgraphs. If these subgraphs can be shown to stay unaffected by the cutting (i.e. propagator poles), then the corresponding parts of the integration can be renormalized and performed independently and later reassembled in the calculation for the entire graph. In the following we prove a theorem implying precisely the latter aspect.
Before examining the connection between the core Hopf algebra and graph cutting, we briefly recapitulate the most important notions collected from the previous chapters in section 5.1.
In chapter 5.2, we give a rigorous proof of a proposition first made by Kreimer in section 5 of [1], claiming that the coproduct Δ_c of the core Hopf algebra $\mathcal{H}_c$ is compatible with the Cutkosky cutting of a graph Γ. In other words, cutting a graph Γ in all possible ways is equivalent to cutting all corresponding cographs Γ/γ in all ways that leaves none of their loops intact.
Further, in chapter 5.3 we discuss a generalized statement, namely Lemma 3 of chapter 3 in [2], which makes an analogous claim for k-cuttings. For each case, we clarify the terminology in which each statement was first given, and finally interpret the results.

5.1. Preliminaries

We briefly recapitulate the most important notions collected from the previous chapters for refreshment. A reader who generally is familiar with the presented background can thus read this chapter independently from the others and look up individual aspects where necessary.

First of all note that, for simplicity, our considerations are restricted to scalar (Feynman) graphs as given in definition 2.2.1. Reintroducing the necessary details for other theories should produce no difficulties.
Remember, all uncut graphs Γ are assumed to be $1PI$, and subgraphs are defined as disjoint unions of 1PI graphs $\gamma = \sqcup_i \gamma_i$. It is exactly this type of subgraph the coproduct Δ_c of the core Hopf algebra $\mathcal{H}_c$ filters for (see chapter 3.3). In this context the operation of shrinking subgraphs $\gamma \subsetneq \Gamma$ is important and assumed to be known from chapter 2.4.1.
All vertices and edges of a graph Γ are labelled uniquely by letters and natural numbers as explained in chapter 2.2. External half edges are not drawn in illustrations, such that figures are clear. Nevertheless, they can be inferred from the fixed valency of each vertex (chapter 2.2). Though cutting external half edges has no physical meaning, and subgraph shrinking does not affect them, in the case of Cutkosky cuts one needs to keep track of them for each component of the cut graph must be connected to at least one of external half edge (see chapter 2.4.2). For k-cuts this restriction is cancelled.
Finally and most importantly, Cutkosky cuts and k-cuts are identified with removing edge sets $E_{\Gamma_F} \subsetneq E_\Gamma$ induced from forests[1] via their corresponding vertex partitioning as explained in chapter 2.4.2 and 2.4.3. For the former this happens under additional physical constraints. The resulting cut graphs Γ_c consist of several (disjoint) connected components $\sqcup_i \Gamma_i, i = 1, ..., k$. In a

[1] Remember, the cutting can just as well be identified with marking edges and defined directly via edge sets or partitioning vertices. Here, we intend to connect to the framework of [2] and make the generalizations to k-cuts for which this definition is practical.

completely cut graph Γ_{cc}, other than in a general cut graph Γ_c, none of its components contains intact subgraphs γ anymore. We will sometimes refer to Cutkosky cuts simply as cuts to be distinguished from the more general notion of a k-cut.

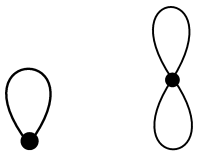


Figure 5.1: Tadopole-like graphs Γ allow for no cuts, i.e. $\Gamma_c = 0$.

Remark 5.1.1. (Cutting $\mathbb{I}$ and tadpole-like graphs)
Note, cutting and k-cutting tadpole graphs or graphs that are tadpole-like (see figure 5.1), as well as cutting the empty graph $\mathbb{I}$, is impossible by the definition of the cutting rules (definitions 2.4.4, 2.4.5, 2.4.7 and 2.4.8) for they can not be separated in components by a forest. These kind of graphs Γ map to zero when cut, i.e. $\Gamma_c = 0$. Consequently, terms with cographs of such shape if produced by the core coproduct Δ_c also give zero.

For the proof of theorem 5.2.1, following in the next chapter, it is important to note that we consider all graphs Γ (including $\Gamma = \mathbb{I}$) vanishing under the cut operation as *equivalent*. Otherwise we would map several cut graphs or tensorproducts to zero.

5.2. Compatibility of the core coproduct and Cutkosky cutting

In this section, we prove a proposition of Kreimer which states that the coproduct Δ_c of the core Hopf algebra $\mathcal{H}_c$ respects the Cutkosky cutting of a graph Γ, precisely this means that cutting a graph Γ in all possible ways is equivalent to cutting the corresponding cographs Γ/γ in all ways that leaves none of their loops intact. Cutkosky cutting and the core Hopf algebra are two very different concepts which hereby receive a linking. The practical implications of this theorem are discussed below.

