DYSON–SCHWINGER EQUATIONS AND QUANTIZATION OF GAUGE THEORIES (SUMMER '21)

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1. General Structure of Combinatorial DSE

1.1. **Operads and sub-Hopf algebras.** Reference (linked on the course homepage): *Hopf algebras in renormalization theory: Locality and Dyson-Schwinger equations from Hochschild cohomology*, Christoph Bergbauer, Dirk Kreimer, IRMA Lect. Math. Theor. Phys. **10** (2006) 133-164, arXiv:hep-th/0506190. .

What is "Operads." an operad? (Jim Stasheff)

The endomorphism operad

$$\mathrm{End}_X := \{\mathrm{Map}(X^n, X)\}$$

together with compositions

$$\circ_i : \mathrm{Map}(X^n, X) \times \mathrm{Map}(X^m, X) \to \mathrm{Map}(X^{n+m-1}, X).$$

For $1 \leq i \leq n$ composition of "place" n:

$$(f \circ_i g)(x_1, \ldots, x_{i-1}, g(x_i, \ldots x_{i+m-1}), x_{i+m}, \ldots)$$

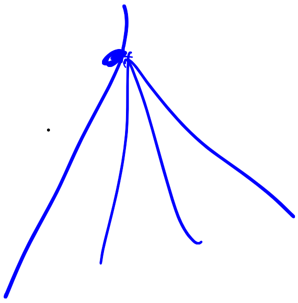
$$(f \circ_i g) \circ_j h$$

$$= f \circ_j (g \circ_{i-j+1} h)$$

for $j \leq i \leq j+n-1$

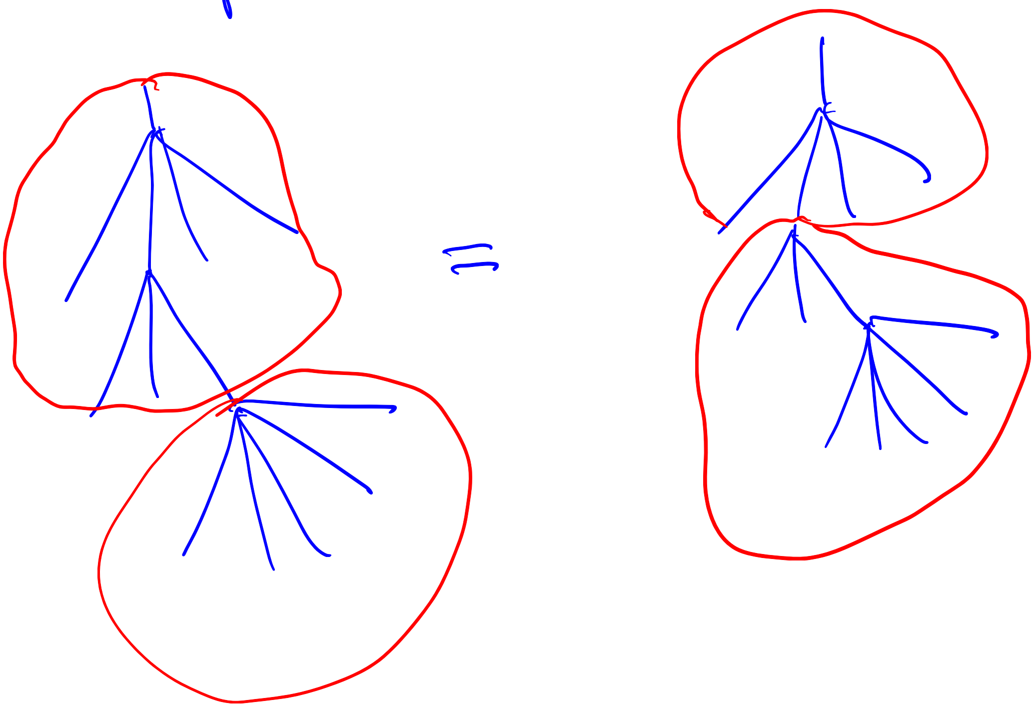
One of the associativity laws of an operad.

