DYSON–SCHWINGER EQUATIONS AND QUANTIZATION OF GAUGE THEORIES (SUMMER '21)

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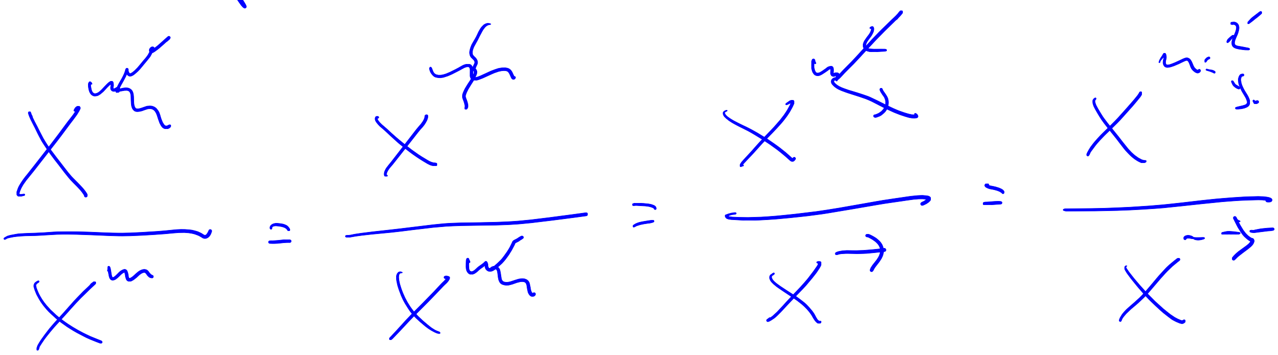
1. Co-ideals and gauge theory

Reference: https://arxiv.org/abs/hep-th/0509135 .

Slavnov-Taylor id's and Ward id's can be captured via co-ideals.

"Anatomy of a gauge theory" identifies those S-T id's

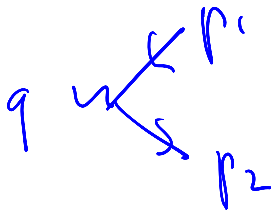


For today: QED

1.1. **QED: the Ward Takahashi identity.** .

The Ward-Takahashi identity:

$q = p_2 - p_1$



$$\Gamma_\mu(\alpha; p_1, m, p_2; \mu^2)$$

$\Sigma(p, m; \mu^2, \alpha)$ fermion self-energy

$\not{p} - m + \Sigma$ is the inverse propagator.

Γ_μ at tree level: $-ie\gamma_\mu$

$\sim \gamma_\mu$ Clifford matrix

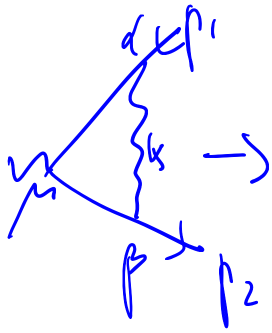
$$q^\mu \gamma_\mu = \not q = \not p_2 - \not p_1$$

$$= (\not p_2 - m) - (\not p_1 - m)$$

WT at tree level:

$$q^\mu \Gamma_\mu = (\not p_2 - m + \Sigma(p_2, \dots)) - (\not p_1 - m + \Sigma(p_1, \dots)).$$

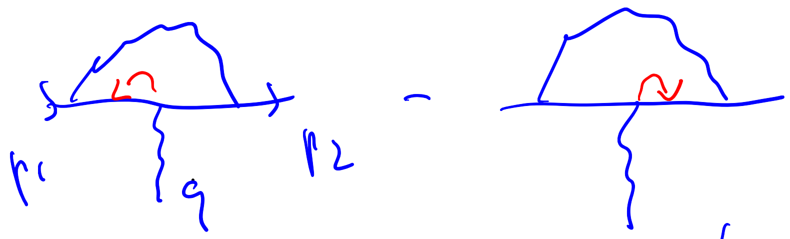
Quick reminder: how do you prove it to all loop orders?



