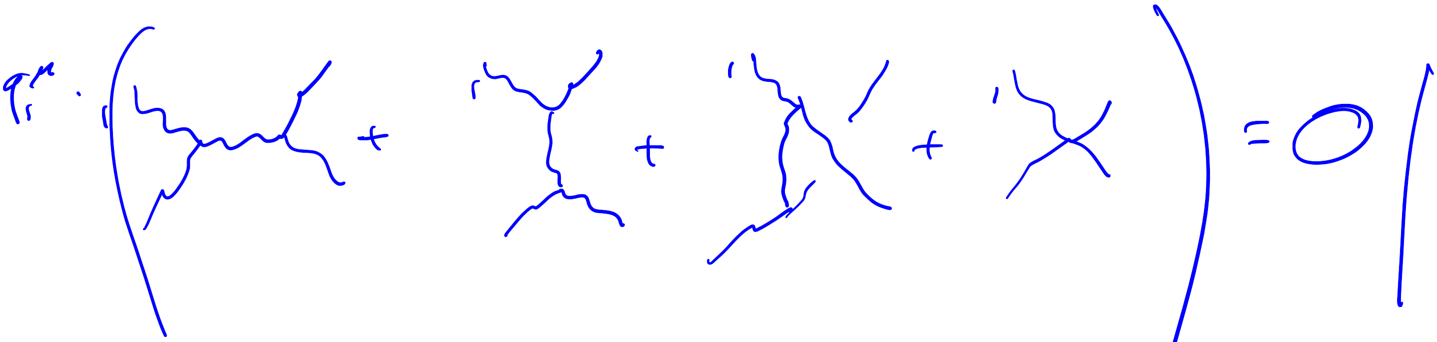
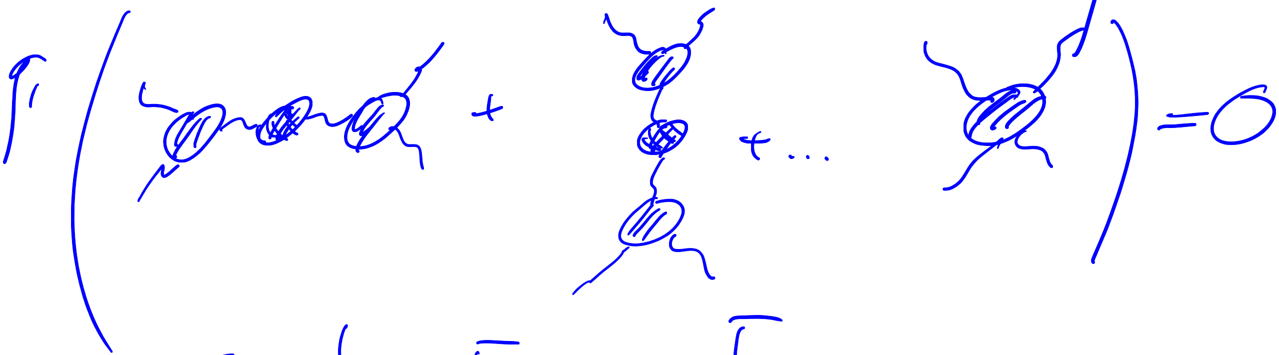
Dyson-Schwinger Eq's and Quantization of gauge theories (Summer '21)

Kreimer, Lect June 1st 2021





$$\Gamma_3 \frac{1}{\Gamma_2} \Gamma_3 = \Gamma_4 \qquad \text{ST}$$

$$\frac{\Gamma_3}{\Gamma_2} = \frac{\Gamma_4}{\Gamma_3}$$

4 Gauge Theory Graphs

We now turn to graphs in gauge theory, as contrasted to 3-regular graphs in scalar field theory. While the latter were graphs which can be regarded as corollas with three half-edges, connected by gluing two half-edges from different corollas to an internal edge e which hence determine a pair of corollas P_e, the former are graphs with 3- and 4-valent vertices.

Again, we can consider them based on corollas, this time corollas which have either three or four half-edges of gauge boson type (indicated by wavy lines), or one gauge-boson half-edge with two half-edges of ghost type (indicated by consistently oriented straight dashed lines), or one gauge-boson half-edge with two half-edges of fermion type (indicated by consistently oriented straight full lines).

We repeat our notational conventions. We mark in such graphs edges and vertices in various ways and we let

$$\mathcal{G}^{r,l}_{n\times, m\circ}$$

be the set of all graphs with external half-edges specifying the amplitude r, with l loops and n 4-gluon vertices, and m ghostloops. Similarly, we will indicate the number of marked edges and other qualifiers as needed.

Still, if we want to leave a qualifier $l, n, m, \cdots$ unspecified (so that we consider the union of all sets with any number of such items), we replace it by $/$. For sums or series of graphs we continue to use

$$X^{r,l}_{n\times, m\circ}$$

which are sums (for fixed l) or series (for $l = /$) of graphs weighted by symmetry, colour and other such factors as defined below.

We now start adopting graph homology to our purposes in gauge theory.

4.1 Marking edges

Recall that the Feynman rule for the 4-valent vertex is

$$\begin{aligned}\Phi\left({}^{1}_{2}\times{}^{4}_{3}\right) = &+ f^{a_1a_2b}f^{a_3a_4b}(g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3})\\ &+ f^{a_1a_3b}f^{a_2a_4b}(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_4}g^{\mu_3\mu_2})\\ &+ f^{a_1a_4b}f^{a_2a_3b}(g^{\mu_1\mu_2}g^{\mu_4\mu_3} - g^{\mu_1\mu_3}g^{\mu_4\mu_2}).\end{aligned}$$

We introduce a new edge type $\text{-}\!\!\vdash\!\!\text{-}$ which has the following Feynman rule:

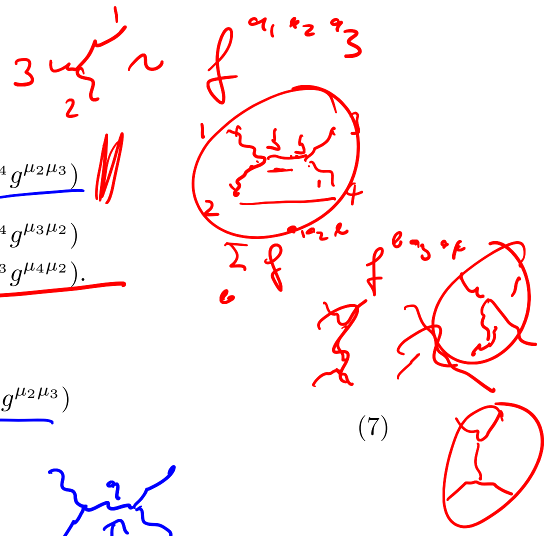
$$\begin{aligned}\Phi\left({}^{1}_{2}\,\rangle\!\overset{e}{-}\!\langle\,{}^{4}_{3}\right) &= f^{a_1a_2b}f^{a_3a_4b}(g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3})\\ &=: \text{colour}\left(\rangle\!-\!\langle\right)W_e,\end{aligned} \tag{7}$$

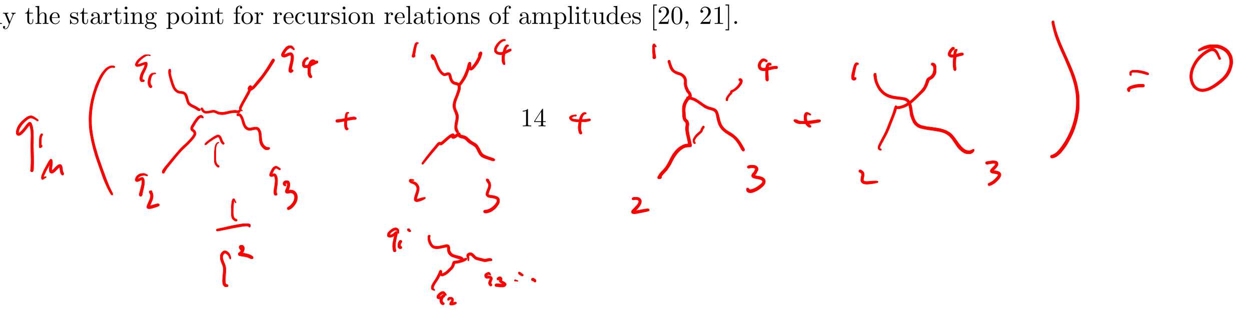
so that we can write the 4-point vertex as

$$\times \sim \rangle\!-\!\langle + \asymp + \rangle\!\mid\!\langle\,. \tag{8}$$

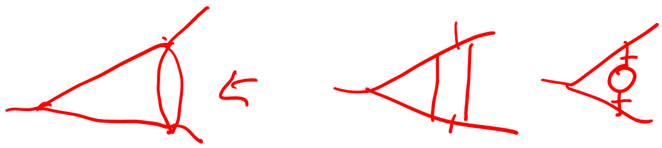
(The relation $\sim$ denotes that the left- and right-hand side have the same Feynman amplitude.) Note that because of this relation, the internal marked edge does not correspond to a propagator. It is just a graphical way of writing the three terms of the 4-valent vertex.

Remark 4.1. The fact that the 4-valent vertex decomposes in such a way into a product of two corollas is actually the starting point for recursion relations of amplitudes [20, 21].





C^2_Γ



For any graph Γ with marked edges, let $\overline{\Gamma}$ be the graph where the marked edges shrink to zero length.[9] The Feynman–Schwinger integrand of a graph with marked edges is given for us by

$$I_\Gamma = \Big(\prod_{e \in \Gamma^{[1]}_{\text{marked}}} W_e \Big) \left(\prod_{v \in V^\Gamma, v \cap \Gamma^{[1]}_{\text{marked}} = \emptyset} V_v \right) I_{\overline{\Gamma}}. \tag{9}$$

Here, V_v is the colour-stripped part of the Feynman rule of a 3-gluon vertex,

$$V_v = \sum_{\text{cycl}(1,2,3)} (\xi_1 - \xi_2)_{\mu_3} g_{\mu_1 \mu_2}. \tag{10}$$

Note that the scalar integrand $I_{\overline{\Gamma}}$ does obviously not contain the edge variables of the $\rightsquigarrow$-marked edges. For future use, we define

$$\mathscr{I}_\Gamma = \Big(\prod_{e \in \Gamma^{[1]}_{\text{marked}}} W_e \Big) \left(\prod_{v \in V^\Gamma, v \cap \Gamma^{[1]}_{\text{marked}} = \emptyset} V_v \right) e^{-\sum_{e \in \Gamma \text{ unmarked}} A_e \xi'^2_e}, \tag{11}$$

which is such that

$$\int \frac{\mathrm{d}\underline{k}_L}{(2\pi)^{dl}} \mathscr{I}_\Gamma = I_\Gamma.$$

Definition 4.2. Define a derivation $\chi_+ : H \to H$ on generators by

$$\chi_+ \Gamma = \sum_{e \in \Gamma^{[1]}_{\text{int}}} \chi^e_+ \Gamma,$$

where

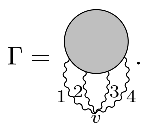
$$\chi^e_+ \Gamma = \begin{cases} 0 & \text{if } e \text{ shares a vertex with a marked, fermion or ghost edge,} \\ \Gamma_{e \rightsquigarrow} & \text{otherwise.} \end{cases}$$

The next lemma shows how symmetry factors relate upon exchanging 4-valent vertices for a pair of corollas with a marked edge in-between. We consider graphs with l loops, k 4-gluon vertices and k' marked edges, for an amplitude r. Also, $\overline{\mathscr{G}}$ denotes unlabelled graphs, in contrast to labelled graphs in $\mathscr{G}$.

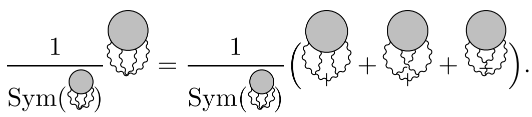
Lemma 4.3. *For any graph* $\Gamma \in \mathscr{G}^{r,l}_{k\times, k'\rightsquigarrow}$ *we have*

$$\frac{1}{\mathrm{Sym}(\Gamma)} \Gamma \sim \frac{1}{k} \sum_{\substack{\Gamma' \in \overline{\mathscr{G}}^{r,l}_{k-1\times, k'+1\rightsquigarrow} \\ \exists e \in \Gamma'^{[1]}_{\text{marked}} : \Gamma'/e = \Gamma}} \frac{1}{\mathrm{Sym}(\Gamma')} \Gamma'. \tag{12}$$

Proof. Let $v \in \Gamma^{[0]}_4$ with adjacent edges $1, 2, 3, 4$:



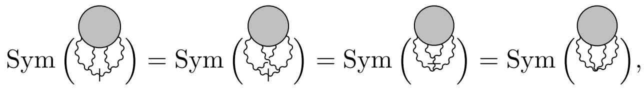
(We do not show Γ's external edges in the diagram.) Apply (8):



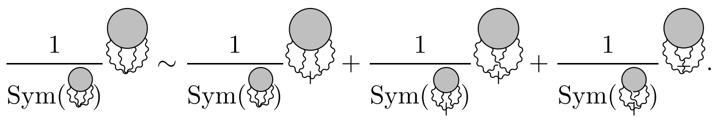
The following three cases can occur:

[9] If we have k marked edges, here are 3^k different graphs Γ which have the same $\overline{\Gamma}$.

1. The four edges adjacent to v are each un-interchangeable. In this case

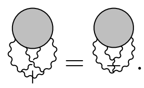


so that

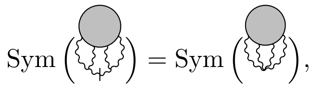


Note that the three graphs at the right-hand side are all non-isomorphic.

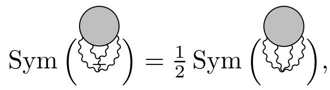
2. Two of v's adjacent edges are interchangeable, say 1 and 2. Then



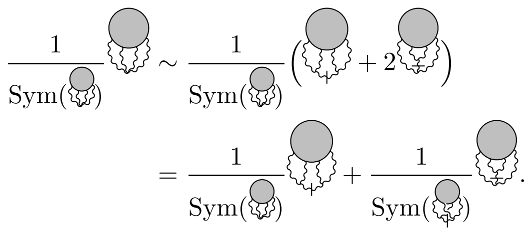
The symmetry factors of the new graphs are



and

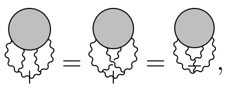


so that

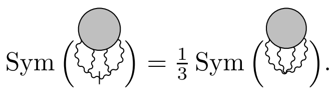


Note that the two graphs at the right-hand side are unequal.