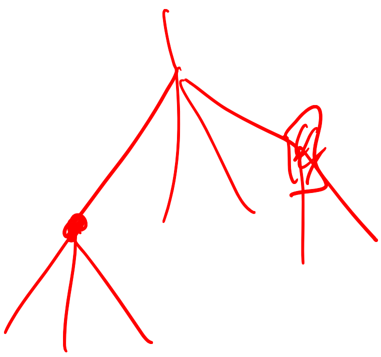
Best described by rooted trees.



stands for $\mathrm{Map}(X^4 \to X)$

4 inputs, one output



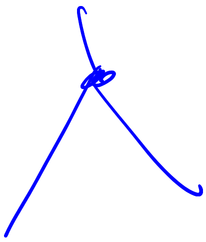


many associativity laws.

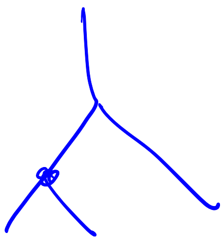
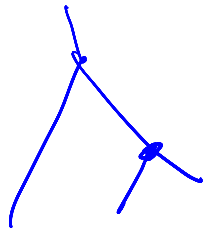
This defines an operad.

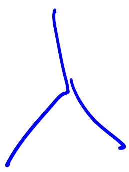
One remark:

Consider composition of just one map $X^2 \to X$



if you impose:

Set  = 

$\to$  is an associative map

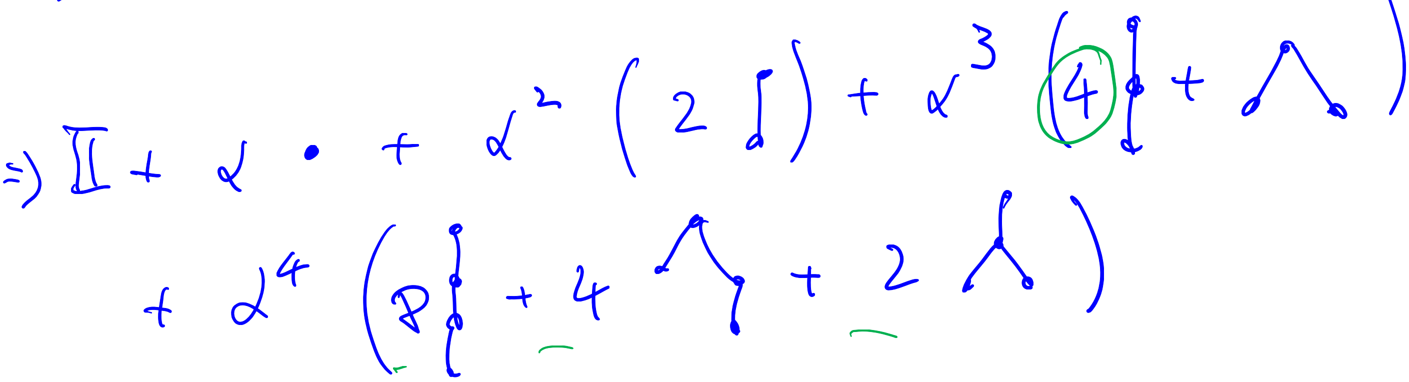
and you generate associative algebras.