In the original terminology of [1], where the proposition was first presented, it reads as follows.

There exists a bijection between

$$C(\Gamma) \quad \textit{and} \quad (id \otimes CC)\Delta_c(\Gamma). \tag{5.1}$$

This theorem can be rewritten in a form more suitable for our purposes, showing the action of the coproduct Δ_c and the cutting explicitly. With $\mathcal{C}(\Gamma)$ denoting the set of all Cutkosky cuts of a graph Γ and $\hat{\mathcal{C}}(\Gamma)$ the set of all its complete cuts, we can write

Theorem 5.2.1.

$$\sum_{\gamma=\sqcup_i\gamma_i} \left[\gamma \otimes \sum_{cc\in\hat{\mathcal{C}}(\Gamma/\gamma)} (\Gamma/\gamma)_{cc} \right] = \sum_{c\in\mathcal{C}(\Gamma)} \Gamma_c. \tag{5.2}$$

To achieve this expression, we defined $C(\Gamma)$ as the sum over all possible Cutkosky cut graphs Γ_c (definition 2.4.4) of a given graph Γ (for an example see figure 5.2a),

$$C(\Gamma) := \sum_{c\in\mathcal{C}(\Gamma)} \Gamma_c. \tag{5.3}$$

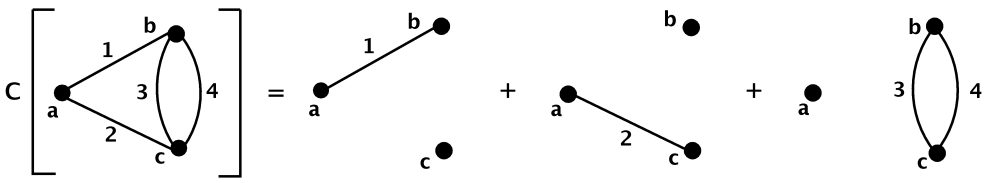
On the other hand, $(id \otimes CC)\Delta_c(\Gamma)$ first sums over all subgraphs $\gamma \subseteq \Gamma$, with $\gamma = \sqcup_i\gamma_i$ and γ_i $1PI$, via the core coproduct Δ_c such that

$$(id \otimes CC)\Delta_c(\Gamma) = \sum_{\gamma=\sqcup_i\gamma_i} \gamma \otimes CC(\Gamma/\gamma), \tag{5.4}$$

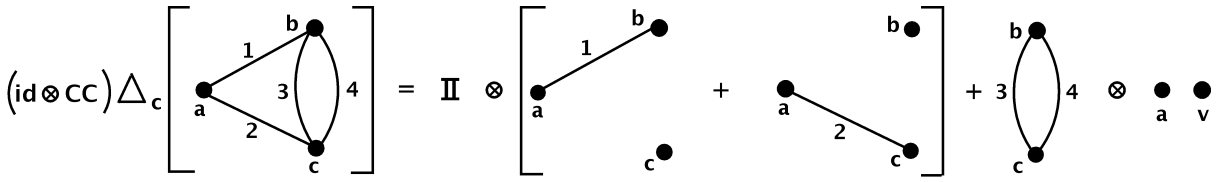
where we define $CC(\Gamma/\gamma)$ as the sum over all completely cut cographs $(\Gamma/\gamma)_{cc}$ (definition 2.4.5) only,

$$CC(\Gamma/\gamma) := \sum_{cc \in \hat{\mathcal{C}}(\Gamma/\gamma)} (\Gamma/\gamma)_{cc}. \tag{5.5}$$

In completely cut graphs Γ_{cc} the only possible subgraph is $\gamma = \mathbb{I}$. For illustration, consider the first two terms on the right side of the equation in figure 5.2a. As an example for the application of $(id \otimes CC)\Delta_c$ on a graph Γ see figure 5.2b.



(a) Summing over all possible ways of (Cutkosky) cutting a graph Γ.



(b) Filtering all subgraphs in a graph Γ with the coproduct Δ_c and subsequently completely cutting the corresponding cographs.

Figure 5.2: Example of theorem 5.2.1 applied to a graph Γ.

Proof of theorem 5.2.1

We now find a proof for theorem 5.2.1. We do so by proving a statement which is slightly different from it, i.e. slightly stronger. We show that there is a bijection between single cut graphs Γ_c and single tensor product terms of subgraphs with completely cut cographs $\gamma \otimes (\Gamma/\gamma)_{cc}$, rather than a bijection between sums over all possibilities to build each of the terms. The original statement for sums in 5.2.1 follows directly. This proceeding captures the underlying structure well, for the correspondence of sums directly emerges from workings on the single graph level. For physical calculations however we are interested in summing[1] over all cuts.