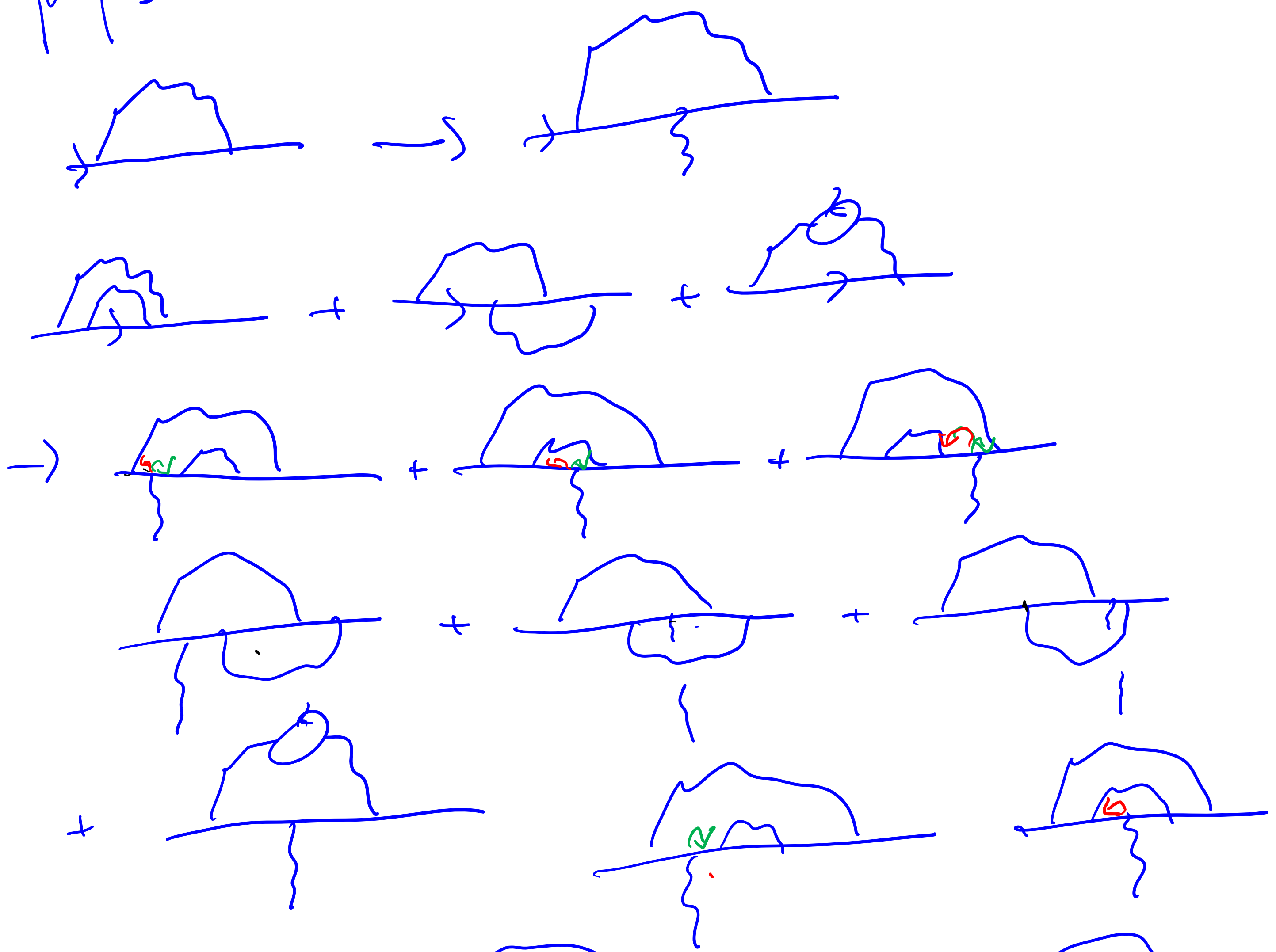
$$\to \int d^4k \, \gamma_\alpha \frac{1}{\not p_1 + \not k - m} \gamma_\mu \frac{1}{\not p_2 + \not k - m} \gamma_\beta \, D_{\alpha\beta}(k^2)$$