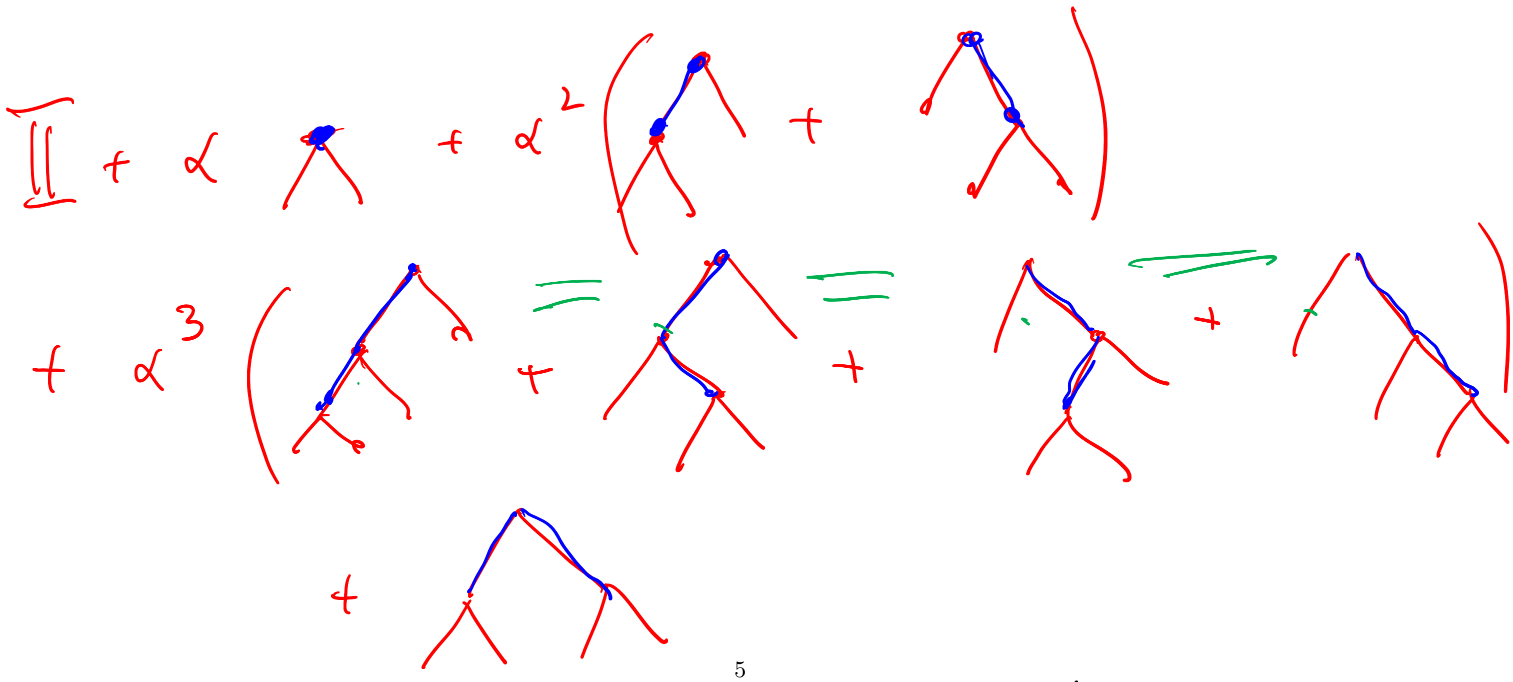
If you demand that

This generates Lie algebras if you demand Jacobi.

So what has this to do with DSE?

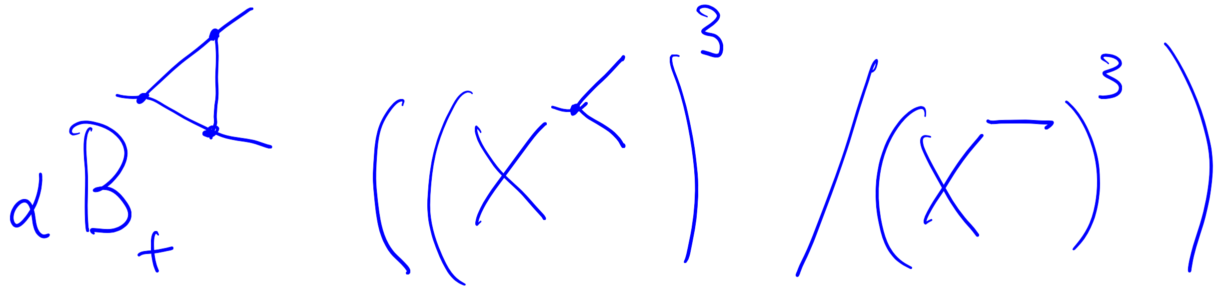
$$X(\alpha) = \mathbb{I} + \alpha B_+\left(X^2(\alpha)\right)$$