First, we fix a graph Γ and define the following two finite sets, with the notation already indicating the cutting of graphs and cographs respectively:

$$\mathcal{G} = \{0\} \cup \{\Gamma_c |\ \Gamma_c \text{ cut graph of } \Gamma\}, \tag{5.6}$$

and

$$\hat{\mathcal{G}} = \{0\} \cup \{\gamma \otimes (\Gamma/\gamma)_{cc} | \gamma \subseteq \Gamma, \gamma = \sqcup_i \gamma_i,\ (\Gamma/\gamma)_{cc} \text{ completely cut graph of } \Gamma/\gamma\}. \tag{5.7}$$

[1] Remember, finally we intend to calculate perturbative scattering amplitudes A_n which are sums of graphs.

Then, the theorem we prove as an intermediate step in order to prove theorem 5.2.1 reads:

Theorem 5.2.2. *For every cut graph $\Gamma_c \in \mathcal{G}$ there exists exactly one $\gamma \otimes (\Gamma/\gamma)_{cc} \in \hat{\mathcal{G}}$, i.e. $|\mathcal{G}| = |\hat{\mathcal{G}}|$.*

Proof: Our approach is to directly show surjectivity between the two (finite) sets $\mathcal{G}$ and $\hat{\mathcal{G}}$ in both directions, for this implies $|\mathcal{G}| \geq |\hat{\mathcal{G}}|$ and $|\mathcal{G}| \leq |\hat{\mathcal{G}}|$, such that $|\mathcal{G}| = |\hat{\mathcal{G}}|$.

$\Rightarrow$: First, let us show that for every combination $\gamma \otimes (\Gamma/\gamma)_{cc} \in \hat{\mathcal{G}}$ there is a $\Gamma_c \in \mathcal{G}$ such that we can map Γ_c onto $\gamma \otimes (\Gamma/\gamma)_{cc}$.

Be $\gamma \otimes (\Gamma/\gamma)_{cc}$ an arbitrary tensor product combination built from Γ. There are three possibilities one can distinguish in this tensor product which need to be reached by our map:

1. $\gamma = \Gamma$ or $(\Gamma/\gamma)_{cc} = 0 \Rightarrow \gamma \otimes (\Gamma/\gamma)_{cc} = 0$
2. $\gamma = \mathbb{I} \Rightarrow \mathbb{I} \otimes (\Gamma)_{cc}$
3. $\gamma = \subsetneq \Gamma \Rightarrow \gamma \otimes (\Gamma/\gamma)_{cc}$

We suggest the the following map $\delta : \mathcal{G} \to \hat{\mathcal{G}}$ as a surjection,

$$\delta(\Gamma_c) := \begin{cases} 0, \text{ if } \Gamma_c = 0 \\ \mathbb{I} \otimes \Gamma_c, \text{ if } \Gamma_c \text{ has no intact subgraph} \\ \gamma_m \otimes \Gamma_c/\gamma_m, \text{ for } \gamma_m \subsetneq \Gamma_c \text{ maximal intact subgraph} \end{cases} \tag{5.8}$$

Let us prove this. Consider an arbitrary cut graph Γ_c. For $\Gamma_c \neq 0$, it consists of exactly two connected components Γ_1 and Γ_2, say, with Γ_1 being connected to the incoming particles, Γ_2 to the outgoing particles. Each component can be a single vertex or spanning tree, or it can contain subgraphs. So generally, in $\Gamma_c = \Gamma_1 \sqcup \Gamma_2$ it is a meaningful operation to search for intact subgraphs $\gamma = \sqcup_k \gamma_k$, where the γ_k are $1PI$ and need not all to be in the same component Γ_i, $i = 1, 2$. Among the possibly many subgraphs, we can find the maximal subgraph γ_m by taking the disjoint union of all subgraphs present. Obviously then, this γ_m is unique in Γ.

To each tensor product case 1.-3. we find a Γ_c as explained in the following.

1. We take $\Gamma_c = 0$ such that $\delta(\{\emptyset\}) = 0$.
2. We take the graph Γ_c to be a completely cut graph, i.e. no loop is left intact and $\gamma_m = \mathbb{I}$. Now, almost identically, we map $\delta(\Gamma_c) = \mathbb{I} \otimes \Gamma_c$.
3. We take a cut graph Γ_c which contains intact subgraphs. Among them we find the maximal subgraph by disjoint union $\gamma_m \subsetneq \Gamma_c = \Gamma_1 \sqcup \Gamma_2$. Now we shrink γ_m the way shrinking is defined in chapter 2.4. We are left with a disjoint union of connected components $\Gamma_c/\gamma_m = \hat{\Gamma}_1 \sqcup \hat{\Gamma}_2$, none of which contains intact subgraphs anymore, i.e it is completely cut. If Γ_c/γ_m had not been completely cut, there was a subgraph in at least one component left, which we did not include in γ_m, thereby contradicting maximality.
   To see Γ_c/γ_m really is a (cut) cograph of Γ, we use $E_\Gamma = E_{\gamma_m} + E_{\Gamma/\gamma_m}$ as obvious from our unique labelling. The primary cutting only happened on edges $e \notin \gamma_m$ for γ_m was left intact, i.e. only edges $e \in E_{\Gamma/\gamma_m}$ were cut. As γ_m is then removed, only cograph edges are left by the above equation. Vertices stay unaffected by edge cutting anyway.
   The respective components of $\Gamma_c/\gamma_m = \Gamma_1 \sqcup \Gamma_2$ are still connected and linked to their respective external edges as shrinking of 1PI subgraphs neither destroys connectivity nor does it affect the linking. Finally, we map Γ_c to the tensor product $\gamma_m \otimes \Gamma_c/\gamma_m$.

