DYSON–SCHWINGER EQUATIONS AND QUANTIZATION OF GAUGE THEORIES (SUMMER 21)

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1. Hopf algebras and DSE

1.1. Definitions.

Recap Hopf algebra.

$\mathbb{Q}$ - algebra A

$$A = \bigoplus_{j=0}^{\infty} A^{(j)} \qquad A^{(0)} \simeq \mathbb{Q}$$

$$A^{(j)} A^{(m)} \subseteq A^{(j+m)} \qquad \text{(multiplicat.)}$$

comultiplication Δ

$$\Delta A^{(n)} \subseteq \bigoplus_{i+j=n} A^{(i)} \otimes A^{(j)}$$

$\mathbb{I}$ is the unit, and by abuse of notation, $\underbrace{\mathbb{I} : \mathbb{Q} \to A}$, $q \to q\mathbb{I}$

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, shift left, "m \otimes id"] \arrow[d, shift left, "id \otimes m"] & A \otimes A \arrow[l, shift left, "\Delta \otimes id"] \arrow[d, shift left, "m"] \\
A \otimes A \arrow[r, shift left, "m"] \arrow[u, shift left, "id \otimes \Delta"] & A \arrow[l, shift left, "\Delta"] \arrow[u, shift left, "\Delta"]
\end{tikzcd}
$$

m is associative.

$\Delta : A \to A \otimes A$ is co-associative.

$A(m, \mathbb{I}, \hat{\mathbb{I}}, \Delta)$ is a bi-algebra

where $\hat{\mathbb{I}} : A \to \mathbb{C}$

$\hat{\mathbb{I}}(\mathbb{I}) = 1$, $\hat{\mathbb{I}}(A^{(j)}) \equiv 0$ $\forall j \geq 1$.

co-unit.

$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$

Δ, m are compatible:

$\Delta\, m(a_1, a_2) = m \otimes m(\Delta(a_1)\, \Delta(a_2))$.

"bialgebra"

The final thing for a Hopf algebra is an antipode: $S : \mathcal{A} \to \mathcal{A}$

$$S(\mathbb{I}) = \mathbb{I}, \quad S(a_1 a_2) = S(a_1) S(a_2)$$

$$m(S \otimes id)\Delta = \mathbb{I} \circ \hat{\mathbb{I}} = m(id \otimes S)\Delta$$

$\mathcal{A}$ be the ring of polynomials in one variable, for example

$$\mathbb{Q}[x] = \mathcal{A}.$$

Set $\Delta x = x \otimes 1 + 1 \otimes x$

Δ, m compatible:

$$\Delta x^2 = (\underline{x} \otimes 1 + \underline{1 \otimes} x)(\underline{x \otimes 1} + \underline{1 \otimes x})$$
$$= x^2 \otimes 1 + 2x \otimes x + 1 \otimes x^2.$$

$$\Delta x^n = \ldots .$$

$$\Delta 1 = 1 \otimes 1 \qquad \mathbb{I} : q \to q 1$$
$$\hat{\mathbb{I}}(1) = 1 \qquad \hat{\mathbb{I}}(x^n) \equiv 0$$

$\mathcal{A}$ is graded with $\mathcal{A}^{(j)}$ the polynomials of degree j.

$$S(x) = -x, \quad S(x^n) = (-1)^n x^n.$$

~~1.2. **Hopf algebra of rooted trees.**~~ .

One interesting map:

$$B_+(x^n) \to \frac{1}{n+1} x^{n+1} \qquad \left(B_+ \sim \int\right)$$

for any polynomial $P(x) \in \mathcal{A}$,

we have $\underline{\Delta B_+(P)} = B_+(P) \otimes 1 + (\mathrm{id} \otimes B_+)\Delta P.$

$$P(x) = 1 + x \qquad B_+ P = \underline{x} + \frac{1}{2} x^2$$

$$\underline{\Delta B_+ P = (x \otimes 1 + 1 \otimes x) + \frac{1}{2}\left(x^2 \otimes 1 + 2\, x \otimes x + 1 \otimes x^2\right)}$$

$$= \left(x + \frac{1}{2} x^2\right) \otimes 1 + \underbrace{(\mathrm{id} \otimes B_+) \underbrace{\Delta(1+x)}_{1 \otimes 1 + 1 \otimes x + x \otimes 1}}_{\underline{1 \otimes x + \frac{1}{2}\, 1 \otimes x^2 + x \otimes x}}$$