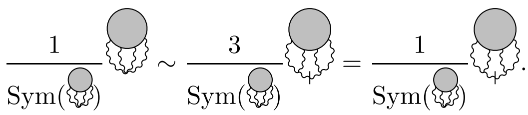
3. Three of v's adjacent edges are interchangeable, say 1, 2 and 3. Then



and



So:



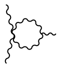
Thus we can conclude that

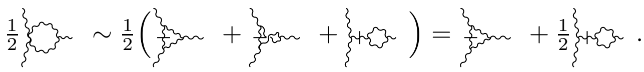
$$\frac{1}{\mathrm{Sym}(\Gamma)}\Gamma = \sum_{\substack{\Gamma' \in \overline{\mathscr{G}}^{n,l}_{k-1\times, k'+1\wedge} \\ \exists e \in \Gamma'^{[1]}_{\mathrm{marked}} : \Gamma'/e = \Gamma \\ \text{where the new vertex is } v}} \frac{1}{\mathrm{Sym}(\Gamma')}\Gamma'.$$

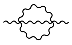
The result follows up summing this over all 4-valent vertices in Γ, giving rise to the factor $\#\Gamma_4^{[0]} = k$:

$$\Gamma \sim \frac{1}{k} \sum_{\substack{\Gamma' \in \overline{\mathscr{G}}^{n,l}_{k-1\times, k'+1\wedge} \\ \exists e \in \Gamma'^{[1]}_{\mathrm{marked}} : \Gamma'/e = \Gamma}} \frac{1}{\mathrm{Sym}(\Gamma')}\Phi(\Gamma').$$

□

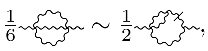
Example 4.4. Take $\Gamma =$  and apply equation (8) to the 4-valent vertex:

 (13)

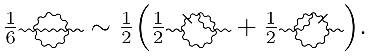
Example 4.5. Take $\Gamma =$  and apply (8) to one of the two vertices:

 (14)

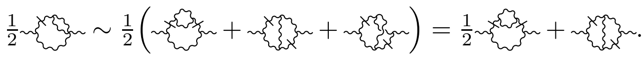
Analogously, we get



so that we can write

 (15)

Example 4.6.



The next lemma is crucial, as it shows that the fundamental relation between a 4-gluon vertex and a pair of 3-gluon vertices, in all three channels, gives a relation between combinatorial Green functions. We would have no chance at getting a well-defined gauge theory without such a relation.

Lemma 4.7. *i. For any k and k', $k' < k$:*

$$\frac{1}{\binom{k}{k'}} X^{n.l}_{k-k'\times, k'\wedge} \sim \frac{1}{\binom{k}{k'+1}} X^{n,l}_{k-k'-1\times, k'+1\wedge}.$$

ii. For any k:

$$X^{n,l}_{k\times} \sim X^{n,l}_{k\wedge}.$$

Proof. For i. we have from lemma 4.3 for the Green's function $X^{n,l}_{k-k'\times,k'\sim}$:

$$
\begin{aligned}
X^{n,l}_{k-k'\times,k'\sim} &= \sum_{\Gamma\in\overline{\mathscr{G}}^{n,l}_{k-k'\times,k'\sim}} \frac{1}{\mathrm{Sym}(\Gamma)}\Gamma \\
&\sim \frac{1}{k-k'} \sum_{\Gamma\in\overline{\mathscr{G}}^{n,l}_{k-k'\times,k'\sim}} \sum_{\substack{\Gamma'\in\overline{\mathscr{G}}^{n,l}_{k-k'-1\times,k'+1\sim} \\ \exists e\in\Gamma'^{[1]}_{\text{marked}}:\Gamma'/e=\Gamma}} \frac{1}{\mathrm{Sym}(\Gamma')}\Gamma' \\
&= \frac{k'+1}{k-k'} \sum_{\Gamma'\in\overline{\mathscr{G}}^{n,l}_{k-k'-1\times,k'+1\sim}} \frac{1}{\mathrm{Sym}(\Gamma')}\Gamma' \\
&= \frac{k'+1}{k-k'} X^{n,l}_{k-k'-1\times,k'+1\sim}.
\end{aligned}
$$

The factor $k'+1$ appears because every graph $\Gamma'\in\overline{\mathscr{G}}^{n,l}_{k-k'-1\times,k+1'\sim}$ can be obtained from $\#\Gamma'^{[1]}_{\text{marked}}=k'+1$ graphs $\Gamma\in\overline{\mathscr{G}}^{n,l}_{k-k'\times,k'\sim}$ by applying (8). Using the identity

$$
\binom{k}{k'+1} = \frac{k-k'}{k'+1}\binom{k}{k'},
$$

it follows that

$$
\frac{1}{\binom{k}{k'}} X^{n,l}_{k-k'\times,k'\sim} \sim \frac{1}{\binom{k}{k'+1}} X^{n,l}_{k-k'-1\times,k'+1\sim}.
$$

For ii. we have

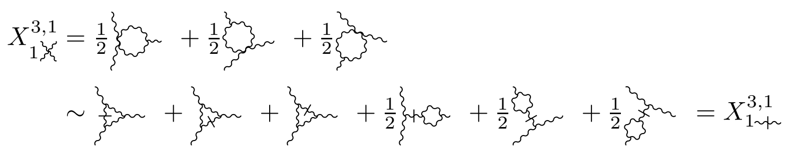
$$
X^{n,l}_{k\times} \sim \frac{1}{\binom{k}{k'}} X^{n,l}_{k-k'\times,k'\sim}.
$$

This is true by induction: it is an equality for $k'=0$ and the inductive step is true by i.. Taking $k'=k$ gives:

$$
X^{n,l}_{k\times} \sim X^{n,l}_{k\sim}. \qquad \square
$$

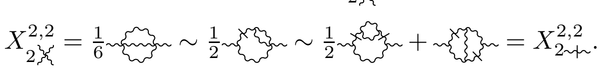
In the following, if r is an n-gluon amplitude, we often replace the subscript r by n, as in this example:

Example 4.8. Take $n=3$, $l=1$ and $k=1$.



where we have used Eq.(13).

Example 4.9. Take $n=2$, $l=2$ and $k=2$. Note that $\overline{\mathscr{G}}^{2,2}_{2\times}$ contains just one graph and use (14) and (15):



Lemma 4.10. *i. For any graph $\Gamma\in\overline{\mathscr{G}}^{n,l}$:*

$$
\frac{1}{\mathrm{Sym}(\Gamma)}\frac{(\chi_+)^k}{k!}\Gamma = \sum_{\substack{\Gamma'\in\overline{\mathscr{G}}^{n,l}_{k\sim} \\ \text{skeleton}(\Gamma')=\Gamma}} \frac{1}{\mathrm{Sym}(\Gamma')}\Gamma'.
$$

where the skeleton of a marked graph is the graph with its markings removed.

ii. For any $k \geq 0$, $\frac{(\chi_+)^k}{k!} X^{n,l} = X^{n,l}_{k\rightsquigarrow}$.

iii. We have $e^{\chi_+} X^{n,l} = X^{n,l}_{/\rightsquigarrow} = X^{n,l}_{/\times}$.

Proof. For i. we have

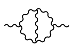
$$\begin{aligned}
\frac{1}{\mathrm{Sym}(\Gamma)} \frac{(\chi_+)^k}{k!}[\Gamma] &= \frac{1}{\mathrm{Sym}(\Gamma)} \sum_{\{e_1,\dots,e_k\}\subset\Gamma^{[1]}_{\mathrm{int}}} [\chi_+^{e_1}\cdots\chi_+^{k_k}\Gamma] \\
&= \frac{1}{\mathrm{Sym}(\Gamma)} \sum_{\substack{\Gamma'\in\overline{\mathscr{G}}^{n,l}_{k\rightsquigarrow} \\ \mathrm{skeleton}(\Gamma')=[\Gamma]}} \#\{\{e_1,\dots,e_k\}\subset\Gamma^{[1]}_{\mathrm{int}} \mid [\chi_+^{e_1}\cdots\chi_+^{k_k}\Gamma]=\Gamma'\}\ \Gamma' \\
&= \sum_{\substack{\Gamma'\in\overline{\mathscr{G}}^{n,l}_{k\rightsquigarrow} \\ \mathrm{skeleton}(\Gamma')=[\Gamma]}} \frac{1}{\mathrm{Sym}(\Gamma')}\Gamma'.
\end{aligned}$$

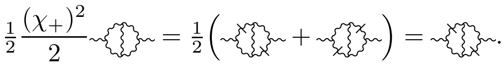
For ii. we apply i. to the combinatorial Green's function $X^{n,l}$, instead of a single graph. Summing over all graphs in $\overline{\mathscr{G}}^{n,l}$ yields

$$\begin{aligned}
\frac{(\chi_+)^k}{k!} X^{n,l} &= \sum_{\Gamma\in\overline{\mathscr{G}}^{n,l}} \frac{1}{\mathrm{Sym}(\Gamma)} \frac{(\chi_+)^k}{k!}\Gamma \\
&= \sum_{\Gamma'\in\overline{\mathscr{G}}^{n,l}_{k\rightsquigarrow}} \frac{1}{\mathrm{Sym}(\Gamma')}\Gamma' = X^{n,l}_{k\rightsquigarrow}.
\end{aligned}$$

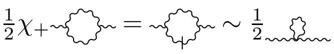
Finally, iii. follows directly from ii. by taking the sum over k.

□

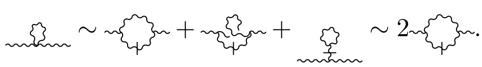
Example 4.11. Take  and $k = 2$; then lemma 4.10.i reads



Remark 4.12. Note that χ_+ has a non-trivial kernel as it can create self-loops graphs, for example:



Here we have used that



This does not influence our results, since self-loops have amplitude zero.

It is now time to study graph homology, again by studying marked edges, but now the labelling plays a crucial role.

4.2 The graph differential s for gauge theory graphs

Definition 4.13. The derivation $s : H \to H$ is given on generators by

$$s\Gamma = \sum_{e\in\Gamma^{[1]}_{\mathrm{int}}} (-)^{\#\{e'\in\Gamma^{[1]}_{\mathrm{marked}} \mid e'<e\}} s_e\Gamma$$

where

$$s_e\Gamma = \begin{cases} 0 & \text{if } e \text{ shares a vertex with a marked or ghost edge} \\ \Gamma_{e \rightsquigarrow \nparallel} & \text{otherwise} \end{cases}$$

In the above, $<$ is a (strict) total ordering on $\Gamma^{[1]}$.

Next, we want to distinguish the markings created by χ_+ and s. Therefore we draw the latter with two lines instead of one. So two lines indicate the action of s, and we denote: $\Gamma^{[1]}_{\rightsquigarrow\!\nmid} \cup \Gamma^{[1]}_{\rightsquigarrow\!\nparallel} = \Gamma^{[1]}_{\mathrm{marked}} \subset \Gamma^{[1]}_{\mathrm{int}}$.

Proposition 4.14. *s is a differential: $s^2\Gamma = 0$.*

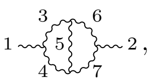
$S \circ S = 0$

Proof. We compute

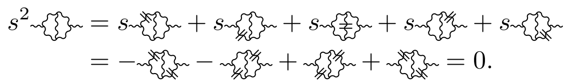
$$\begin{aligned} s^2\Gamma &= \sum_{e_1,e_2 \in \Gamma^{[1]}_{\mathrm{int}}} (-)^{\#\{e_1' \in \Gamma^{[1]}_{\mathrm{marked}} \mid e_1' < e_1\} + \#\{e_2' \in \Gamma^{[1]}_{\mathrm{marked}} \cup \{e_1\} \mid e_2' < e_2\}} s_{e_1} s_{e_2} \Gamma \\ &= \sum_{\substack{e_1,e_2 \in \Gamma^{[1]}_{\mathrm{int}} \\ e_1 < e_2}} (-)^{\#\{e \in \Gamma^{[1]}_{\mathrm{marked}} \mid e_1 < e < e_2\}+1} s_{e_1} s_{e_2} \Gamma \\ &\quad + \sum_{\substack{e_1,e_2 \in \Gamma^{[1]}_{\mathrm{int}} \\ e_1 > e_2}} (-)^{\#\{e \in \Gamma^{[1]}_{\mathrm{marked}} \mid e_2 < e < e_1\}} s_{e_1} s_{e_2} \Gamma \\ &= 0. \end{aligned}$$

□

Example 4.15. We work with labelled graphs, eg.

$\Gamma =$  ,

for which



4.3 The differential S for gauge theories

Marking edges, which corresponds upon summation of connected diagrams to shrinking pairs of two 3-gluon vertices to 4-gluon vertices, should match with the graphs with 4-gluon vertices present in the theory. This can be phrased homologically.