Thus δ is a surjection and $|\hat{\mathcal{G}}| \leq |\mathcal{G}|$.

$\Leftarrow$: Reversely, let us show that for every cut graph $\Gamma_c \in \mathcal{G}$ there exists a combination $\gamma \otimes (\Gamma/\gamma)_{cc} \in \hat{\mathcal{G}}$, such that we can map $\gamma \otimes (\Gamma/\gamma)_{cc}$ onto Γ_c.

Let Γ_c be a cut graph of Γ. As noted before, we have the following different cases Γ_c our map needs to reach.

1. $\Gamma_c = 0$.
2. $\Gamma_c = \Gamma_{cc}$, i.e. it is completely cut.
3. Γ_c is cut with subgraphs γ intact.

For this direction, we suggest the the following map $\mathcal{M} : \hat{\mathcal{G}} \to \mathcal{G}$ as a surjection,

$$\mathcal{M}(\gamma \otimes (\Gamma/\gamma)_{cc}) := \begin{cases} 0, & \text{if } \gamma = \Gamma \text{ or } (\Gamma/\gamma)_{cc} = 0 \\ \tilde{m}(\mathbb{I} \otimes \Gamma_{cc}), & \text{if } \gamma = \mathbb{I} \\ \tilde{m} \circ (\tilde{c} \otimes \mathcal{I}^{\gamma}) (\gamma \otimes (\Gamma/\gamma)_{cc}), & \text{if } \gamma \subsetneq \Gamma \ . \end{cases} \tag{5.9}$$

Letting $\mathcal{C}$ denote the set of all cut graphs and $\hat{\mathcal{C}}$ the set of all completely cut graphs ($\hat{\mathcal{C}} \subsetneq \mathcal{C}$), we define the insertion of an uncut graph γ' into a completely cut graph γ''_{cc} to give a cut graph with intact subgraph Γ_c as $\mathcal{I} : \hat{\mathcal{C}} \to \mathcal{C}, \mathcal{I}^{\gamma'}(\gamma''_{cc}) := \Gamma_c$. Further, we define the constant map $\tilde{c} : \mathcal{H}_c \to \mathcal{H}_c, \Gamma \mapsto \mathbb{I}$ to erase the subgraph γ in the tensor product and finally we multiply everything together by $\tilde{m} : \mathcal{H}_c \otimes \mathcal{C} \to \mathcal{C}, \mathbb{I} \otimes \Gamma_c \mapsto \mathbb{I}\Gamma_c = \Gamma_c$.

To show surjectivity of $\mathcal{M}$, we first list again the (above mentioned) different cases in the tensor product $\gamma \otimes (\Gamma/\gamma)_{cc}$. It can be zero, $\mathbb{I} \otimes \Gamma_{cc}$, or γ can be a proper disjoint union of $1PI$ subgraphs. For this direction of the proof, the essential operation is to reinsert γ into the cut cograph $(\Gamma/\gamma)_{cc}$ in order to obtain the respective cut graph $\Gamma_c = \Gamma - E_{\Gamma_F}$, witfh the inserted subgraph intact. Generally, this is possible because from building the tensor product in the first place, we are given all information about the original (labelled) Γ and Γ/γ.

To each cut graph case Γ_c 1.-3. we find a $\gamma \otimes (\Gamma/\gamma)_{cc}$ explained as follows.