$$\mathcal{A} = \bigoplus_{j=0}^{\infty} A^{(j)} \qquad A^{(0)} = \mathbb{Q}$$

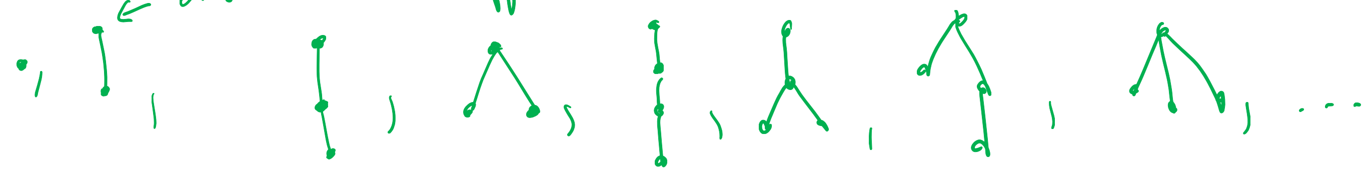
$$\mathrm{Aug}_{\mathcal{A}} = \bigoplus_{j \geq 1}^{\infty} A^{(j)} \qquad B_+ : \mathcal{A} \to \mathrm{Aug}_{\mathcal{A}}.$$

B_+ is a Hochschild 1-cocycle.

Hopf algebra of rooted trees

rooted trees are graphs which are
i) connected, ii) simply connected,
made from edges and vertices,
with one distinguished vertex,
and each edge connecting two vertices.

$\mathbb{I}$ is the empty tree,
← dist. vertex (uppermost)



those are rooted trees.
Make this into a $\mathbb{Q}$-algebra.

Product is disjoint union:

$$m(\mid, \wedge) = \mid\wedge$$

$$m(\emptyset, T) = T \qquad \mathbb{I} \simeq \emptyset$$

$$m(3\,\bullet, 17 \mid + 4 \wedge) = 51 \bullet \mid + 12 \bullet \wedge$$

$$\hat{\mathbb{I}}(\mathbb{I}) = 1 \in \mathbb{Q} \qquad \hat{\mathbb{I}}(\cancel{t}^{(j)}) \equiv 0$$

where $\mathcal{A}^{(j)}$ is the vector space of products of trees with j vertices all together.

$$m(\mathcal{A}^{(j)}\,\mathcal{A}^{(k)}) \subseteq \mathcal{A}^{(j+k)}$$

$$\mathcal{A} = \bigoplus_{j=0}^{\infty} \mathcal{A}^{(j)}.$$ This is a graded $\mathbb{Q}$ algebra.

Now we need $\Delta : \underline{\mathcal{A}} \to \underline{\mathcal{A} \otimes \mathcal{A}}$

$$\Delta \underline{\mathbb{I}} = \underline{\mathbb{I}} \otimes \underline{\mathbb{I}}.$$

Two equivalent definitions of Δ.

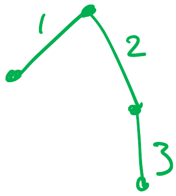
i) admissible cuts.

$$\underline{\Delta T = T \otimes \underline{\mathbb{I}} + \underline{\mathbb{I}} \otimes T} + \sum_{\substack{\text{adm.}\\ \text{cuts } C}} P^{C}(T) \otimes R^{C}(T)$$

admissible cut C : edges of T

subset of edges $C \subseteq E_T$ such that any path from any vertex to the root traverses at most one element of C

~~1.3. **Hopf algebra of graphs.**~~ .