Definition 4.16. A derivation $S : H \to H$ is given by $S = s + \sigma$ where

$$\sigma\Gamma = (-)^{\#\Gamma^{[1]}_{\mathrm{marked}}} \sum_{e \in \Gamma^{[1]}_{\rightsquigarrow\!\nmid}} (-)^{\#\{e' \in \Gamma^{[1]}_{\rightsquigarrow\!\nmid} \mid e' > e\}} \sigma_e \Gamma,$$

and

$$\sigma_e\Gamma = \Gamma_{e \rightsquigarrow \nparallel}.$$

Proposition 4.17. *S is a differential: $S^2\Gamma = 0$.*

Proof. A calculation shows that

$$
\begin{aligned}
s\sigma\Gamma &= (-)^{\Gamma^{[1]}_{\text{marked}}} \sum_{e_1 \in \Gamma^{[1]}_{\leftrightsquigarrow}} \sum_{e_2 \in \Gamma^{[1]}_{\text{int}}} (-)^{\#\{e_1' \in \Gamma^{[1]}{\leftrightsquigarrow} \mid e_1' > e_1\} + \#\{e_2' \in \Gamma^{[1]}_{\text{marked}} \mid e_2' < e_2\}} s_{e_2}\sigma_{e_1}\Gamma \\
\sigma s\Gamma &= (-)^{\Gamma^{[1]}_{\text{marked}}+1} \sum_{e_2 \in \Gamma^{[1]}_{\text{int}}} \sum_{e_1 \in \Gamma^{[1]}{\leftrightsquigarrow}} (-)^{\#\{e_2' \in \Gamma_{\text{marked}} \mid e_2' < e_2\} + \#\{e_1' \in \Gamma^{[1]}{\leftrightsquigarrow} \mid e_1' > e_1\}} s_{e_2}\sigma_{e_1}\Gamma \\
&= -s\sigma\Gamma,
\end{aligned}
$$

and also

$$
\begin{aligned}
\sigma^2\Gamma &= \sum_{e_1 \in \Gamma^{[1]}{\leftrightsquigarrow}} \sum_{e_1 \in \Gamma^{[1]}{\leftrightsquigarrow} \setminus \{e_1\}} (-)^{\#\{e_1' \in \Gamma^{[1]}{\leftrightsquigarrow} \mid e_1' > e_1\} + \#\{e_2' \in \Gamma^{[1]}{\leftrightsquigarrow} \setminus \{e_1\} \mid e_2' > e_2\}} \sigma_{e_1}\sigma_{e_2}\Gamma \\
&= \sum_{\substack{e_1, e_2 \in \Gamma^{[1]}{\leftrightsquigarrow} \\ e_1 < e_2}} (-)^{\#\{e \in \Gamma^{[1]}{\leftrightsquigarrow} \mid e_1 < e < e_2\}+1} \sigma_{e_1}\sigma_{e_2}\Gamma \\
&\quad + \sum_{\substack{e_1, e_2 \in \Gamma^{[1]}{\leftrightsquigarrow} \\ e_1 > e_2}} (-)^{\#\{e \in \Gamma^{[1]}{\leftrightsquigarrow} \mid e_1 < e < e_2\}} \sigma_{e_1}\sigma_{e_2}\Gamma \\
&= 0.
\end{aligned}
$$

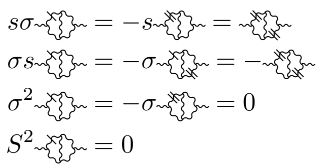
so that

$$S^2\Gamma = s^2\Gamma + s\sigma\Gamma + \sigma s\Gamma + \sigma^2\Gamma = 0.$$

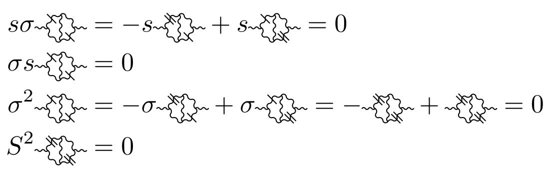
□

Remark 4.18. Note that upon summing the markings in a 3-valent corolla, and identifying such a sum with a 4-valent vertex, the operators s, S here reduce to the operators $\widetilde{s}$ and $\widetilde{S}$ we had before in Eq.(2).

Example 4.19.



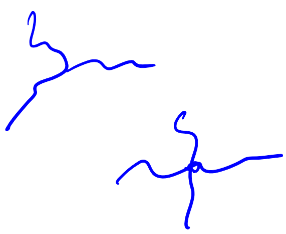
Example 4.20.



The cancellations between 3-gluon and 4-gluon vertices necessary to obtain a unitary and covariant gauge theory demand that shrinking internal edges in graphs with k_3 3-gluon vertices and k_4 4-gluon vertices matches with the graphs having $(k_3 - 2)$ 3-gluon vertices, and $(k_4 + 1)$ 4-gluon vertices. Rephrased in terms of our marked edges and using our sign conventions, that precisely is captured by

Proposition 4.21. *Let* Γ *be a graph without marked edges. Then:*

$$Se^{\chi_+}\Gamma = 0. \tag{16}$$



Proof. By definition:

$$e^{\chi_+}\Gamma = \sum_{k\geq 0} \sum_{\substack{e_1,\ldots,e_k \in \Gamma^{[1]}_{\mathrm{int}} \\ e_1<\cdots<e_k}} \chi_+^{e_1}\cdots\chi_+^{e_k}\Gamma$$

on which

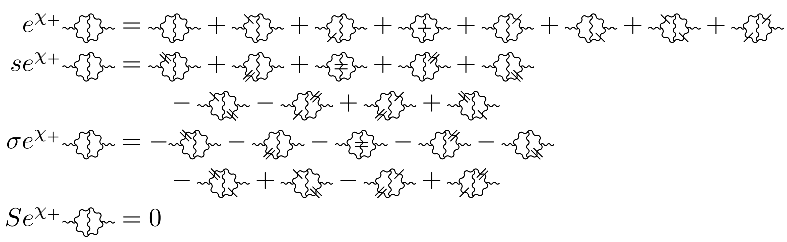
$$\begin{aligned}
se^{\chi_+}\Gamma &= \sum_{k\geq 0} \sum_{\substack{e_1,\ldots,e_{k+1} \in \Gamma^{[1]}_{\mathrm{int}} \\ e_1<\cdots<e_{k+1}}} \sum_{l=1}^{k+1} (-)^{l-1}\chi_+^{e_1}\cdots s_{e_l}\cdots\chi_+^{e_{k+1}}\Gamma \\
&= \sum_{k\geq 1} \sum_{\substack{e_1,\ldots,e_k \in \Gamma^{[1]}_{\mathrm{int}} \\ e_1<\cdots<e_k}} \sum_{l=1}^{k} (-)^{l-1}\chi_+^{e_1}\cdots s_{e_l}\cdots\chi_+^{e_k}\Gamma \\
\sigma e^{\chi_+}\Gamma &= \sum_{k\geq 1} \sum_{\substack{e_1,\ldots,e_k \in \Gamma^{[1]}_{\mathrm{int}} \\ e_1<\cdots<e_k}} (-)^k \sum_{l=1}^{k} (-)^{k-l}\chi_+^{e_1}\cdots s_{e_l}\cdots\chi_+^{e_k}\Gamma \\
&= -se^{\chi_+}\Gamma.
\end{aligned}$$

We conclude that

$$Se^{\chi_+}\Gamma = (s+\sigma)e^{\chi_+}\Gamma = 0.$$

□

Example 4.22.



This finishes our considerations of graph homology; we have proved Theorem 1.1.

4.4 The ghost cycle generator δ_+^C

We now turn to an investigation of the ghost sector through cycle homology.

Definition 4.23. Let $\mathscr{C}_\Gamma$ be the set of cycles in Γ. We write $\mathscr{C}_\Gamma = \{C_1, C_2, \ldots\}$.

$$\delta_+\Gamma = \sum_{C\in\mathscr{C}_\Gamma} \delta_+^C\Gamma,$$

where

$$\delta_+^C\Gamma = \begin{cases} 0 & \text{if } C \text{ has a vertex which has an adjacent marked or ghost edge} \\ \Gamma_{C\rightsquigarrow\cdots\cdots} & \text{otherwise.} \end{cases}$$

Note that an (unoriented) ghost cycle is the short-hand notation for the sum of the two orientations:

Lemma 4.24. *i. For any graph $\Gamma \in \overline{\mathscr{G}}^{n,l}$:*

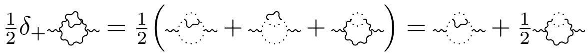
$$\frac{1}{\mathrm{Sym}(\Gamma)} \frac{(\delta_+)^k}{k!} \Gamma = \sum_{\substack{\Gamma' \in \overline{\mathscr{G}}^{n,l}_{k\bigcirc} \\ \mathrm{skeleton}(\Gamma') = \Gamma}} \frac{1}{\mathrm{Sym}(\Gamma')} \Gamma'.$$

ii. For any $k \geq 0$, $\frac{(\delta_+)^k}{k!} X^{n,l} = X^{n,l}_{k\bigcirc}$.

iii. We have $e^{\delta_+} X^{n,l} = X^{n,l}_{/\bigcirc}$.

Proof. Analogous to Lemma 4.7. □

Example 4.25. An example for Lemma 4.24.i is



Remark 4.26. The operators χ^e_+ and δ^C_+ commute, hence so do χ_+ and δ_+.

Corollary 4.27. *i. The combinatorial Green's functions $X^{n,l}_{k\leadsto,\tilde{l}\bigcirc}$ can be written as*

$$X^{n,l}_{k\leadsto,\tilde{l}\bigcirc} = \frac{\chi^k_+ \delta^{\tilde{l}}_+}{k!\tilde{l}!} X^{n,l}_{0\leadsto,0\bigcirc}.$$

ii. The full combinatorial Green's function can be written as

$$X^{n,l}_{/\leadsto,/\bigcirc} = e^{\chi_+} e^{\delta_+} X^{n,l}_{0\leadsto,0\bigcirc}.$$

$e^{\delta^R_+}$

4.5 The cycle differential t

Definition 4.28. For a graph Γ choose a labelling of the cycles $C_1, C_2, \ldots \in \mathscr{C}_\Gamma$. We define a derivation $t : H \to H$ acting on graphs as:

$$t\Gamma = \sum_{C_i \in \mathscr{C}_\Gamma} (-)^{\#\{C_{i'} \in \mathscr{C}_{\Gamma\mathrm{gh}} \mid i' < i\}} t_{C_i} \Gamma$$

where

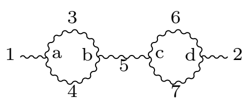
$$t_C \Gamma = \begin{cases} 0 & \text{if } C \text{ has a vertex which has an adjacent marked or ghost edge} \\ \Gamma_{C\leadsto\,\cdots\cdots} & \text{otherwise.} \end{cases}$$

Next, we want to distinguish the markings created by δ_+ and t. Therefore we draw the former with little circles instead of dots. We denote: $\mathscr{C}_{\Gamma\circ\circ\circ} \cup \mathscr{C}_{\Gamma\cdots} = \mathscr{C}_{\Gamma\mathrm{gh}} \subset \mathscr{C}_\Gamma$.

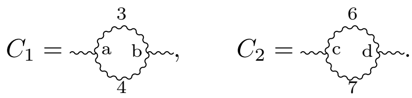
Proposition 4.29. *t is a differential: $t^2\Gamma = 0$.*

Proof. Analogous to Proposition 4.14. □

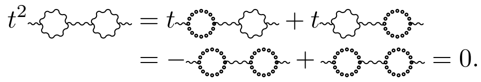
Example 4.30. Consider the graph

$\Gamma =$ 

and label the two cycles:



Then:



4.6 The differential T

The T-homology checks that the longitudinal degrees of freedom in a loop through 3-gluon vertices are appropriately matched by ghost loops, so that physical amplitudes are in the kernel of T. Hence, we define

Definition 4.31. A derivation $T : H \to H$ is given by $T = t + \tau$ where

$$\tau\Gamma = (-)^{\#\mathscr{C}_{\Gamma_{\mathrm{gh}}}} \sum_{C_i \in \mathscr{C}_{\Gamma}^{\cdots}} (-)^{\#\{C_i' \in \mathscr{C}_{\Gamma}^{\cdots} \mid i' > i\}} \tau_{C_i}\Gamma$$

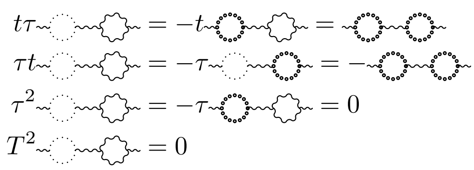
and

$$\tau_{C_i}\Gamma = \Gamma_{C_i \rightsquigarrow \cdots}.$$

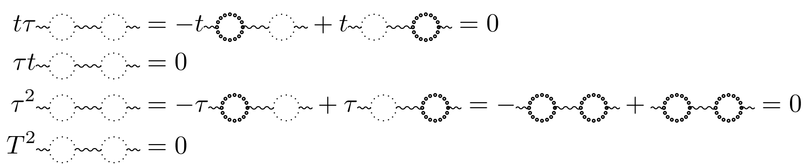
Proposition 4.32. *T is a differential: $T^2\Gamma = 0$.*

Proof. As in Proposition 4.17 this follows from $t\tau\Gamma = -\tau t\Gamma$ and $\tau^2\Gamma = 0$. □

Example 4.33.



Example 4.34.

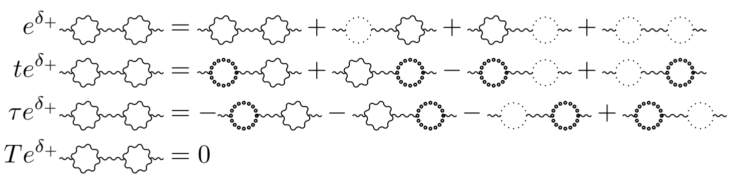


Proposition 4.35. *Let Γ be a graph without ghost edges. Then $Te^{\delta_+}\Gamma = 0$.*

Proof. Analogous to Proposition 4.21. □

Symmetry factors are no issue in the following example as we sum over both orientations for the two ghost lines.

Example 4.36.



This homology ensures that longitudinal degrees of freedom propagating in loops cancel. We summarize:

Theorem 4.37. *Let Γ be a graph without marked and ghost edges. Then*

$$Se^{\delta_+}e^{\chi_+}\Gamma = 0, \quad \textit{and} \quad Te^{\delta_+}e^{\chi_+}\Gamma = 0.$$

4.7 The bicomplex

As $[s,t] = [S,T] = 0$, we get a double complex:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \\
\cdots & \xrightarrow[t]{} & H_{k,\tilde{l}} & \xrightarrow[t]{} & H_{k,\tilde{l}+1} & \xrightarrow[t]{} & \cdots \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \\
\cdots & \xrightarrow[t]{} & H_{k+1,\tilde{l}} & \xrightarrow[t]{} & H_{k+1,\tilde{l}+1} & \xrightarrow[t]{} & \cdots \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

Here, $H_{\ldots,\ldots}$ are to be regarded as reflecting the relevant vector space structure only of these spaces. The corresponding Hopf algebras and combinatorial Green functions are discussed now. This bicomplex above and its relation to gauge symmetry and BRST cohomology will be the study of future work.

5 Combinatorial Green functions

The Hopf algebras on scalar graphs straightforwardly generalize to gauge theory graphs. In particular, the coproduct acts on the sum of all graphs contributing to a given amplitude —the combinatorial Green function— filtered by the number of 4-valent vertices and the number of ghost loops. Let us make this more precise.

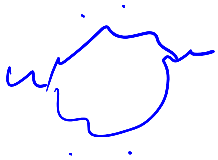
5.1 Gradings on the Hopf algebra

Recall that the Hopf algebra H is graded by the loop number, since the number of loops in a subgraph $\gamma \subset \Gamma$ and in the graph Γ/γ add up to $|\Gamma| \equiv n(\Gamma)$. Another (multi)grading is given by the number of vertices. In order for this to be compatible with the coproduct —creating an extra vertex in the quotient Γ/γ— we say a graph Γ with $E_E(\Gamma)$ external edges, is of multi-vertex-degree $(j_3, j_4, \ldots)$ if the number of m-valent vertices is equal to $j_m + \delta_{m,E_E(\Gamma)}$. One can check that this grading is compatible with the coproduct. Moreover, the two degrees are related via $\sum_m (m-2) j_m(\Gamma) = 2|\Gamma|$. This grading can be extended to involve other types of vertices —such as $\tilde{j}$ ghost-gluon vertices— cf. [22] for full details.