1. We take $\Gamma \otimes (\Gamma/\gamma)_{cc} = 0$ such that $\mathcal{M}(0) = 0$.
2. We take $\mathbb{I} \otimes \Gamma_{cc}$ and map, almost identically, $\mathcal{M}(\mathbb{I} \otimes \Gamma_{cc}) = \tilde{m}(\mathbb{I} \otimes \Gamma_{cc}) = \Gamma_{cc}$.
3. We start off at $\gamma \otimes (\Gamma/\gamma)_{cc}$, with $\gamma \neq \mathbb{I}$, $(\Gamma/\gamma)_{cc} \neq 0$, and $\gamma \subsetneq \Gamma$. We reinsert γ into $(\Gamma/\gamma)_{cc}$ to obtain Γ_c with $\gamma \subsetneq \Gamma$ intact. How, precisely, is this done?
   As indicated above, we know the original graphs Γ and Γ/γ, with all their vertices and edges uniquely labelled as described in chapter 2.2. Remember, Γ/γ contains l new vertices at the places of the shrunken $1PI$ subgraph components of $\gamma = \sqcup_k^l \gamma_k$ (especially also for propagator correction subgraphs). Thanks to the labelling, there is exactly one possibility (w.r.t place and orientation) to reinsert the shrunken subgraph $\gamma \subsetneq \Gamma$ (rather its $1PI$ components) into the cograph Γ/γ as to obtain the full graph Γ.
   Now take the new cut information into account, how can we be rebuild $\Gamma_c = \Gamma - E_{\Gamma_F}$, with E_{Γ_F} the set of cut edges induced by a forest of Γ? The key observation here is, that cutting is an operation on edges, whereas subgraph inserting is an operation on vertices. Consider again $E_\Gamma = E_\gamma + E_{\Gamma/\gamma}$. All subgraph information on edges and vertices is safely stored on the left side of the tensor product $\gamma \otimes (\Gamma/\gamma)_{cc}$. Cutting the cograph Γ/γ on the right can only affect (i.e. remove) edges $e \in E_{\Gamma/\gamma}$. Reinserting the uncut subgraph γ into the completely cut cograph $(\Gamma/\gamma)_{cc}$ though, can take place at vertices only which have been unaffected by the beforehand cutting. Especially, the vertex v from shrinking γ remains untouched.

Finally, we can uniquely rebuild Γ_c from our knowledge of Γ by reinserting γ into $v \in (\Gamma/\gamma)_{cc}$ in a way that gives $\Gamma_c := \Gamma - E_{(\Gamma/\gamma)_F}$, where our labelling uniquely identifies the edges $e \in E_{(\Gamma/\gamma)_F}$ with certain edges $e \in E_\Gamma$. This means we are left with a cut graph $\Gamma_c = \Gamma - E_{\Gamma_F}$ s.t. $\gamma \subsetneq \Gamma_c$ is intact.
Inserting $1PI$ subgraphs in vertices of $(\Gamma/\gamma)_{cc}$ obviously leaves Γ_c separated in exactly two connected components and preserves the primary connection to external edges.
Then, we map $\gamma \otimes (\Gamma/\gamma)_{cc}$ to Γ_c as explicitly given in (5.9).

This proves $\mathcal{M}$ really is a surjection and $|\mathcal{G}| \leq |\hat{\mathcal{G}}|$.

Together with the first part, we showed that $|\mathcal{G}| = |\hat{\mathcal{G}}|$, s.t. the existence of a bijection is proved. In fact, we actually gave the explicit bijection δ with its inverse $\mathcal{M} = \delta^{-1}$. □

To complete the proof of theorem 5.2.1, we sum over all tensor products $\gamma \otimes (\Gamma/\gamma)_{cc} \in \hat{\mathcal{G}}/\{0\}$ to achieve the left side of equation (5.2) and sum over all cut graphs $\Gamma_c \in \mathcal{G}/\{0\}$ to achieve the right side. The bijection is proved for each summand by theorem 5.2.2, the statement for sums follows directly. Thus, theorem 5.2.1 is proved.

Interpretation

In chapter 4 we saw that Cutkosky rules cut certain edges of a graph Γ when their corresponding propagators in the Feynman integral can be put on shell simultaneously. Thus, they pictorially illustrate the situation where the integrand experiences branch cuts from propagator poles and gives rise to imaginary parts.
Now, for practical calculations theorem 5.2.1 shows that if a Cutkosky cut leaves certain subgraphs $\gamma \subsetneq \Gamma$ intact, then the integration for the corresponding part of the integral can be renormalized and performed separately, regardless of the pole structure of the integrand, and later reassembled with the remaining integration to give the calculation for the entire graph Γ.
Apart from this, theorem 5.2.1 allows to restrict proofs on cut graphs to proofs on the smaller set of completely cut graphs (no loops intact) for the former can be rebuilt by the explicitly given bijection. In the case of cut graphs with loops intact the coproduct also reduces the considerations to a lower loop order.
On the more conceptual level, the validity of theorem 5.2.1 shows that the application of the coproduct and Cutkosky cutting of graphs are concepts that respect each other in a certain way. As seen in chapter 4.4 the Cutting rules work on the graph level, but can be shown to prove unitarity perturbatively. The core coproduct also is an operation working on single graphs, but is embedded in the bigger framework of the Hopf algebra.
This encourages to pursue the goal of understanding the global connection between the Hopf algebraic approach to quantum field theories and unitarity, together with analyticity properties, via the core Hopf algebra and Cutkosky cutting rules.