$\{1\}$ ✓

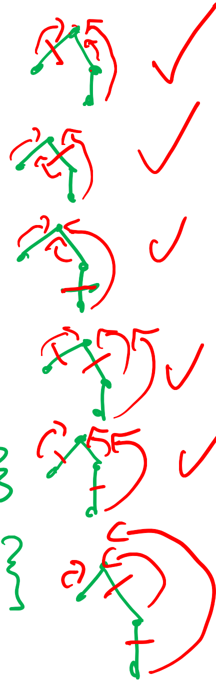
$\{2\}$ ✓

$\{3\}$ ✓

$\{1,2\}$ ✓

$\{1,3\}$ ✓

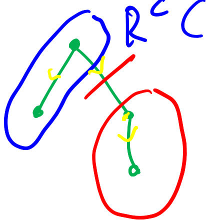
$\{2,3\}$



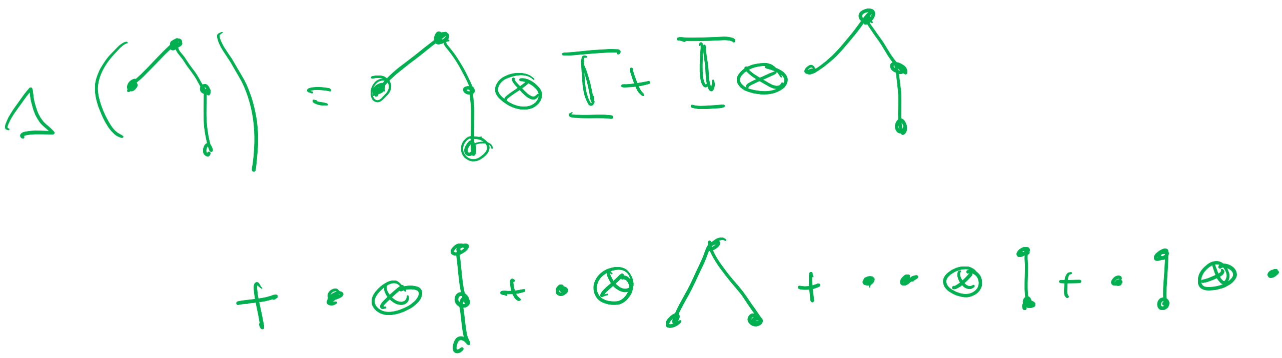
2 elements

What is $P^c(T)$?

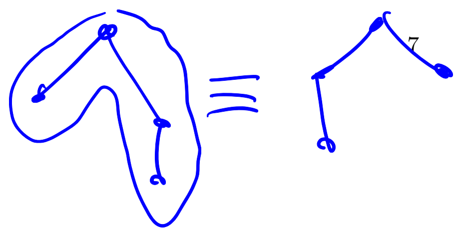
$R^c(T)$



$P^c(T)$



Note: we take our rooted tree non-planar, so that



$$\Delta(T_1 \dots T_k) = \Delta(T_1) \dots \Delta(T_k)$$

$$\Delta(\bullet\, \ell) = \Delta(\bullet)\,\Delta(\ell) = (\bullet \otimes \mathbb{I} + \mathbb{I} \otimes \bullet)(\ell \otimes \mathbb{I} + \mathbb{I} \otimes \ell + \bullet \otimes \bullet)$$

$$\left.\begin{aligned} &= \bullet\,\ell \otimes \mathbb{I} + \bullet \otimes \ell + \bullet\bullet \otimes \bullet \\ &+ \ell \otimes \bullet + \mathbb{I} \otimes \ell\,\bullet + \bullet \otimes \bullet\bullet \end{aligned}\right\} (*)$$

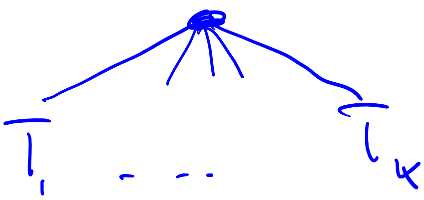
Thm. $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$

That was first definition of Δ.

Next:

ii) Consider $B_+ : \mathcal{A} \to \mathcal{A}$, $f \to \bigoplus_{j=1}^{\infty} \mathcal{A}^{(j)}$

$B_+(\mathbb{I}) = \bullet \qquad B_+(T_1 \dots T_k) =$

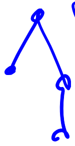


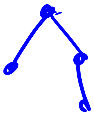
$B_+(\bullet) = \ell$

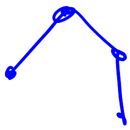
$B_+(\bullet\,\bullet) = \wedge \qquad B_+(\ell) = \ell_3, \quad \dots$

$$\Delta T_1 \dots T_k = \Delta T_1 \dots \Delta T_k$$

$$\Delta T = \Delta B_+(X) = B_+(X) \otimes \mathbb{I} + (id \otimes B_+)\,\Delta(X).$$

$\Delta\big($  $\big) = \Delta\big(B_+(\bullet\ \mathrm{I})\big)$

$=$  $\otimes\, \mathbb{I} + (id \otimes B_+)\,\Delta(\bullet\ \mathrm{I})$

$=$  $\otimes\, \mathbb{I} + (id \otimes B_+)\,(*)$

$\Rightarrow$ the same result as before.

Thm. both definitions agree.

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1.4. **combinatorial DSE.** .

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Humboldt U. Berlin