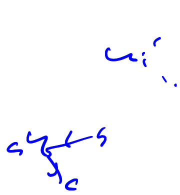
5.2 Series of graphs

As said, from a physical point of view, it is not so interesting to study individual graphs; rather, one considers whole sums of graphs with the same number of external lines. In this section, we will study series of 1PI graphs in the Hopf algebra H:

$$G_0^{k,n} = \sum_{|\Gamma|=n, |E_E(\Gamma)|=k} \Gamma \frac{\mathrm{colour}(\Gamma)}{\mathrm{sym}(\Gamma)},$$



$f^{abc}\delta_{cd}\delta_{be}f^{def}$



f^{abc}

T^a_{bc}

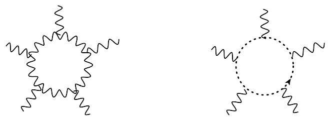


Figure 1: A 3-regular gluon cycle (left) and an oriented ghost cycle (right)

This is the sum of all 1PI 3-regular (0 4-valent vertices) graphs with first Betti number n and k external gluon edges (which fixes the amplitude r under consideration), normalized by their symmetry factors $\mathrm{sym}(\Gamma)$, the rank of their automorphism groups, in the denominator, and also weighted in the numerator by the corresponding colour factor $\mathrm{colour}(\Gamma)$:

$$\mathrm{colour}(\Gamma) := \prod_{v\in V^\Gamma} R_v \prod_{e\in E_I^\Gamma} \delta_{s(e),t(e)}.$$

Here, R_v is determined by a choice of a representation of th gauge group at v, and $s(e), t(e)$ are the vertex labels for source and target of the internal edge e. Typical, R_v is the adjoint representation for gluon self-interactions or the fundamental representation for a gluon interacting with fermionic matter fields.

Similarly, we write $G_j^{k,n}$ for series of graphs which have j 4-valent vertices, with all other vertices 3-valent. Also, we consider external ghost edges and loops. We let $G_{j;\widetilde{n}}^{k,\widetilde{k},n}$ denote the sum of graphs which have k external gluon edges, $\widetilde{k}$ external ghost edges, j 4-valent vertices, with all other vertices 3-valent, and $\widetilde{n}$ ghost cycles (Figure 1). We let $\widehat{G}_{j;\widetilde{n}}^{k,n}$ be the same sum where we consider all j 4-valent vertices, and all $\widetilde{n}$ ghost cycles as marked.

Summarizing, the superscript on G always indicate the external structure of the graphs in the series, whereas the subscripts indicate the 4-vertex degree, the loop number, or the ghost cycle degree.

We have shown in [23] that we can impose the Slavnov–Taylor identities on the Hopf algebra H, compatibly with the coproduct, equating all of the following formal elements:

$$Q^{k,\widetilde{k}} := \left(\frac{G_{/}^{k,\widetilde{k},/}}{(G_{/}^{2,0,/})^{k/2} (G_{/}^{0,\widetilde{2},/})^{\widetilde{k}/2}} \right)^{1/(k+\widetilde{k}-2)}, \tag{17}$$

independent of the numbers k and $\widetilde{k}$ of external gluon and ghost edges, respectively. The thus-defined single formal series $Q \equiv Q^{k,\widetilde{k}}$ will play the role of a 'charge' element in the Hopf algebra.

Proposition 5.1. *The coproduct on the Green's functions read*

$$\Delta(G_{j_3 j_4;\widetilde{n}}^{k,n}) = \sum_{\substack{j_m = j'_m + j''_m \\ n=n'+n'' \\ \widetilde{n}=\widetilde{n}'+\widetilde{n}''}} (G^{k,n'} Q^{2n''})_{j'_3 j'_4;\widetilde{n}'} \otimes G_{j''_3 j''_4;\widetilde{n}''}^{k,n''}$$

with $G_{j_3 j_4;\widetilde{n}}^{k,n}$ the above series of graphs of vertex multidegree (j_3, j_4), first Betti number n and $\widetilde{n}$ ghost cycles.

After taking the Slavnov–Taylor identities (17) into account, the coproduct reads on the above series of graphs

$$\Delta(G^{k,n}) = \sum_{n=n'+n''} (G^k Q^{2n''})_{n'} \otimes G^{k,n''}.$$

Remark 5.2. Note that neither the lhs nor the rhs depend on $\widetilde{k}$ in the above proposition, as $Q \equiv Q^{k,\widetilde{k}}$, $\forall k, \widetilde{k}$.

Remark 5.3. The inclusion of fermions is parallel to the study of ghost edges and loops, and a mere notational exercise.

Another way to describe the Green's function G^k is in terms of so-called grafting operators, defined in terms of 1PI primitive graphs. We start by considering maps $B_+^\gamma : H \to \mathrm{Aug}$, with Aug the augmentation ideal, which will soon lead us to non-trivial one co-cycles in the Hochschild cohomology of H. They are defined as follows.

$$B_+^\gamma(h) = \sum_{\Gamma \in \langle \Gamma \rangle} \frac{\mathbf{bij}(\gamma, h, \Gamma)}{|h|_\vee} \frac{1}{\mathrm{maxf}(\Gamma)} \frac{1}{(\gamma|h)} \Gamma,$$

where $\mathrm{maxf}(\Gamma)$ is the number of maximal forests of Γ, $|h|_\vee$ is the number of distinct graphs obtainable by permuting edges of h, $\mathbf{bij}(\gamma, h, \Gamma)$ is the number of bijections of external edges of h with an insertion place in γ such that the result is Γ, and finally $(\gamma|h)$ is the number of insertion places for h in γ [24]. $\sum_{\Gamma \in <\Gamma>}$ indicates a sum over the linear span $\langle \Gamma \rangle$ of generators of H.

The sum of the B_+^γ over all primitive 1PI Feynman graphs at a given loop order and with given residue will be denoted by $B_+^{l;n}$, as in [24]. More precisely,

$$B_+^{k;n} = \sum_{\substack{\gamma \text{ prim} \\ |\gamma|=n \\ E_E(\gamma)=k}} \frac{1}{\mathrm{Sym}(\gamma)} B_+^\gamma.$$

With this and the above Proposition, we can show [24, Theorem 5]:

$$G^k = \sum_{l=0}^{\infty} B_+^{k;n}(G^k Q^{2n}); \tag{18}$$

$$\Delta(B_+^{k;n}(G^k Q^{2n})) = B_+^{k;n}(G^k Q^{2n}) \otimes \mathbb{I} + (\mathrm{id} \otimes B_+^{k;n}) \Delta(G^k Q^{2n}). \tag{19}$$

Equation (18) is known as the combinatorial Dyson–Schwinger equation, while (19) shows that $B_+^{k;n}$ is a Hochschild cocycle for the Hopf algebra H.

5.3 The generator of ghost loops

We again consider the map $\delta_+ : H \to H$ that replaces gluon loops in a Feynman graph by ghost loops.

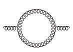
Remark 5.4. In accordance with our previous definition of δ_+, it becomes an algebra derivation $\delta_+ : H \to H$ by the assignment

$$\delta_+(\Gamma) = (\widetilde{l} + 1) \sum_{g \subset \Gamma} \Gamma_{g \mapsto \widetilde{g}}.$$

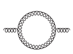
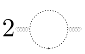
for a 1PI Feynman graph Γ at ghost loop order $\widetilde{l}$. The sum is over all oriented 3-regular gluon cycles g, and $\Gamma_{g \mapsto \widetilde{g}}$ denotes the graph Γ with the 3-regular gluon cycle g replaced by a ghost cycle $\widetilde{g}$ (cf. Figure 1), of the same orientation.

The notation δ_+ suggests that there is also a δ_-. In fact, such an operator can be defined and would replace a ghost loop by a gluon loop. We will not further study such an operator, since our interest lies in generating physical amplitudes from zero-ghost-loop amplitudes.

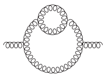
Example 5.5. Consider the following one-loop gluon self-energy graph:

$\Gamma =$  .

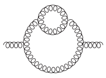
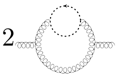
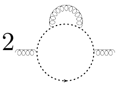
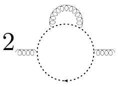
Its symmetry factor is $\mathrm{Sym}(\Gamma) = 2$ so that

$\delta_+ \Big($  $\Big) = 2$  .

A two loop example is given by the graph

$$\Gamma' =$$ 

for which $\mathrm{Sym}(\Gamma')=2$. Now,

$$\delta_+\Big($$  $$\Big) = 2$$  $$+\,2$$  $$+\,2$$  .

The first graph on the rhs obtains a factor of two because the two orientation of the ghost loop both reproduce this graph when the little gluon loop is replaced by a ghost loop.

The other two possible gluon cycles give the same graphs with again a coefficient of two for each of them, with the two orientations of the ghost cycle now resulting in those two remaining graphs on the rhs. In full accordance with Lemma 4.24, the ratio of the symmetry factor of a graph on the left by the symmetry factor of a graph on the right counts such multiplicities.

With that lemma we conclude:

$B_+^{k,n} = \Sigma\, B_+^{\delta^T\gamma}$

Proposition 5.6. *When acting on series of graphs with no ghost cycles ($\widetilde{n}=0$):*

$$e^{\delta^+}\circ B_+^{k;n} = \sum_{\substack{\gamma \text{ prim}\\ |\gamma|=n,\widetilde{n}(\gamma)=0\\ E_E(\gamma)=k}} \frac{1}{\mathrm{Sym}(\gamma)} B_+^{e^{\delta^+}(\gamma)} \circ e^{\delta_+}$$

where the sum is over graphs γ with no ghost cycles.

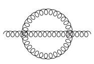
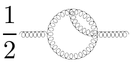
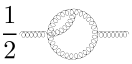
Remark 5.7. There is a similar result for connected graphs on the exponentiation of χ_+. We give it here without proof. It follows directly though from extending the definition of graph Hopf algebras and their Hochschild cohomology from 1PI to connected graphs. When acting on series of graphs with no marked edges:

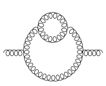
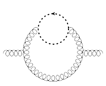
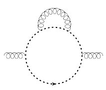
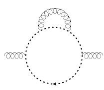
$$e^{\chi^+}\circ B_+^{k;n} = \sum_{\substack{\gamma \text{ prim}\\ |\gamma|=n,j(\gamma)=0\\ E_E(\gamma)=k}} \frac{1}{\mathrm{Sym}(\gamma)} B_+^{e^{\chi^+}(\gamma)} \circ e^{\chi_+}$$

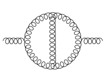
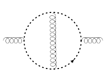
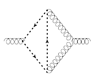
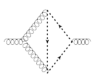
where the sum is over graphs γ with no marked edges and $j(\gamma)$ is the number of 4-valent vertices.

Together, the two results on the interplay of Hochschild cohomology and exponentiation show that gauge invariant combinatorial Green functions are obtained from gauge invariant skeleton graphs into which gauge invariant subgraphs are inserted.

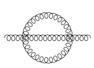
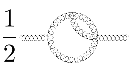
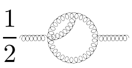
Example 5.8. Let us consider the example of the gluon self-energy at two loops:

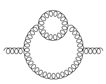
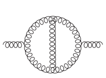
$$G^2_{n=2} = +\frac{1}{6}$$  $$+\frac{1}{2}$$  $$+\frac{1}{2}$$  (20)

$$+\frac{1}{2}$$  $$+$$  $$+$$  $$+$$ 

$$+\frac{1}{2}$$  $$+$$  $$+$$  $$+$$ 

whose zero-ghost-loop part is

$$G^2_{n=2,\widetilde{n}=0} = \frac{1}{6}$$  $$+\frac{1}{2}$$  $$+\frac{1}{2}$$  (21)

$$+\frac{1}{2}$$  $$+\frac{1}{2}$$ 

One readily checks that $(1+\delta_+)G^2_{n=2,\widetilde{n}=0} = G^2_{n=2}$.

Theorem 5.9. *Let $\widetilde{H}$ be the Hopf subalgebra of H generated by $G^{k,n}$ for all $n \geq 0$ and $k = 2, 3, 4$. Then $\exp \delta_+$ is an automorphisms of the graded Hopf algebra $\widetilde{H}$:*

$$\exp \delta_+(x_1 x_2) = \exp \delta_+(x_1) \exp \delta_+(x_2); \qquad \Delta(\exp \delta_+(x)) = (\exp \delta_+ \otimes \exp \delta_+)\Delta(x).$$

for $x_1, x_2, x \in \widetilde{H}$.

Proof. By definition, δ_+ is an algebra derivation so that $\exp \delta_+$ is an algebra automorphism. Note that at a given loop order l, the exponential terminates at that power n and is thus well-defined on the graded algebra underlying $\widetilde{H}$.

Let us then consider the compatibility of δ_+ with the coproduct structure. Recall from [22] the formula

$$\Delta(G^k) = \sum_{j_3, j_4, \widetilde{j} \geq 0} G^k (Q^3)^{j_3} (Q^4)^{2j_4} (Q^{1,\widetilde{2}})^{\widetilde{n}} \otimes G^k_{j_3 j_4 \widetilde{j}}.$$

which holds even without the Slavnov–Taylor identities. It continues to hold when restricting to graphs with zero ghost loops:

$$\Delta(G^k_{\widetilde{n}=0}) = \sum_{j_3, j_4} G^k_{\widetilde{n}=0} (Q^3_{\widetilde{n}=0})^{j_3} (Q^4_{\widetilde{n}=0})^{2j_4} \otimes G^r_{j_3, j_4; \widetilde{n}=0}.$$

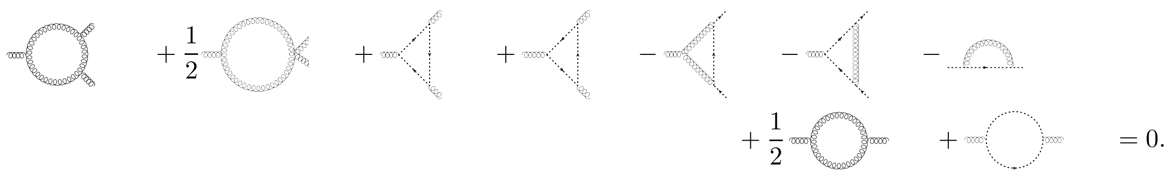
We now apply $\exp \delta_+ \otimes \exp \delta_+$ to this equation to obtain after imposing the Slavnov–Taylor-identities $Q^3 = Q^4$:

$$\begin{aligned}(\exp \delta_+ \otimes \exp \delta_+)\Delta(G^k_{\widetilde{n}=0}) &= \sum_{j_3, j_4} G^k (Q^3)^{j_3} (Q^4)^{2j_4} \otimes \exp \delta_+ \left(G^k_{j_3 j_4; \widetilde{n}=0}\right) \\ &= \sum_{n \geq 0} G^k Q^{2n} \otimes \exp \delta_+ \left(G^k_{n, \widetilde{n}=0}\right)\end{aligned}$$

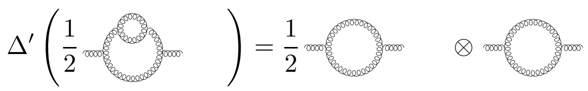
since in the absence of ghost vertices $j_3 + 2j_4 = 2n$ in terms of the first Betti number n. Lemma(4.24) then yields $\exp \delta_+ (G^k_{n, \widetilde{n}=0}) = G^k_n$, which completes the proof. □

This can be extended to the *connected* Green's functions $X_{k,n}$, where also a similar result can be shown for $\exp \chi_+$.