5.3. Generalization to k-cutting

Further, let us discuss a proposition from Kreimer closely related to theorem 5.2.1, namely Lemma 3 of chapter 3 in [2], which makes an analogous claim but this time for general k-cuttings.
After definition 2.4.7 and 2.4.8 (complete) k-cuts separate the graph in k components which do not necessarily need to be connected to external half-edges. So the subsequently presented statement is identical to the above theorem 5.2.1 when sums over all Cutkosky cuts (2-cuts) are replaced with sums over all possible k-cuts.

The theorem is originally given in a notation very different from the one used above, and therefore explained in detail below. Now it becomes clear, why we introduced the concept of forests in

chapter 2.4.2 and 2.4.3. As mentioned before, the considerations so far would have been possible without forests but in order to understand the below theorem and its implications this concept is very convenient and we benefit from our previous work.

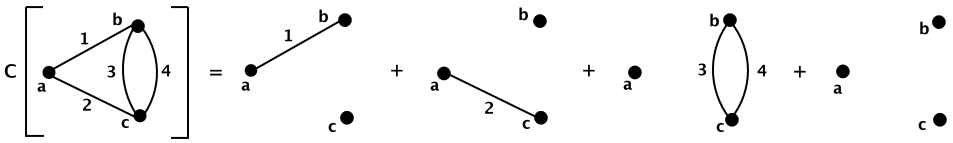
Theorem 5.3.1.

$$\sum_{T|\Gamma'} \left(\Gamma, T \cup \sum_{k=1}^{v_\Gamma} \sum_{F \in \mathcal{F}_k(\Gamma''), |\mathcal{G}^F|=0} F \right) = \sum_{k=1}^{v_\Gamma} \sum_{F \in \mathcal{F}_k(\Gamma)} (\Gamma, F) \tag{5.10}$$

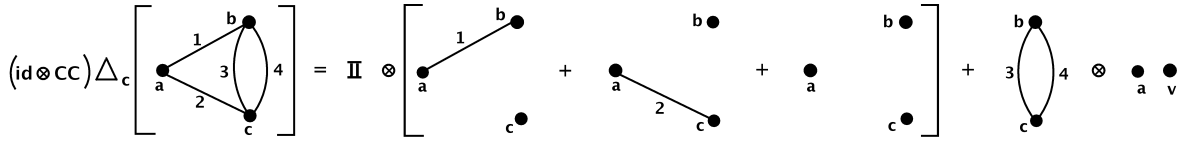
Here, we have the following ingredients from left to right (see also chapter 2.4.2):

- Γ' and Γ'' denote a subgraph and cograph of Γ respectively in the style of Sweedler's notation.
- $T|\Gamma'$ denotes a tree T spanning Γ'. As subgraphs are allowed to be disjoint unions of 1PI graphs $\Gamma' = \sqcup_i \Gamma'_i$, it is possible that T is a disjoint union of trees, one for each subgraph component, i.e. $T = \sqcup_i T_i$.
- $\mathcal{F}_k(\Gamma)$ is the set of all k-forests of given graph Γ.
- $\mathcal{G}^F = \{\Gamma_i, ..., \Gamma_k\}$ denotes the set of graphs induced by a forest F via $\Gamma - E_{\Gamma_F}$.
- $(\Gamma, F) = \Gamma - E_{\Gamma_F} = \sqcup_i \Gamma_i$, $i = 1, ..., k$, denotes the cut graph given by the original graph Γ, together with the cut information delivered by the forest F.

The above proof of theorem 5.2.1 on Cutkosky cuts can directly be transferred to a proof of theorem 5.3.1 for general k-cuts.
The reasoning is identical, we only use definitions 2.4.7 and 2.4.8 for k-cut graphs and complete k-cut graphs instead of definitions 2.4.4 and 2.4.5 for (complete) Cutkosky cuts.
Our argumentation is not affected, for the coproduct does not "see" the number of connected components in a graph and keeps filtering all cut graphs depending on the loop number of the subgraphs which are possibly left intact. Figures 5.3a and 5.3b give an example, where special attention is to be paid on the 3-cut graph appearing in the third term on the right side of the equation in figure 5.3b.



(a) Summing over all possible ways of k-cutting a graph Γ.



(b) Filtering all subgraphs in a graph Γ with the coproduct Δ_c and completely k-cutting the corresponding cographs.

Figure 5.3: Example for theorem 5.3.1 applied to a graph Γ.

The main difference in the setup is that, other than before, the several induced graph components do not necessarily have to be connected to external edges. However, this does not affect the two key operations of our proof above, namely searching for maximal subgraphs γ_m in a cut graph Γ_c or rebuilding of graphs Γ_c from cographs $(\Gamma/\gamma)_{cc}$ by subgraph insertion with the help of Γ.