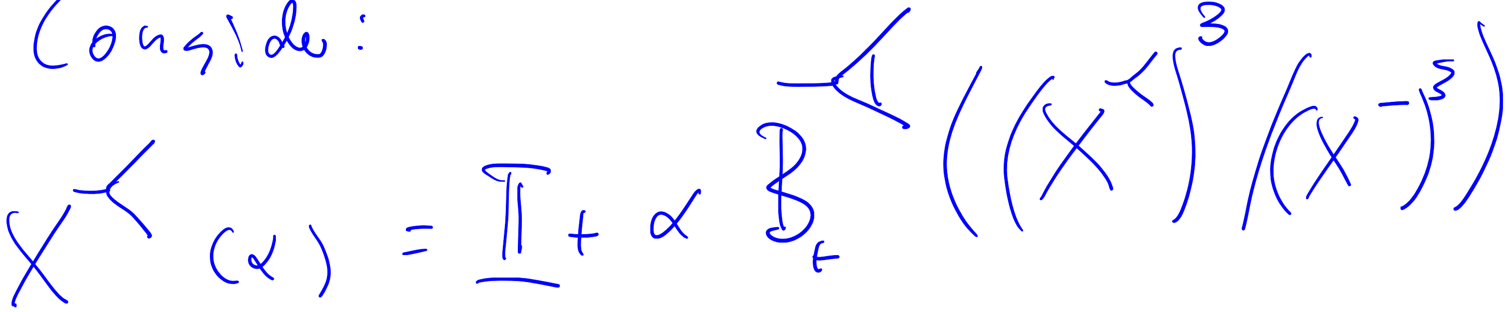


This immediately proves that you get sub-Hopf algebras.

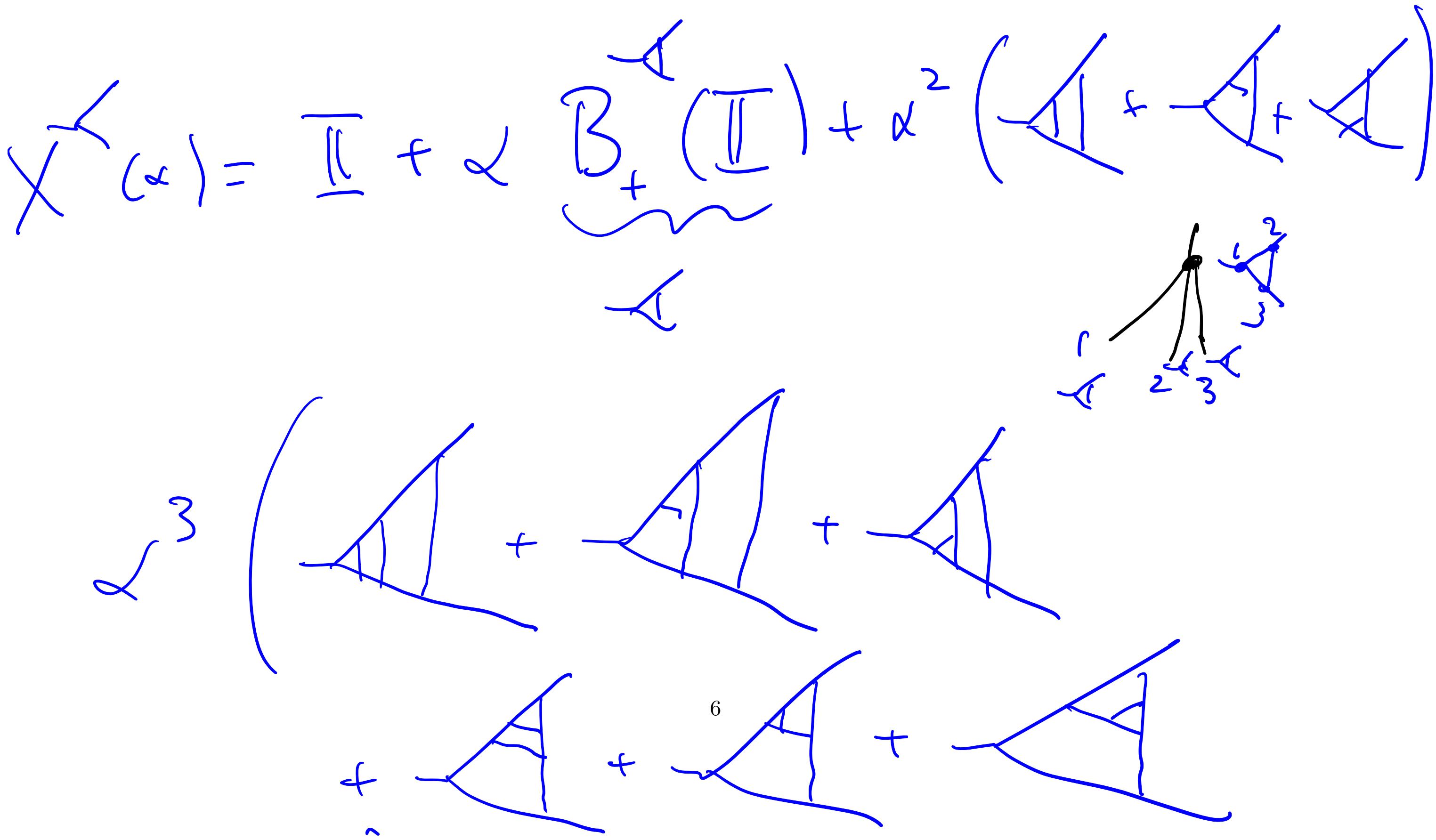
1.2. **Insertion places.** .