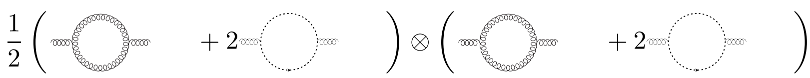
Example 5.10. First, recall the Slavnov–Taylor identities $G^3 G^{\widetilde{2}} = G^{1,\widetilde{2}} G^2$ which at one-loop order become:



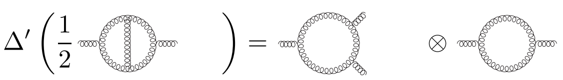
We compute $\Delta'(G^{2,n=2}_{\widetilde{n}=0})$ with $G^{2,n=2}_{\widetilde{n}=0}$ given in Eq.(21). For the first graph on the last line, we have



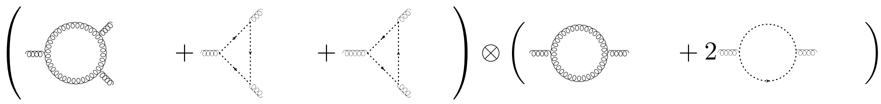
If we apply $\exp \delta_+ \otimes \exp \delta_+$ to this expression, we obtain



For the coproduct on the last graph in Eq.(21) we have

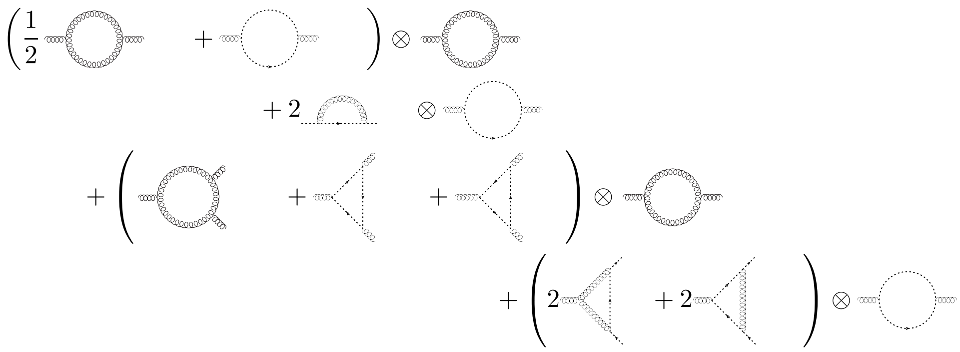


and applying $\exp\delta_+ \otimes \exp\delta_+$ to this expression yields



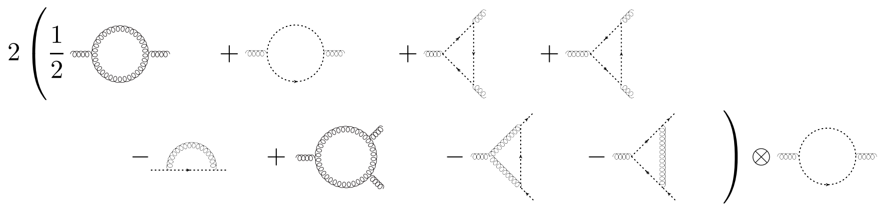
On the other hand, $\Delta'(\exp\delta_+ G^{2,n=2}_{\tilde{n}=0}) = \Delta'(G^{2,n=2})$ is computed from Eq.(20):

$\Delta'(G^{2,n=2}) =$



We conclude that

$$(\exp\delta_+ \otimes \exp\delta_+)\Delta'(G^{2,n=2}_{\tilde{n}=0}) - \Delta'(\exp\delta_+(G^{2,n=2}_{\tilde{n}=0})) =$$



~~which vanishes by the Slavnov–Taylor identities upon adding the contribution of 4-valent vertices.~~

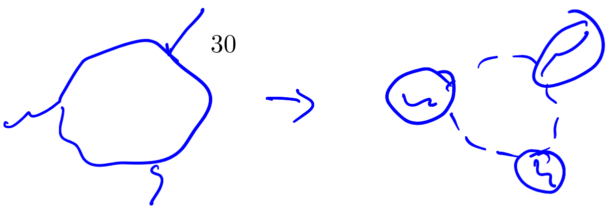
6 The corolla polynomial and differentials

6.1 The Corolla Polynomial

Finally, we introduce the Corolla Polynomial ([10]). It is a polynomial based on half-edge variables $a_{v,j}$ assigned to any half-edge (v,j) determined by a vertex v and an edge j. We need the following definitions:

- For a vertex $v \in V$ let $n(v)$ be the set of edges incident to v (internal or external).
- For a vertex $v \in V$ let $D_v = \sum_{j\in n(v)} a_{v,j}$.
- Let $\mathscr{C}$ be the set of all cycles of Γ (cycles, not circuits). This is a finite set.
- For C a cycle and v a vertex in V, since Γ is 3-regular, there is a unique edge of Γ incident to v and not in C, let v_C be this edge.
- For $i \geq 0$ let

$$C^i = \sum_{\substack{C_1,C_2,\ldots C_i\in\mathscr{C} \\ C_j \text{pairwise disjoint}}} \left(\left(\prod_{j=1}^{i} \prod_{v\in C_j} a_{v,v_C} \right) \prod_{v\notin C_1\cup C_2\cup\cdots\cup C_i} D_v \right)$$



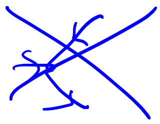
- Let

$$C = \sum_{j \geq 0} (-1)^j C^j$$

For any finite graph Γ, this is a polynomial $C = C(\Gamma)$ —the corolla polynomial— because $C^i = 0$ for $i > |\mathscr{C}|$.

Theorem 6.1. *([10]) Let $\mathscr{T}^\Gamma$ be the set of sets T of half edges of Γ with the property that*

- *every vertex of Γ is incident to exactly one half edge of T*
- *$\Gamma \setminus T$ has no cycles*



Then

$$C(\Gamma) = \sum_{T \in \mathscr{T}^\Gamma} \prod_{h \in T} a_h$$

Remark 6.2. This shows that the corolla polynomial is strictly positive. As it applies in this form as a corolla differential to pure Yang–Mills theory, this results in a positivity statement on Yang–Mills theory which does not hold for gauge fields coupled to matter fields. Accordingly, the sign of the β-function in gauge theory becomes dependent on the number of fermion families, and their representations.

Remark 6.3. For a graph Γ, let E be a set of pairwise disjoint internal edges of Γ. For $i \geq 0$ let

$$C_E^i(\Gamma) = \sum_{\substack{C_1, C_2, \ldots C_i \in \mathscr{C} \\ C_j \text{ pairwise disjoint} \\ C_j \cap E = \emptyset}} \left(\left(\prod_{j=1}^{i} \prod_{v \in C_j} a_{v, v_C} \right) \prod_{v \notin C_1 \cup C_2 \cup \cdots \cup C_i \cup E} D_v \right)$$

where the sum forbids cycles from sharing either vertices or edges with E.

Let

$$C_E(\Gamma) = \sum_{j \geq 0} (-1)^j C_E^j(\Gamma).$$

Then,

$$C_E(\Gamma) = C(\Gamma - E)$$

where $\Gamma - E$ is the graph with the edges and vertices involved in E removed. Removing a vertex removes all its incident half-edges so that $2|E|$ new external edges are generated. Note that $C_\emptyset(\Gamma) = C(\Gamma)$.

Define

$$C^{\text{fr}}(\Gamma) := \sum_E \left(C_E(\Gamma) \prod_{e \in E} W_e \right),$$

where W_e is defined in (7).

Corollary 6.4.

$$C^{\text{fr}}(\Gamma) = \sum_E \left(\sum_{T \in \mathscr{T}^E} (\prod_{h \in T} a_h) \prod_{e \in E} W_e \right).$$

Proof. Immediate. □

Remark 6.5. Consider a 3-regular graph Γ which has j, $j \geq 2$, 3-valent vertices, and let $\mathscr{P}$ be a set of m paths, $2m \leq j$, on internal edges and 3-valent vertices in Γ which each connect two external 3-valent vertices (a 3-valent vertex v is external if $n(v)$ contains an external edge) with $p_i \cap p_j = \emptyset$, $\forall p_i, p_j \in \mathscr{P}$.

Consider for chosen set E and $\mathscr{P}$ as above, with $E \cap \mathscr{P} = \emptyset$, for $i \geq 0$,

$$
\begin{aligned}
C^i_{E,\mathscr{P}}(\Gamma) \quad = \quad & \sum_{\substack{C_1, C_2, \ldots C_i \in \mathscr{C} \\ C_j \text{ pairwise disjoint} \\ C_j \cap E = \emptyset \\ C_j \cap \mathscr{P} = \emptyset}} \left(\left(\prod_{j=1}^{i} \prod_{v \in C_j} a_{v, v_C} \right) \prod_{v \notin C_1 \cup C_2 \cup \cdots \cup C_i \cup E \cup \mathscr{P}} D_v \right) \times \\
& \times \left(\prod_{p \in \mathscr{P}} \prod_{v \in p} a_{v, v_p} \right),
\end{aligned}
$$

where the sum forbids cycles from sharing either vertices or edges with E, and v_p is the unique half-edge at v not in p.

Let

$$
C_{E,\mathscr{P}}(\Gamma) = \sum_{j \geq 0} (-1)^j C^j_{E,\mathscr{P}}(\Gamma).
$$

Finally, we set

$$
C^{\mathrm{fr}}_{\mathscr{P}}(\Gamma) := \sum_{E} \left(C_{E,\mathscr{P}}(\Gamma) \prod_{e \in E} W_e \right). \tag{22}
$$

6.2 Corolla differentials

Our main use of the corolla polynomial is to construct differential operators with it. These operators differentiate with respect to momenta ξ_e assigned to edges e of a graph, and act on the second Kirchhoff polynomial written for generic edge momenta ξ_e, that is on $|N|_{\mathrm{Pf}}$.

Only at the end of the computation will we employ the map

$$
Q : \xi_e \to \xi_e + q(e).
$$

We then set $\xi_e = 0$ after we have applied the corolla differentials so that we obtain the standard second Symanzik polynomial for specific external momenta as prescribed by gauge theory amplitudes.

For a half edge $h \equiv (w, f) \in H^\Gamma$, we let $e(h) = f$ and $v(h) = w$. We remind the reader that h_+ and h_- are the successor and the precursor of h in the oriented corolla at $v(h)$, and that we assign to a graph Γ:

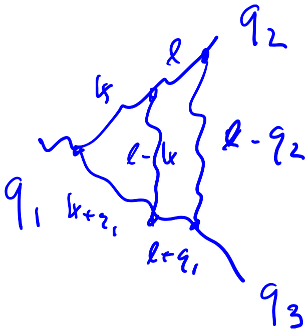
i. to each (possibly external) edge e, a variable A_e and a 4-vector ξ_e;

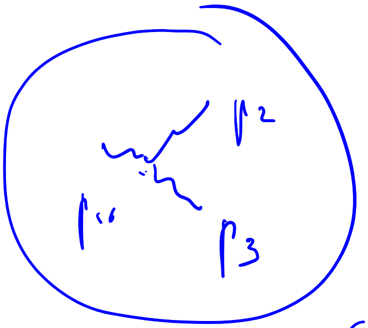
ii. to each half edge h, a Lorentz index $\mu(h)$;

iii. a factor colour(Γ).

6.3 The differential D^0

The corolla polynomial is an alternating sum over terms C^i, where i counts the number of loops. Similarly, the corolla differentials are a sum of terms D^i. We start with D^0.





$$\to \cdots \sum_{\substack{\text{cycl}\\ 1,2,3}} \left((p_1 - p_2)_{\mu_3}\, g_{\mu_1 \mu_2} \right)$$

$$\cancel{\frac{1}{A_3}} \sum_{\text{cycl}} \left(\frac{1}{A_1}\frac{\partial}{\partial p_1} - \frac{1}{A_2}\frac{\partial}{\partial p_2} \right) p_3 \quad e^{-A_1 p_1^2 - A_2 p_2^2 - A_3 p_3^2}$$

$$\frac{\partial}{\partial l_\mu} \quad \frac{\partial}{\partial l_\nu}$$

$$\frac{\partial}{\partial p_{1\nu}} \left(2\, p_{1\mu}\, e^{-A_1 p_1^2} \right)$$

$$\frac{1}{A_1} \frac{\partial}{\partial p_{1\nu}} p_{1\mu} \quad \underline{-A_1 g_{\mu\nu}} \qquad p_{1\mu} \frac{\partial}{\partial p_{1\nu}} e^{\cdots}$$

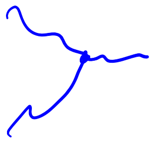
$$C = \sum c_i (-1)^i$$

C_0 no ghost loop

$\sum D_v$

Let $\Gamma \in \mathscr{G}^{n,l}$:

$$U^0(\Gamma) = \int_E \frac{\mathrm{d}\underline{k}_L}{(2\pi)^{dl}} C^0_\Gamma(\underline{D}) e^{-\sum_{e\in\Gamma^{[1]}} A_e \xi_e'^2},$$



where

$$C^0_\Gamma(\underline{D}) = \prod_{v\in\Gamma^{[0]}} D_v, \qquad D_v = D_{v1} + D_{v2} + D_{v3}$$

(the edges incident on v are labelled $1, 2, 3$),

$$D_{v1} = -\tfrac{1}{2} g^{\mu_2\mu_3}\Big(\varepsilon_{v2}\frac{1}{A_2}\frac{\partial}{\partial \xi_{2\mu_1}} - \varepsilon_{v3}\frac{1}{A_3}\frac{\partial}{\partial \xi_{3\mu_1}}\Big).$$

Using that all corollas are oriented, we can write this as

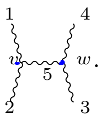
$$D_g(h) := -\tfrac{1}{2} g^{\mu_{h+}\mu_{h-}}\Big(\varepsilon_{h_+}\frac{1}{A_{e(h_+)}}\frac{\partial}{\partial \xi(h_+)_{\mu_h}} - \varepsilon_{h_-}\frac{1}{A_{e(h_-)}}\frac{\partial}{\partial \xi(h_-)_{\mu_h}}\Big),$$

for any half-edge h. The operator D_v is such that if it acts on $e^{-\sum_{e\in\Gamma^{[1]}} A_e \xi_e'^2}$, it gives the 3-vertex Feynman rule of Eq.(10):

$$D_v e^{-\sum_{e\in\Gamma^{[1]}} A_e \xi_e'^2} = V_v^{(3)} e^{-\sum_{e\in\Gamma^{[1]}} A_e \xi_e'^2}.$$

In order to calculate $C^0_\Gamma(\underline{D}) \equiv C^0_\Gamma(h \to D_g(h))$, we also need to know the Leibniz terms $D_v V_w^{(3)}$, where $v, w \in \Gamma^{[0]}$.