Just like theorem 5.2.1 before, theorem 5.3.1 implies that certain subgraph integrations can be performed independently from the remaining part of the integration affected by the cutting operation. Referring to [2], we further note that the forest approach in a very simple manner captures the fact that the difference of a k-cut and a $(k+1)$-cut defines a Cutkosky cut on some subgraph. This observation suggests that the optical theorem may also be applied iteratively.

6. Conclusion and future prospects

In this thesis we rigorously proved that the coproduct of the core Hopf algebra $\mathcal{H}_c$ respects the Cutkosky cutting of a graph Γ. This fact can be employed to simplify practical calculations and indicates a more general linking between Hopf algebras and the unitarity property of the S-matrix in a quantum field theory.

Having set up a clear graph theoretic terminology for Feynman diagrams Γ as labelled graphs as well as for subgraph shrinking and graph cutting (via forests) in chapter 2, we introduced the core Hopf algebra $\mathcal{H}_c$ in chapter 3. The core Hopf algebra is built on all $1PI$ (2-connected) graphs and can be regarded as the Hopf algebra for a field theory in infinite dimension. It is linked to the well known renormalization Hopf algebra $\mathcal{H}$, which it contains as a quotient (in the sense of Hopf algebras).
In chapter 4, we learned that for a theory with a hermitian Lagrangian $\mathcal{L}$, Cutkosky cutting rules are intimately related to the unitarity of the S-matrix via the optical theorem. They even prove unitarity perturbatively. Also, they illustrate the case when certain propagators in a Feynman integral are put on shell simultaneously, thereby giving rise to branch cuts in the integrand, which consequently result in imaginary parts of the integral. Luckily, the real part along the branch cut can be traced back from these imaginary parts via dispersion relations, also known as Hilbert transforms.
Finally, in chapter 5, we dealt with the connection between the core Hopf algebra $\mathcal{H}_c$ and Cutkosky cutting rules by rigorously proving a concrete proposition from Kreimer, stating that the core coproduct Δ_c is compatible with the Cutkosky cutting of a graph. In other words, cutting a graph Γ in all possible ways is equivalent to cutting its corresponding cographs Γ/γ all ways that leaves none of their loops intact.
Our approach was to suggest two maps working on the level of single terms and prove the bijection by showing surjectivity in both directions. The result for sums over all possibilities of cutting follows directly from this.
For practical calculations, our result implies that if a Cutkosky cut leaves certain subgraphs $\gamma \subsetneq \Gamma$ intact, the corresponding renormalization and integration can be performed unaffected by the pole structure of the integrand, and later reassembled with the remaining integration for the entire graph Γ. Further, proofs on cut graphs can be restricted to proofs on the smaller set of completely cut graphs for the former can be rebuilt by the explicit bijection. In the case of cut graphs with loops intact, the coproduct even reduces the considerations to a lower loop order.

On the more conceptual level, the proved compatibility of the core coproduct Δ_c and Cutkosky cutting of graphs encourages to pursue the goal of understanding the global connection between the the beautiful theory of Hopf algebras, which did us a great service in understanding renormalization, and unitarity, coming with a link to the analytic structure of Feynman graphs, as two fundamental features of a quantum field theory.
We gave a short outlook on current work in this direction by briefly looking at a very similar statement for more general k-cuttings from [2]. Within this context the primarily introduced concept of forests proved useful, for it readily captures the fact that the difference of a k-cut and a $(k+1)$-cut defines a Cutkosky cut on some subgraph. In principle, this fact allows for iterating cut operations to higher loop orders, offering a new tool for further research.

A. Appendix

A.1. Integral (4.6)

We give an explicit calculation of the integral in equation (4.6). This can also be taken as a brief reminder for performing contour integrals and calculating residues. Consider

$$\int_{-\infty}^{\infty} \frac{\epsilon}{(p^2-m^2)^2+\epsilon^2} dp^2. \tag{A.1}$$

Substitute $x = p^2 - m^2$ to write

$$\int_{-\infty}^{\infty} \frac{\epsilon}{x^2+\epsilon^2} dx. \tag{A.2}$$

To evaluate this integral, consider the integrand as a complex valued function

$$f(z) = \frac{\epsilon}{z^2+\epsilon^2}, \tag{A.3}$$

with singularities at $\pm i\epsilon$. We pick a contour $\mathcal{C}$ in the complex plane that contains the real-valued integral. The semicircle on the upper half plane with boundary diameter on the real line from $-\alpha$ to α is a suitable choice. Note, that by Cauchy's integral formula

$$\oint_{\mathcal{C}} f(z)dz = \int_{-a}^{a} f(z)dz + \int_{arc} f(z)dz, \tag{A.4}$$

where we are interested in the middle integral.
Now, factorize the integrand and put the only (first order) pole in the contour $\mathcal{C}$ in a convenient place for direct application of *Cauchy's integral formula.*

$$\frac{\epsilon}{z^2+\epsilon^2} = \frac{\epsilon}{(z+i\epsilon)(z-i\epsilon)} = \frac{\frac{\epsilon}{(z+i\epsilon)}}{(z-i\epsilon)}. \tag{A.5}$$