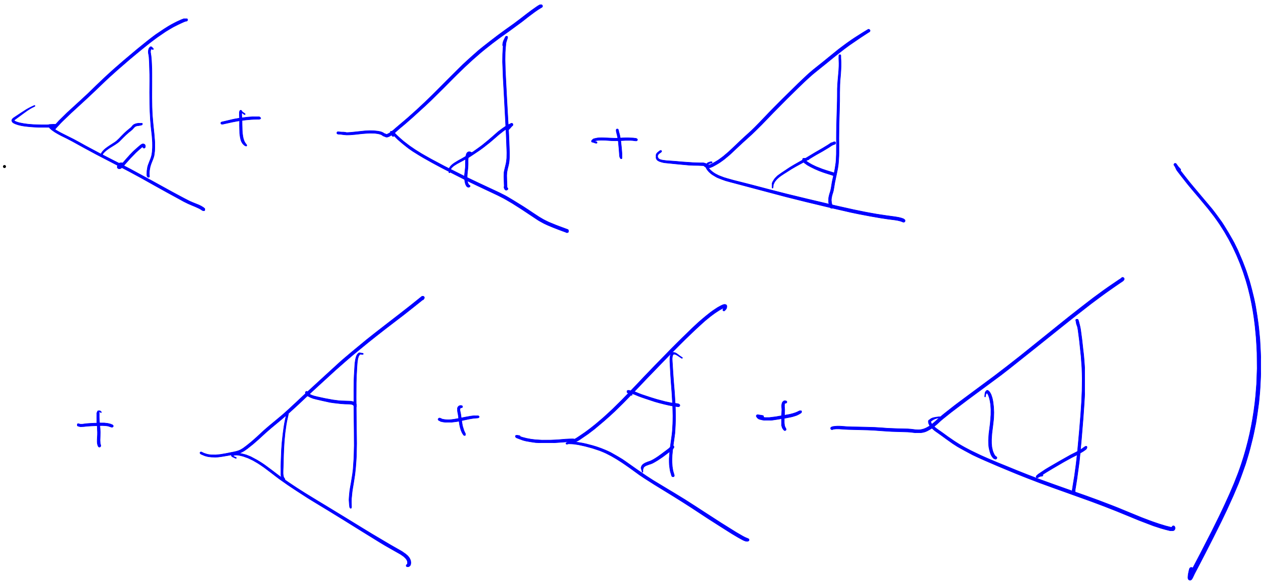


Consider:



For simpl. set $X^{-} = \mathbb{I}$.