- If v and w do not share an edge, $D_v V_w^{(3)} = 0$.
- Suppose they share exactly one edge; we give it label 5. Let 1 and 2 be the other edges at v and 3 and 4 the other ones at w:

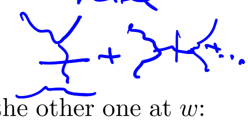
  

  Then:

  $$D_v V_w^{(3)} = \frac{1}{A_5}(g^{\mu_4\mu_2}g^{\mu_3\mu_1} - g^{\mu_2\mu_3}g^{\mu_4\mu_1}) \equiv \frac{W_e}{A_e}$$

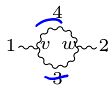
  4. vertex Feynman rule

  where W_e is the Feynman rule for a marked edge (equation (7)). Note that thus

  $$D_v V_w^{(3)} = D_w V_v^{(3)}.$$

- Suppose that v and w share two edges, 3 and 4. Let 1 be the other edge at v and 2 the other one at w:

  Then:

  $$\begin{aligned} D_v V_w^{(3)} &= \Big(\frac{1}{A_3}(g^{\mu_4\mu_4}g^{\mu_2\mu_1} - g^{\mu_2\mu_4}g^{\mu_4\mu_1}) + \frac{1}{A_4}(g^{\mu_3\mu_3}g^{\mu_2\mu_1} - g^{\mu_3\mu_2}g^{\mu_3\mu_1})\Big) \\ &= \frac{W_3}{A_3} + \frac{W_4}{A_4}. \end{aligned}$$

  where we have used equation (7). Note that also in this case

  $$D_v V_w^{(3)} = D_w V_v^{(3)}.$$

  Contracting the indices further gives self-loops which can be omitted:

  $$D_v V_w^{(3)} = 3C_2 \delta^{a_1 a_2} g^{\mu_1\mu_2}\Big(\frac{1}{A_3} + \frac{1}{A_4}\Big).$$

6.4 Regular terms and residues

We can now compute immediately the application of D^0 to the scalar integrand I_Γ, that is, $U^0(\Gamma) := D^0 I_\Gamma$.

$$\begin{aligned}
U^0(\Gamma) &= \int \frac{\mathrm{d}\underline{k}_L}{(2\pi)^{dl}} \Bigg(\prod_{v\in\Gamma^{[0]}} D_v \Bigg) e^{-\sum_{e\in\Gamma^{[1]}} A_e \xi_e'^2} \\
&= \int \frac{\mathrm{d}\underline{k}_L}{(2\pi)^{dl}} \Bigg[\Bigg(\prod_{v\in\Gamma^{[0]}} V_v^{(3)} \Bigg) \\
&\qquad + \sum_{\substack{w,w'\in\Gamma^{[0]} \\ w \text{ and } w' \text{ share an edge} \\ w<w'}} (D_w V_{w'}^{(3)}) \Bigg(\prod_{\substack{v\in\Gamma^{[0]} \\ v\neq w,w'}} V_v^{(3)} \Bigg) \\
&\qquad + \sum_{\substack{w,w',x,x'\in\Gamma^{[0]} \\ w \text{ and } w' \text{ and } x \text{ and } x' \text{ share an edge} \\ w<w'<x<x'}} (D_w V_{w'}^{(3)})(D_x V_{x'}^{(3)}) \Bigg(\prod_{\substack{v\in\Gamma^{[0]} \\ v\neq w,w',x,x'}} V_v^{(3)} \Bigg) \\
&\qquad + \cdots \Bigg] e^{-\sum_{e\in\Gamma^{[1]}} A_e \xi_e'^2}.
\end{aligned}$$

With the result of the previous subsection we get (recall that we exclude graphs with self-loops):

$$\begin{aligned}
U^0(\Gamma) &= \int \frac{\mathrm{d}\underline{k}_L}{(2\pi)^{dl}} \Bigg[\Bigg(\prod_{v\in\Gamma^{[0]}} V_v^{(3)} \Bigg) + \sum_{e\in\Gamma^{[1]}_{\text{int}}} \frac{W_e}{A_e} \Bigg(\prod_{\substack{v\in\Gamma^{[0]} \\ v \text{ not adj. to } e}} V_v^{(3)} \Bigg) \\
&\qquad + \sum_{\substack{\{e_1,e_2\}\subset\Gamma^{[1]}_{\text{int}} \\ e_1 \text{ and } e_2 \text{ do not share a vertex}}} \frac{W_{e_1} W_{e_2}}{A_{e_1} A_{e_2}} \Bigg(\prod_{\substack{v\in\Gamma^{[0]} \\ v \text{ not adj. to } e_1,e_2}} V_v^{(3)} \Bigg) \\
&\qquad + \cdots \Bigg] e^{-\sum_{e'\in\Gamma^{[1]}} A_{e'} \xi_{e'}'^2}. \\
&= \sum_{k\geq 0} \sum_{\substack{\{e_1,\ldots,e_k\}\subset\Gamma^{[1]}_{\text{int}} \\ e_1,\ldots,e_k \text{ do not share a vertex}}} \frac{W_{e_1}\cdots W_{e_k}}{A_{e_1}\cdots A_{e_k}} \int \frac{\mathrm{d}\underline{k}_L}{(2\pi)^{dl}} \\
&\qquad \times \Bigg(\prod_{\substack{v\in\Gamma^{[0]} \\ v \text{ not adj. to } e_1,\ldots,e_k}} V_v^{(3)} \Bigg) e^{-\sum_{e\in\Gamma^{[1]}} A_e \xi_e'^2}.
\end{aligned}$$

The first term we recognise as the Feynman-Schwinger integrand of Γ. The other terms we can write as the integrands of marked versions of Γ (equation (9)). More precisely,

$$U^0(\Gamma) = \sum_{k\geq 0} \sum_{\{e_1,\ldots,e_k\}\subset\Gamma^{[1]}_{\text{int}}} \frac{1}{A_{e_1}\cdots A_{e_k}} \int \frac{\mathrm{d}\underline{k}_L}{(2\pi)^{dl}} \mathscr{I}(\chi_+^{e_1}\cdots\chi_+^{e_k}\Gamma) e^{-A_{e_1}\xi_{e_1}'^2 - \cdots - A_{e_k}\xi_{e_k}'^2},$$

where $\mathscr{I}(\Gamma)$ is given in equation (11). Recall that in the exponent in the integrand only the unmarked edges are included. That is why the factor $e^{-A_{e_1}\xi_{e_1}'^2-\cdots-A_{e_k}\xi_{e_k}'^2}$ appears. This factor does not change the residue along $\prod_{e\in\Gamma^{[1]}_{\text{int}}} A_e = 0$.

Each subset of edges here is accompanied by a corresponding set of poles. By construction, the residues along these poles correspond to integrands where the edges shrink to form 4-valent vertices with the correct Feynman rules.

Using the χ_+^e-operator, we can write the integral $\widetilde{U}^0(\Gamma)$ as:

$$\widetilde{U}^0(\Gamma) = \sum_{k\geq 0} \sum_{\{e_1,\ldots,e_k\}\subset\Gamma^{[1]}_{\mathrm{int}}} \int \mathrm{d}\underline{A}_{\Gamma^{[1]}\setminus\{e_1,\ldots,e_k\}} I(\chi_+^{e_1}\cdots\chi_+^{e_k}\Gamma).$$

χ_+

In terms of Feynman amplitudes, this is

$$\widetilde{U}^0(\Gamma) = \sum_{k\geq 0} \sum_{\{e_1,\ldots,e_k\}\subset\Gamma^{[1]}_{\mathrm{int}}} \Phi(\chi_+^{e_1}\cdots\chi_+^{e_k}\Gamma) = \Phi(e^{\chi_+}\Gamma). \tag{23}$$

Instead of applying $\widetilde{U}^0$ to a single graph, we can do this to the combinatorial Green's function $X^{n,l}$. This gives us the Green's function for all graphs in Yang–Mills theory without the ghosts, but including the 4-valent vertices:

Proposition 6.6. *Collecting residues as above produces the evaluation by the Feynman rules of all 3- and 4-valent graphs in gauge theory without internal ghost or fermion edges:*

$$\widetilde{U}^0(X^{n,l}) = \Phi(e^{\chi_+}X^{n,l}) = \Phi(X^{n,l}_{/\times}).$$

Proof. The above equation (23) is used, together with Lemma 4.10.iii. $\square$

6.5 Exponentiating residues

Let us discuss the pairing between the integrand with poles along the boundaries of the simplex, with boundaries given by $\sigma_\Gamma : \prod_{i=1}^{|\Gamma_I^{[1]}|} A_i = 0$, and the Feynman integrand $U^0(\Gamma)$ in more detail.

The amplitude $\widetilde{U}^0(\Gamma)$ can be obtained from $U^0(\Gamma)$ by taking residues along hypersurfaces $\prod_{e\in E} A_e = 0$ and regular parts and integrating:

$$\widetilde{U}^0(\Gamma) = \sum_{k\geq 0} \sum_{\{e_1,\ldots,e_k\}\subset\Gamma^{[1]}_{\mathrm{int}}} \int \mathrm{d}\underline{A}_{\Gamma^{[1]}\setminus\{e_1,\ldots,e_k\}} \underset{A_1,\ldots,\widehat{A_{e_1}},\ldots,\widehat{A_{e_k}},\ldots=0}{\mathrm{Reg}} \ \underset{A_{e_1},\ldots,A_{e_k}=0}{\mathrm{Res}} U^0(\Gamma).$$

For a function $f = f(\{A_e\})$ of graph polynomial variables $A_e, e \in \gamma_I^{[1]}$ with at most simple poles at the origin localized in disjoint sets of edges E, we can write

$$f = \sum_E f^E,$$

where the sum is over all such sets and f_E is the part of f which is regular upon setting variables $A_e, e \in (\Gamma_I^{[1]} - E)$ to zero.

For any set E of mutually disjoint internal edges of Γ, consider $\prod_{e\in E}\oint_{\gamma_e} f$, and let f^E be its regular part. For any finite graph Γ, let $\mathscr{E}^\gamma$ be the set of all sets of mutually disjoint edges ($\emptyset$ included).

Consider the differential form

$$J_\Gamma^f := \Big(f^E \bigwedge_{e\in(\Gamma_I^{[1]}-E)} dA_e\Big)_{E\in\mathscr{E}^\Gamma}.$$

Let M_Γ^E be the hypercube

$$M_\Gamma^E := \mathbb{R}_+^{|\Gamma_I^{[1]}|-|E|},$$

and the corresponding vector

$$H_\Gamma(M_\Gamma^E)_{E\in\mathscr{E}^\Gamma}.$$

Then, there is a natural pairing

$$\int H_\Gamma \cdot J_\Gamma^f := \sum_{E\in\mathscr{E}^\Gamma} \int_{M_\Gamma^E} \Big(f^E \bigwedge_{e\in(\Gamma_I^{[1]}-E)} dA_e\Big).$$

6.6 Graph homology and the residue map

Note that in parametric integration we integrate against the simplex $\sigma \equiv \sigma_\Gamma$ with boundary $\prod_{e\in\Gamma^{[1]}} A_e = 0$. We have co-dimension k-hypersurfaces given by

$$A_{i_1} = \cdots = A_{i_k} = 0.$$

The Feynman integrand we have constructed above comes from regular parts, and residues along these hypersurfaces. It can be described by the following commutative diagram.

$$\begin{array}{ccc} \mathcal{G}_{j\times,/\circ} \ni \Gamma & \xrightarrow{\chi_+} & \chi_+(\Gamma) \in \mathcal{G}_{(j+1)\times,/\circ} \\ \downarrow \Phi & & \downarrow \Phi \\ \Phi(\Gamma) & \xrightarrow{\sum_e \mathrm{Res}_e} & \Phi(\chi_+(\Gamma)) \end{array}$$

→ associahedra
permutohedra
Stasheff
Gerstenhaber

The underlying geometry will be interpreted elsewhere.

6.7 Covariant gauges

For an edge e, let

$$G^\rho_{\mu\nu}(e) := \frac{g_{\mu\nu}}{{\xi'_e}^2} - 2\rho \frac{\xi'_{e\mu}\xi'_{e\nu}}{{\xi'_e}^4}$$

the corresponding gluon propagator in a covariant gauge ($\rho = 1/2$ being the transversal Landau gauge, $\rho = 0$ the Feynman gauge). One computes

$$G^\rho_{\mu\nu}(e) = \int\limits_0^\infty \frac{-1}{2A\rho} \frac{\partial}{\partial \xi'_{e\mu}} \frac{\partial}{\partial \xi'_{e\nu}} e^{-\rho A {\xi'_e}^2} dA =: \int\limits_0^\infty g^\rho_{\mu\nu}(e) dA.$$

We set

$$G^\rho_\Gamma := \prod_{e\in\Gamma_I^{[1]}} G^\rho_{\mu_{(s(e),e)}\mu_{(t(e),e)}}(e),$$

for half-edges $(s(e), e)$ and $(t(e), e)$, and g^ρ_Γ accordingly.

We let $I_\Gamma(\rho)$ be the corresponding scalar integrand obtained by substituting $A_e \to \rho A_e$ for each internal edge e.

G^ρ_Γ acts as a differential operator so that

$$G^\rho_\Gamma I_\Gamma(\rho) = F_G(\rho) I_\Gamma(\rho),$$

with $F_G(\rho)$ a polynomial in ρ, edge variables A_e and 4-momenta ξ_e.