By calculating the *residuum* of the first order pole, the left side of the formula gives

$$\oint_{\mathcal{C}} \frac{\frac{\epsilon}{(z+i\epsilon)}}{(z-i\epsilon)} dz = 2\pi i (z-i\epsilon) \left(\frac{\frac{\epsilon}{(z+i\epsilon)}}{(z-i\epsilon)} \right) \Bigg|_{z\to i\epsilon} = 2\pi i \frac{\epsilon}{z+i\epsilon} \Bigg|_{z\to i\epsilon} = \pi. \tag{A.6}$$

It remains to show that the integral over the semicircle *arc* tends to zero as $\alpha \to \infty$. We use the the *estimation lemma*, whit $\left(\frac{\epsilon}{\alpha^2+\epsilon^2}\right)$ as the upper bound along the *arc*, and $(\alpha\pi)$ the length of arc. It follows

$$\left| \int_{arc} f(z)dz \right| \leq \left(\frac{\epsilon}{\alpha^2+\epsilon^2} \right) (\alpha\pi) \to 0, \text{ for } a \to \infty, \tag{A.7}$$

such that finally

$$\int_{-\infty}^{\infty} \frac{1}{z^2+\epsilon^2} dz = \int_{-\infty}^{\infty} f(z)dz = \lim_{a\to+\infty} \int_{-a}^{a} f(z)dz = \frac{\pi}{2}. \tag{A.8}$$

B. Bibliography

References

[1] D. Kreimer, *The core Hopf algebra*, Clay Mathematics Proceedings, Volume 10, 2008, arXiv:0902.1223v1 [hep-th] (2009)

[2] S. Bloch and D. Kreimer, *Cutkosky rules and Outer space*, arXiv:1512.01705v1 [hep-th] (2015)

[3] B. Bollobás, *Modern Graph Theory*, p.1-14, Springer Verlag New York Inc. (2000)

[4] R.W. Robinett, *Recursive relations in the core Hopf algebra*, arXiv:0903.2849v1 [hep-th] (2009)

[5] D. Manchon, *Hopf algebras, from basics to applications to renormalization*, arXiv:math/04084052 [math.QA] (Updated version 2006)

[6] D. Kreimer, *On the Hopf algebra structure of perturbative quantum field theories*, arXiv:0705.3897v1 [hep-th] (2007)

[7] S. Bloch and D. Kreimer, *Mixed Hodge structures and renormalization in physics*, arXiv:0804.4899v3 [hep-th] (2008)

[8] K. Ebrahimi-Fard and D. Kreimer, *Hopf algebra approach to Feynman diagram calculations*, J. Phys. A: Math. Gen., 38, (2005), R385-R406 , arXiv:hep-th/0510202v2

[9] D. Kreimer, *A remark on quantum gravity*, Ann. Phys. 323 (2008) 49-60 , arXiv:0804.4899v3 [hep-th] (2008)

[10] G. t'Hooft and M. Veltman, *Diagrammar*, CERN European organization for nuclear research, Geneva (1973)

[11] M. le Bellac, *Quantum and Statistical Field Theory*, p. 377-387, Oxford University Press (1991)

[12] R. J. Eden, P. V. Landshoff, D.I. Olive, J.C. Polkinghorne *The analytic S-matrix*, p.26-36, Cambridge University Press (1966)

[13] B. A. Kniehl, *Dispersion Relations in Loop Calculations*, arXiv:hep-ph/9607255v1 (1996)

[14] G. Sterman *An Introduction to Quantum Field Theory*, p.247-252 and p.411-415, Cambridge University Press (1993)

[15] M. Schwartz, *Introduction to Quantum Field Theory*, p.255-258, Cambridge University Press (2014)

[16] T. Kinoshita, *Quantum Electrodynamics*, p.190-192, World Scientific Publishing Co. Pte. Ltd. (1990)

[17] M. E. Peskin and D. V. Schroeder *An introduction to quantum field theory*, p.64-71, Westview Press (1995)

C. Acknowledgement

For their support with this master's thesis I would like to sincerely thank the following people. They have all contributed significantly to making this thesis succeed.

First of all, I would like to thank Prof. Dirk Kreimer for entrusting me with this fascinating subject. Furthermore I would like to thank him for his friendly supervision while writing this thesis and actively integrating me into the group.

Another thanks goes to Dr. Christian Bogner for being the second supervisor.

A special thanks goes to the whole group for their welcoming attitude and their humorous atmosphere, everyday again, which allowed to survive well such moments in the middle of a proof that would otherwise have let one's hair stand on end. Special thanks goes to Henry Kissler, Marco Berghoff and Lucas Delage for some fun and illuminating blackboard discussions or coffees on the topic.