$+ \ldots$

So this can be generalized

to $\sum B_+^\gamma$

$n = \#$ insertion places

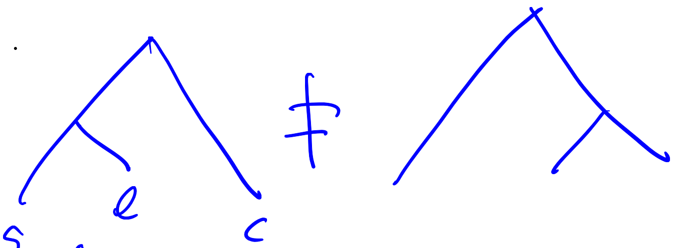
$\mathrm{Map}_\gamma(H_{FG}^n, H_{FG})$

One thing operads do

is homotopy theory.

Stasheff polytopes.

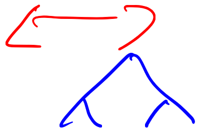
Consider

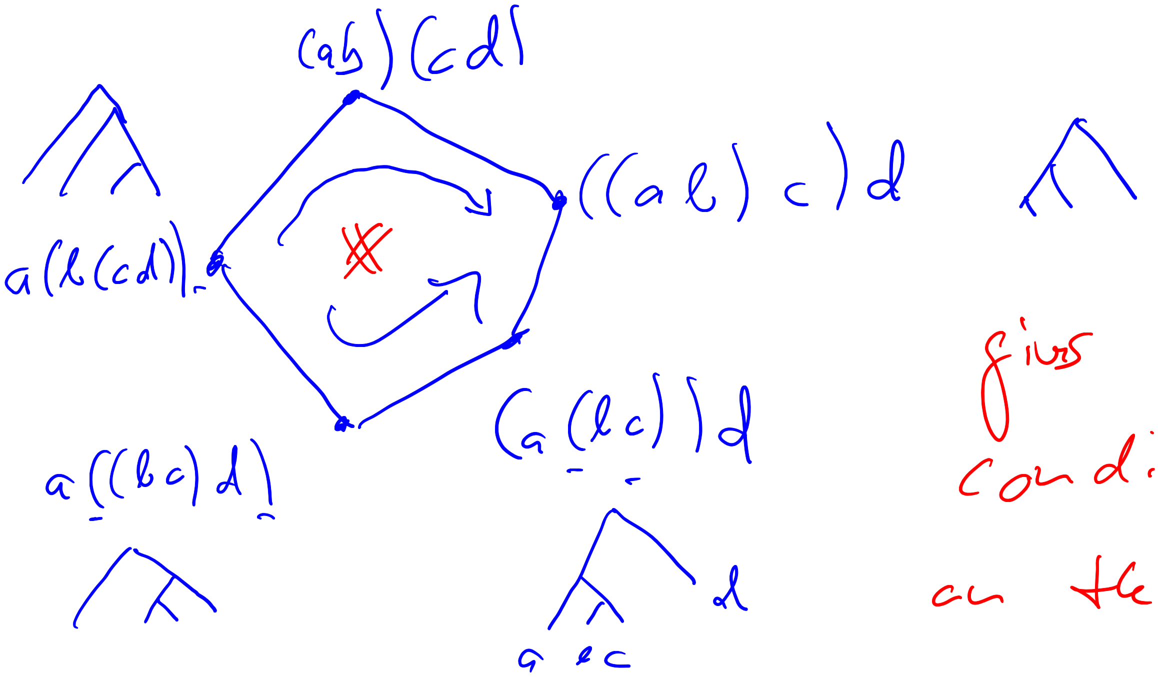


but almost the same.