Similarly, the corolla differential D_Γ acts as a differential operator so that

$$D_\Gamma I_\Gamma(1) = F_D I_\Gamma(1),$$

with F_D a polynomial in the 4-momenta ξ_e and a rational function in the edge variables A_e.

To compute in an arbitrary covariant gauge, we then work with

$$F_G(\rho) F_D I_\Gamma(1).$$

6.8 Yang–Mills theory

Consider a cycle C through 3-valent vertices in a graph Γ, and consider

$$\prod_{v\in C} D_g(v_C).$$

This is a differential operator with coefficients which are monomials in variables $1/A_e$, where $e \in C$.

Let D_C be the part in this differential operator which is linear in all variables $1/A_e$, for $e \in C$. Let ϕ_C be the Feynman rule for a ghost loop on C, summed over both orientations.

Lemma 6.7.

$$\phi_C := D_C e^{-\sum_{e\in C} A_e {\xi'_e}^2}.$$

Proof. This follows directly from the Feynman rules for ghost propagators and ghost-gluon vertices. Linearization eliminates all poles with residues corresponding to 4-valent 2-ghost-2-gluon vertices. ☐

Now consider the corolla polynomial $C(\Gamma)$ and replace each half-edge variable h by the differential $D_g(h)$. This defines a differential operator

$$d(\Gamma)^{\mathrm{YM}} := C(\Gamma)(h \to D_g(h)),$$

We consider $d(\Gamma)^{\mathrm{YM}}(I_\Gamma)$ where

$$I_\Gamma := \frac{e^{-\frac{|N_\Gamma|_{\mathrm{Pf}}}{\psi_\Gamma}}}{\psi_\Gamma^2},$$

is the scalar integrand for a graph Γ.

Proposition 6.8. *All poles in $d(\Gamma)^{\mathrm{YM}} I_\Gamma$ are located along co-dimension $|E|$ hypersurfaces $A_e = 0$, $e \in E$ for subsets E of mutually disjoint edges are simple poles.*

Proof. Corollary 3.6 ensures that poles are at most of first order and appear only when two derivatives act on the same edge. By the definition of the corolla polynomial this can only appear in mutually disjoint ordered pairs of corollas. All poles coming from divergent subgraphs are located along subsets of connected edges, as divergent subgraphs have more than a single edge. ☐

By our previous results on the Leibniz terms we can summarize now for the parametric integrand:

Corollary 6.9. *The residues of these poles correspond to graphs where each corresponding pair of corollas P_e is replaced by a 4-valent vertex.*

Proof. Setting an edge variable to zero shrinks that edge in the two Symanzik polynomials by the standard contraction-deletion identities [2, 3, 25]. ☐

The Leibniz terms serve the useful purpose to shrink an edge between two 3-gluon vertices. They provide a residue which corresponds to the integrand where the corresponding edge is a marked edge in our conventions. it is hence part of the integrand for a graph with a corresponding 4-valent vertex. As we have checked before, when summing over all connected 3-regular graphs, we correctly reproduce the Feynman integrand for all gluon self-interactions.

We stress that in doing so we want to shrink edges only between pairs of corollas which both are corollas for 3-gluon vertices, and will not mark edges between other type of vertices. This leads us to

Definition 6.10. We let $D(\Gamma)^{\mathrm{YM}} I_\Gamma$ be the part $d(\Gamma)^{\mathrm{YM}} I_\Gamma$ which is linear in all variables $1/A_e$.

This eliminates all poles in $D(\Gamma)^{\mathrm{YM}} I_\Gamma$ of the form $1/A_e$. We can regain then the contribution of 4-valent 4-gluon vertices by using Theorem 6.1 together with Remark 6.3:

Lemma 6.11. *Let*

$$U_\Gamma = g_\Gamma^\rho C^{\mathrm{fr}}(\Gamma)(a_h \to D_g(h))I_\Gamma.$$

Then $\overline{U}_\Gamma$ (cf. Eq.(5)) generates the integrand for the complete contribution of Γ to the full Yang–Mills theory amplitude. $\overline{U}_\Gamma^R$ generates the corresponding integrand for the renormalized contribution.

Proof. Immediate application of Lemma 6.7 and Theorem 1.3. □

This also proves Theorem 1.5 in the context of Yang–Mills theory.

Remark 6.12. If we were to work with non-linear gauges, we could avoid this linearization and use the Leibniz terms for the graphs with 2-gluon 2-ghost and 4-ghost vertices. Also, note that $\overline{U}_\Gamma^R = \overline{U}_\Gamma^R(\rho)$ depends on the gauge parameter.

6.9 Amplitudes with open ghost or fermion lines

For k open ghost lines we have a straightforward generalization of these differentials by using $C_{\mathscr{P}}^{\mathrm{fr}}(G)$, see Eq.(22), where each half edge h is again replaced by $D_g(h)$ and linearization is understood as before. For fermion lines, see below.

6.10 Gauge Theory

If we include matter fields, we need to add a second differential in particular for fermion fields:

$$\begin{aligned} D_f(h) \quad := \quad & \left(\frac{1}{A(e(h_+))} \frac{\partial}{\partial \xi(e(h_+))_{\mu(h_+)}} \gamma_{\mu(h_+)}\gamma_{\mu(h)} \right. \\ & \left. - \frac{1}{A(e(h_-))} \frac{\partial}{\partial \xi(e(h_-))_{\mu(h_-)}} \gamma_{\mu(h)}\gamma_{\mu(h_-)} \right). \end{aligned}$$

γ_μ

Now we must carefully distinguish between fermion and ghost cycles.

For a collection of cycles $C_1, \cdots, C_j$ contributing to C^j, consider partitions of this set into two subsets I_f, I_g containing $|I_f| + |I_g| = j$ cycles. Replace $a_{v,v_C} \to b_{v,v_C}$ for each $C \in I_f$. This defines $C^{I_g,I_f}(\Gamma)(a_h, b_h)$. Upon summing over all possible partitions I_g, I_l of the cycles for each j, this gives a further corolla polynomial for which we write in slight abuse of notation $C(\Gamma)(a_h, b_h)$. Assign a differential operator as follows:

$$U_\Gamma = g_\Gamma^\rho \sum_{j\geq 0} \sum_{|I_g|+|I_f|=j} C^{I_g,I_f}(\Gamma)(D_g(h), D_f(h))\mathrm{colour}^{I_g,I_f}(\Gamma),$$

where in C^{I_g,I_f}, for $I_g \cup I_f \neq \emptyset$, we keep only terms which are linear in variables $1/A_e$ for edges $e \in C_1 \cup \cdots \cup C_j$. We can now proceed with $\overline{U}_\gamma$ as before.

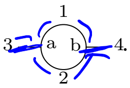
Note that the restriction to $I_l = \emptyset$ gives back the corresponding operator for Yang–Mills theory. From here on, Theorem 1.5 follows for gauge theory as before for Yang–Mills theory.

Remark 6.13. Note that all this can be turned into a projective integrand, illuminating the slots in the period matrix which are filled in a gauge theory as compared to a scalar field theory. In particular, one hopes that the geometry of Eq.(6.6) is helpful to explain appearances and disappearances of periods in gauge theory.

Remark 6.14. Putting fermions into the same colour rep as gauge bosons allows for immediate cancellations between D_g and D_f. This can be illuminating in studying the simplifications for supersymmetric gauge theories.

6.11 Examples: QED and Yang–Mills theory

In the following two examples, we compute the one-loop vacuum polarization in quantum electrodynamics, and then the one-loop gluon vacuum polarization in Yang–Mills theory. Both examples can be obtained from corolla differentials acting on the simplest possible 3-regular graph:

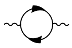
$\Gamma :=$ 

We label its two internal edges $1, 2$, and the external edges $3, 4$. We also label the two vertices a, b. Edge 3 is oriented from vertex a to vertex b, and edge 4 vice versa, say.

We have six half-edges: $h_1 := (a,3), h_2 := (a,2), h_3 := (a,1)$ and $h_4 := (b,1), h_5 := (b,2), h_6 := (b,4)$, with corresponding half-edge variables a_{a3}, a_{a2} etc.

We have four 4-vectors $\xi_1, \xi_2, \xi_3, \xi_4$, with $\xi_e \in \mathbb{M}^4$, Minkowski space, with scalar product $\xi_e^2 \equiv \xi_e \cdot \xi_e = {\xi_e}_0^2 - {\xi_e}_1^2 - {\xi_e}_2^2 - {\xi_e}_3^2$.

Example 6.15. In order to compute the one-loop vacuum polarisation in massless QED,

$\Pi_1 =$ 

we proceed as follows. We have for the corolla polynomial

$$C_1(\Gamma) = a_{a3} a_{b4}.$$

$$\frac{e^{-\frac{q}{4}}}{4^2}$$

The scalar integrand is

$$I(\Gamma) = \frac{1}{2} {\xi_3}^2 {\xi_4}^2 \frac{e^{-\frac{(\xi_1-\xi_2)^2 A_1 A_2 + (A_3 {\xi_3}^2 + A_4 {\xi_4}^2)(A_1+A_2)}{A_1+A_2}}}{(A_1+A_2)^2} dA_1 dA_2 dA_3 dA_4.$$

We can directly integrate A_3, A_4 eliminating any appearance of ${\xi_3}^2, {\xi_4}^2$ as in this example no derivatives with respect to external edges appear in the corolla differential.

Indeed, replacing the two half-edge variables in $C_1(\Gamma)$ by the fermion differential and using the linearized corolla differential (we symmetrize below in $\mu(3), \mu(4)$ when allowed)

$$\frac{1}{4A_1A_2}\left(\frac{\partial}{\partial {\xi_3}_{\mu(3)}} \frac{\partial}{\partial {\xi_4}_{\mu(4)}} + \frac{\partial}{\partial {\xi_4}_{\mu(4)}} \frac{\partial}{\partial {\xi_3}_{\mu(3)}}\right) = \frac{1}{2A_1A_2} \frac{\partial}{\partial {\xi_3}_{\mu(3)}} \frac{\partial}{\partial {\xi_4}_{\mu(4)}}$$

delivers π_1, the integrand for Π_1:

$$\begin{aligned}
\pi_1 \quad := \quad & -\frac{1}{4} \mathrm{Tr}(\gamma_{\mu(3)}\gamma_{\mu(2)}\gamma_{\mu(4)}\gamma_{\mu(1)}) \frac{\partial}{A_1 \partial {\xi_1}_{\mu(1)}} \frac{\partial}{A_2 \partial {\xi_2}_{\mu(2)}} I(\Gamma) \\
= \quad & -\mathrm{Tr}(\gamma_{\mu(3)}\gamma_{\mu(2)}\gamma_{\mu(4)}\gamma_{\mu(1)})(\xi_1-\xi_2)_{\mu(1)}(\xi_2-\xi_1)_{\mu(2)} A_1 A_2 \times \\
& \times \frac{e^{-\frac{(\xi_1-\xi_2)^2 A_1 A_2}{A_1+A_2}}}{(A_1+A_2)^4} dA_1 dA_2 \left(A_\gamma^{F_1} = \frac{A_1 A_2}{(A_1+A_2)^4}, \, |A_\gamma^{F_1}|_\gamma = 0 \right) \\
& + \mathrm{Tr}(\gamma_{\mu(3)}\gamma_{\mu(2)}\gamma_{\mu(4)}\gamma_{\mu(1)}) \frac{1}{2} g_{\mu(1)\mu(2)} \times \\
& \times \frac{e^{-\frac{(\xi_1-\xi_2)^2 A_1 A_2}{A_1+A_2}}}{(A_1+A_2)^3} dA_1 dA_2 \left(A_\gamma^{F_2} = \frac{1}{(A_1+A_2)^3}, \, |A_\gamma^{F_2}|_\gamma = 2 \right).
\end{aligned}$$

Partially integrating the metric tensor term (equivalently, multiplying $A_\gamma^{F_2}$ by $\frac{A_1 A_2}{(A_1+A_2)A_4}$ before integrating

A_4, see Eq.(5)) gives

$$\begin{aligned}\pi_1 \quad = \quad & \mathrm{Tr}(\gamma_{\mu(3)}\gamma_{\mu(2)}\gamma_{\mu(4)}\gamma_{\mu(1)})(\xi_1-\xi_2)_{\mu(1)}(\xi_2-\xi_1)_{\mu(2)}A_1A_2\times\\ & \times\frac{e^{-\frac{(\xi_1-\xi_2)^2A_1A_2}{A_1+A_2}}}{(A_1+A_2)^4}dA_1dA_2\\ & +\mathrm{Tr}(\gamma_{\mu(3)}\gamma_{\mu(2)}\gamma_{\mu(4)}\gamma_{\mu(1)})\frac{1}{2}g_{\mu(1)\mu(2)}(\xi_1-\xi_2)^2A_1A_2\times\\ & \times\frac{e^{-\frac{(\xi_1-\xi_2)^2A_1A_2}{A_1+A_2}}}{(A_1+A_2)^4}dA_1dA_2.\end{aligned}$$

Evaluating the trace, contracting indices and integrating delivers (Q replaces $\xi_1-\xi_2$ by q, subtraction at $q^2=\mu^2$ understood, *i.e.* Q_0 replaces $\xi_1-\xi_2$ by μ)

$$\Pi_1 = 8(q^2g_{\mu(3)\mu(4)}-q_{\mu(3)}q_{\mu(4)})\int\frac{A_1A_2e^{-\frac{q^2A_1A_2}{A_1+A_2}}}{(A_1+A_2)^4}-\cdots_{|q^2=\mu^2}\,dA_1dA_2$$

which can be written projectively

$$\Pi_1 = 8(q^2g_{\mu(3)\mu(4)}-q_{\mu(3)}q_{\mu(4)})\ln\frac{q^2}{\mu^2}\int_{\mathbb{P}^1(\mathbb{R}_+)}\frac{A_1A_2}{(A_1+A_2)^4}(A_1dA_2-A_2dA_1)$$

and which correctly evaluates to the expected transversal result

$$\Pi_1 = \frac{4}{3}(q^2g_{\mu(3)\mu(4)}-q_{\mu(3)}q_{\mu(4)})\ln\frac{q^2}{\mu^2}.$$

Next, we turn to Yang–Mills theory.