$$(ab)c \simeq a(bc)$$

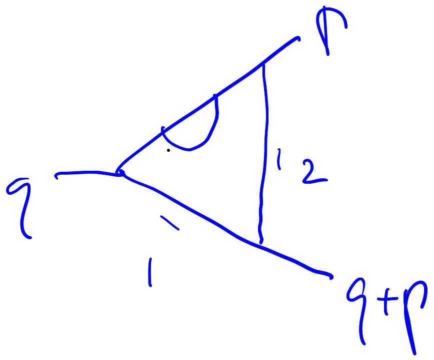
difference is an "associator"

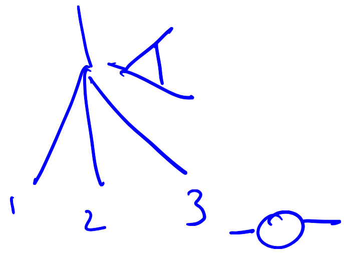


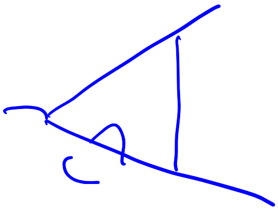


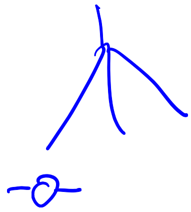
gives condition on the associator

Similar problem in QFT.

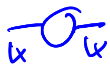






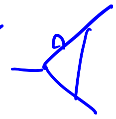


$$\sim (k^2)^{-s}$$



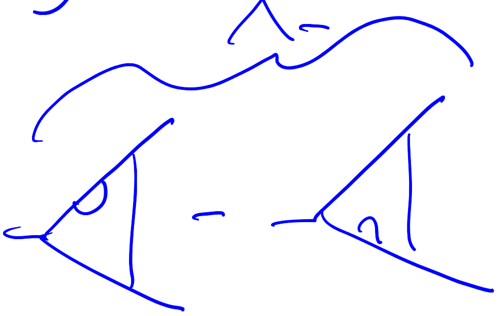
Analytically: (massless theory)

$$\int d^6k \, \frac{1}{k^2 (k+q)^2 (k-p)^2} \qquad \frac{1}{(k^2)^s} \simeq$$



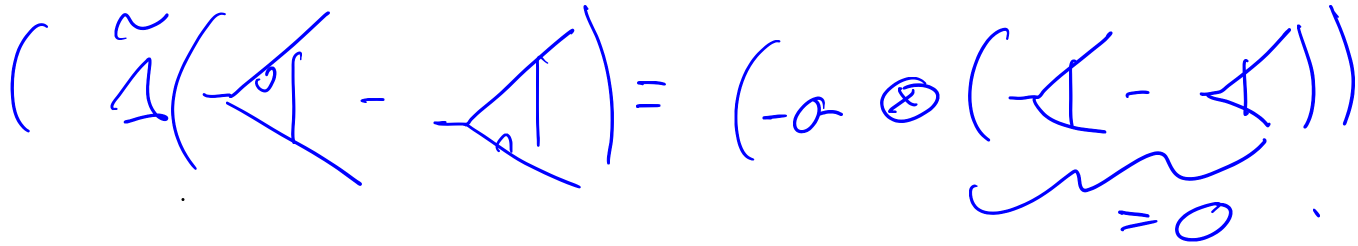
$$\int d^6k \, \frac{1}{k^2 (k+q)^2 (k-p)^2} \frac{1}{((k+q)^2)^s}$$

X_-



is not zero.

But: $\Delta(X_-) = X_- \otimes \mathbb{I} + \mathbb{I} \otimes X_-$



What this is saying is that we can work with one insertion place instead of two. Price: new primitive

$$B_+^{X_-} \text{ in our DSE.}$$

This can be systematized and allows you to work "with one insertion place!

↳ ↳ one variable Mellin transf

↳ RGE can be represented as in our toy Yukawa model by a differential operator

in $\frac{\partial}{\partial \zeta}$ for a single ζ.