Example 6.16. We have

$$\frac{|N_{\text{-O-}}|_{\mathrm{Pf}}}{\psi_{\text{-O-}}} = -\xi_3^2A_3-\xi_4^2A_4-\frac{(\xi_1-\xi_2)^2A_1A_2}{A_1+A_2}$$

while the corolla polynomials read

$$\begin{aligned}C^0_{\text{-O-}}(\underline{a}) &= (a_{\mathrm{a}3}+a_{\mathrm{a}1}+a_{\mathrm{a}2})(a_{\mathrm{b}4}+a_{\mathrm{b}1}+a_{\mathrm{b}2})\\ C^1_{\text{-O-}}(\underline{a}) &= a_{\mathrm{a}3}a_{\mathrm{b}4}\end{aligned}$$

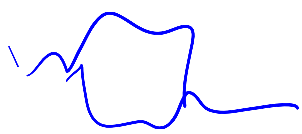
The corresponding differentials then become

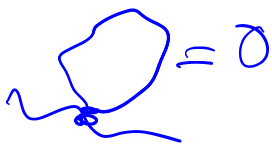
$$
\begin{aligned}
C^0_{\text{-}\circ\text{-}}(\underline{D}) &= (D_{\mathrm{a}3}+D_{\mathrm{a}1}+D_{\mathrm{a}2})(D_{\mathrm{b}4}+D_{\mathrm{b}2}+D_{\mathrm{b}1}) \\
&= \big(-\tfrac{1}{2}\big)^2\Big(g^{\mu_1\mu_2}\Big(-\frac{1}{A_1}\frac{\partial}{\partial\xi_{1\mu_3}}-\frac{1}{A_2}\frac{\partial}{\partial\xi_{2\mu_3}}\Big) \\
&\qquad + g^{\mu_2\mu_3}\Big(\frac{1}{A_2}\frac{\partial}{\partial\xi_{2\mu_1}}-\frac{1}{A_3}\frac{\partial}{\partial\xi_{3\mu_1}}\Big) \\
&\qquad + g^{\mu_3\mu_1}\Big(\frac{1}{A_3}\frac{\partial}{\partial\xi_{3\mu_2}}+\frac{1}{A_1}\frac{\partial}{\partial\xi_{1\mu_2}}\Big)\Big) \\
&\qquad \times\Big(g^{\mu_2\mu_1}\Big(-\frac{1}{A_2}\frac{\partial}{\partial\xi_{2\mu_4}}-\frac{1}{A_1}\frac{\partial}{\partial\xi_{1\mu_4}}\Big) \\
&\qquad + g^{\mu_1\mu_4}\Big(\frac{1}{A_1}\frac{\partial}{\partial\xi_{1\mu_2}}-\frac{1}{A_4}\frac{\partial}{\partial\xi_{4\mu_2}}\Big) \\
&\qquad + g^{\mu_4\mu_2}\Big(\frac{1}{A_4}\frac{\partial}{\partial\xi_{4\mu_1}}+\frac{1}{A_2}\frac{\partial}{\partial\xi_{2\mu_1}}\Big)\Big) \\
C^1_{\text{-}\circ\text{-}}(\underline{D}) &= D_{\mathrm{a}3}D_{\mathrm{b}4} \\
&= 4\big(-\tfrac{1}{2}\big)^2\Big(-\frac{1}{A_1}\frac{\partial}{\partial\xi_{1\mu_3}}-\frac{1}{A_2}\frac{\partial}{\partial\xi_{2\mu_3}}\Big) \\
&\qquad \times\Big(-\frac{1}{A_2}\frac{\partial}{\partial\xi_{2\mu_4}}-\frac{1}{A_1}\frac{\partial}{\partial\xi_{1\mu_4}}\Big)
\end{aligned}
$$

for which the linear part, without the factor 4 (the space-time dimension), is

$$
\widetilde{C}^1_{\text{-}\circ\text{-}}(\underline{D}) = \big(-\tfrac{1}{2}\big)^2\frac{1}{A_1A_2}\Big(\frac{\partial^2}{\partial\xi_{1\mu_3}\partial\xi_{2\mu_4}}+\frac{\partial^2}{\partial\xi_{2\mu_4}\partial\xi_{1\mu_3}}\Big)
$$

We compute





$$
\begin{aligned}
U^0_{\text{-}\circ\text{-}} &= C^0_{\text{-}\circ\text{-}}(\underline{D})\frac{e^{\overline{\phi}_{\text{-}\circ\text{-}}/\psi_{\text{-}\circ\text{-}}}}{\psi^2_{\text{-}\circ\text{-}}} \\
&= \frac{1}{(A_1+A_2)^4}\Big((A_1-A_2)g^{\mu_1\mu_2}q^{\mu_3}-(2A_1+A_2)g^{\mu_2\mu_3}q^{\mu_1} \\
&\qquad + (A_1+2A_2)g^{\mu_3\mu_1}q^{\mu_2}\Big)\Big((A_1-A_2)g^{\mu_2\mu_1}q^{\mu_4} \\
&\qquad + (A_1+2A_2)g^{\mu_1\mu_4}q^{\mu_2}-(2A_1+A_2)g^{\mu_4\mu_2}q^{\mu_1}\Big) \\
&\qquad \times e^{-q^2\left(\frac{A_1A_2}{A_1+A_2}+A_3+A_4\right)} \\
&\qquad + \frac{3}{(A_1+A_2)^3}\Big(1-\overbrace{\frac{A_1}{A_2}-\frac{A_2}{A_1}}^{\to 0,\ \text{as residues are scale-independent self-loops}}\Big)g^{\mu_1\mu_2}e^{-q^2\left(\frac{A_1A_2}{A_1+A_2}+A_3+A_4\right)}
\end{aligned}
$$

and so

$$
\begin{aligned}
\widetilde{U}^0_{\text{-}\circ\text{-}} &= \Big(\frac{1}{(A_1+A_2)^4}\big(-(2A_1^2+2A_2^2+14A_1A_2)q^{\mu_3}q^{\mu_4} \\
&\qquad +(5A_1^2+5A_2^2+8A_1A_2)q^2g^{\mu_3\mu_4}\big)\Big)_{\text{hence }|A^{F_1}_\gamma|_\gamma=0}\, e^{-q^2\frac{A_1A_2}{A_1+A_2}}\,\mathrm{d}A_1\mathrm{d}A_2 \\
&\qquad + \Big(\frac{3}{(A_1+A_2)^3}g^{\mu_3\mu_4}\Big)_{\text{hence }|A^{F_2}_\gamma|_\gamma=2}\, e^{-q^2\frac{A_1A_2}{A_1+A_2}}\,\mathrm{d}A_1\mathrm{d}A_2
\end{aligned}
$$

Similarly,

$$\widetilde{U}^{1}_{-\!\bigcirc\!-} = \Big(\frac{2A_1A_2}{(A_1+A_2)^4}q^{\mu_3}q^{\mu_4} - \frac{1}{(A_1+A_2)^3}g^{\mu_3\mu_4}\Big)e^{-q^2\frac{A_1A_2}{A_1+A_2}}\mathrm{d}A_1\mathrm{d}A_2$$

We thus obtain for the corresponding integrals:

$$\int \widetilde{U}^{0\mathrm{R}}_{-\!\bigcirc\!-} = \left(\tfrac{11}{3}q^{\mu_3}q^{\mu_4} - \tfrac{25}{6}q^2g^{\mu_3\mu_4}\right)\ln\left(\frac{q^2}{\mu^2}\right),$$

which corresponds to the gauge boson loop:

,

and

$$\int \widetilde{U}^{1\mathrm{R}}_{-\!\bigcirc\!-} = -\left(\tfrac{1}{3}q^{\mu_3}q^{\mu_4} + \tfrac{1}{6}q^2g^{\mu_3\mu_4}\right)\ln\left(\frac{q^2}{\mu^2}\right), \tag{24}$$

which corresponds to the ghost loop:

.

They combine to a transversal result:

$$\int \widetilde{U}^{\mathrm{R}}_{-\!\bigcirc\!-} = \int \widetilde{U}^{0\mathrm{R}}_{-\!\bigcirc\!-} - \int \widetilde{U}^{1\mathrm{R}}_{-\!\bigcirc\!-}$$
$$= 4(q^{\mu_3}q^{\mu_4} - q^2g^{\mu_3\mu_4})\ln\left(\frac{q^2}{\mu^2}\right).$$

Multiplying with $\frac{\mathrm{colour}(\gamma)}{\mathrm{sym}(\gamma)} = \frac{1}{2}f^{h_1h_2h_3}f^{h_2h_3h_6}$, this is the result for the 1-loop gluon self-energy in Yang–Mills theory. The gauge theory result is immediate from including the previous example with a suitable colour factor for the fermion loop.

7 Conclusion

7.1 Covariant quantization without ghosts

Consider $\overline{U_\Gamma}^R =: \sum_{i=0}^{\infty}(-1)^i\overline{U_\Gamma}^{i,R}$ in a notation which reflects the alternating structure of the corolla polynomial. Set $\overline{U^{\mathrm{gh}}}_\Gamma^R := \sum_{i=1}^{\infty}(-1)^i\overline{U_\Gamma}^{i,R}$.

Covariant quantization delivers naively the integrand $\overline{U_\Gamma}^{0,R}$. Let P_L be a projector onto longitudinal degrees of freedom so that a physical amplitude is in the kernel of P_L, P_T the corresponding projector such that $P_L + P_T = \mathrm{id}$.

Summing over connected graphs contributing to a physical amplitude $X^{r,n}$ at n loops, we know that

$$P_L\left(\overline{U}^{0,R}_{X^{r,n}}\right) = -P_L\left(\overline{U^{\mathrm{gh}}}^R_{X^{r,n}}\right).$$

The undesired longitudinal part of the ghost free sector determines the longitudinal part of the ghost contribution by definition.

But also, to compute the ratio

$$\frac{P_L\left(\overline{U^{\mathrm{gh}}}^R_{X^{r,n}}\right)}{P_T\left(\overline{U^{\mathrm{gh}}}^R_{X^{r,n}}\right)}$$

is a combinatorial exercise in determining the interplay of these projectors with the Leibniz terms originating from the corolla differentials in the various topologies. These longitudinal and transversal differentials are determined by the same scalar integrand, and hence are not independent. Eq.(24) with the ratio two between the qq and g form-factor is a typical example.

So the transversal part of the ghost sector is determined by the combinatorics of scalar graphs and the longitudinal part. It hence is implicitly determined by the ghost free sector.

7.2 Slavnov–Taylor Identities

Slavnov–Taylor identities are treated here as originating from co-ideals in the corresponding Hopf algebras. We reproduce the Feynman rules in four dimensions as renormalized integrands, and can similarly reproduce them in dimensional regularization, and checked that our renormalized Feynman integrand vanishes on the corresponding co-ideals, as required.

In future work, we will directly demonstrate the validity of Slavnov–Taylor identities from the structure of the corolla polynomial.

References

[1] S. Bloch, H. Esnault and D. Kreimer, *On Motives associated to graph polynomials,* Commun. Math. Phys. **267** (2006) 181 [math/0510011 [math-ag]].

[2] F. Brown, *The Massless higher-loop two-point function,* Commun. Math. Phys. **287** (2009) 925 [arXiv:0804.1660 [math.AG]].

[3] F. C. S. Brown, *On the periods of some Feynman integrals,* arXiv:0910.0114 [math.AG].

[4] F. Brown and O. Schnetz, *A K3 in ϕ^4,* Duke Math. J. Volume **161**, Number 10 (2012), 1817-1862; arXiv:1006.4064 [math.AG].

[5] F. Brown and O. Schnetz, *Proof of the zig-zag conjecture,* arXiv:1208.1890 [math.NT].

[6] F. Brown, O. Schnetz and K. Yeats, *Properties of c_2 invariants of Feynman graphs,* arXiv:1203.0188 [math.AG].

[7] D. Broadhurst, see `http://www.mathematik.hu-berlin.de/~maphy/QPPIIIBroadhurst.pdf` and *Feynman's sunshine numbers,* arXiv:1004.4238 [physics.pop-ph].

[8] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, *Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes,* Phys. Rev. D **85** (2012) 105014 [arXiv:1201.5366 [hep-th]].

[9] D. A. Kosower, R. Roiban and C. Vergu, *The Six-Point NMHV amplitude in Maximally Supersymmetric Yang-Mills Theory,* Phys. Rev. D **83** (2011) 065018 [arXiv:1009.1376 [hep-th]].

[10] D. Kreimer and K. Yeats, *Properties of the corolla polynomial of a 3-regular graph,* arXiv:1207.5460 [math.CO].

[11] S. Weinberg, *Feynman Rules for Any Spin,* Phys. Rev. **133** (1964) B1318.

[12] S. Weinberg, *Feynman Rules for Any Spin. 2. Massless Particles,* Phys. Rev. **134** (1964) B882.

[13] S. Weinberg, *Feynman rules for any spin. iii,* Phys. Rev. **181** (1969) 1893.

[14] P. Cvitanovic, *Field Theory,* `ChaosBook.org/FieldTheory`, Niels Bohr Institute (Copenhagen 2004), RX-1012 (NORDITA).

[15] J. Conant, K. Vogtmann, *On a theorem of Kontsevich,* Algebr. Geom. Topol.**3** (2003) 1167-1224, arXiv:math/0208169v2.

[16] D. Kreimer, *Algebraic Structures in local QFT,* Nucl. Phys. Proc. Suppl. **205-206** (2010) 122 [arXiv:1007.0341 [hep-th]], and references here.

[17] F. Brown and D. Kreimer, *Angles, Scales and Parametric Renormalization,* arXiv:1112.1180 [hep-th].

[18] D. Kreimer and E. Panzer, *Renormalization and Mellin transforms,* arXiv:1207.6321 [hep-th].

[19] S. Bloch and D. Kreimer, *Feynman amplitudes and Landau singularities for 1-loop graphs,* Commun. Num. Theor. Phys. **4** (2010) 709 [arXiv:1007.0338 [hep-th]].

[20] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory,* Phys.Rev.Lett. **94**:181602 (2005).

[21] D. Kreimer and W. D. van Suijlekom, *Recursive relations in the core Hopf algebra,* Nucl. Phys. B **820**, 682 (2009) [arXiv:0903.2849 [hep-th]].

[22] W. D. van Suijlekom, *The structure of renormalization Hopf algebras for gauge theories I: Representing Feynman graphs on BV-algebras,* Commun. Math. Phys. **290** (2009) 291-319. arXiv:0807.0999 [math-ph]

[23] W. D. van Suijlekom, *Renormalization of gauge fields: A Hopf algebra approach,* Commun. Math. Phys. **276** (2007) 773 [hep-th/0610137].

[24] D. Kreimer, *Anatomy of a gauge theory,* Annals Phys. **321** (2006) 2757 [hep-th/0509135].

[25] P. Aluffi, M. Marcolli, *Feynman motives and deletion-contraction relations,* arXiv:0907.3225v1 [math-ph].

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DSE. $\Phi_R\left(B_+^{\gamma}(\mathbb{I})\right) = c_\gamma \ln \frac{q^2}{\mu^2}$

?