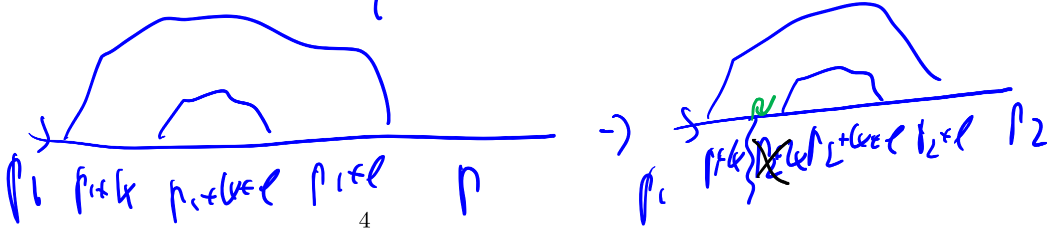
$$q^\mu \cdots \to \int d^4k \, \gamma_\alpha \frac{1}{\dots} \big((\not p_2 + \not k - m) - (\not p_1 + \not k - m)\big) \frac{1}{\not p_2 + \not k - m} \gamma^\beta$$

$$= \Sigma'(p_2) - \Sigma'(p_1)$$



All 1PI vertex graphs can be obtained from all self energy graphs:



$\int d^4 l \, d^4 k$ 

Proof by "telescoping"

1.1.1. *the WT identity as an ideal.* Reference: Walter van Suijlekom, *The Hopf Algebra of Feynman Graphs in Quantum Electrodynamics* Letters in Mathematical Physics (2006) **77**:265–281 .

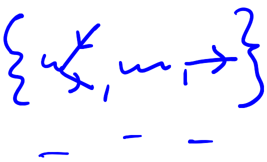
$$\frac{\partial}{\partial q_\mu} \Sigma(q) = \Gamma_\mu(q, q, m, \alpha, \mu^2)$$

Ward id.

There is a wonderful co-ideal in the Hopf alg. of QED graphs.

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma$$

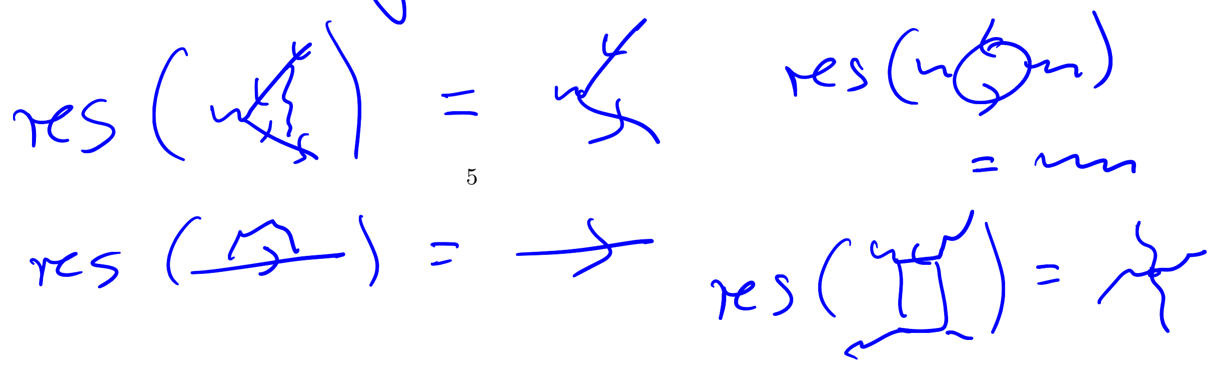
$$\gamma = \bigcup_i \gamma_i$$

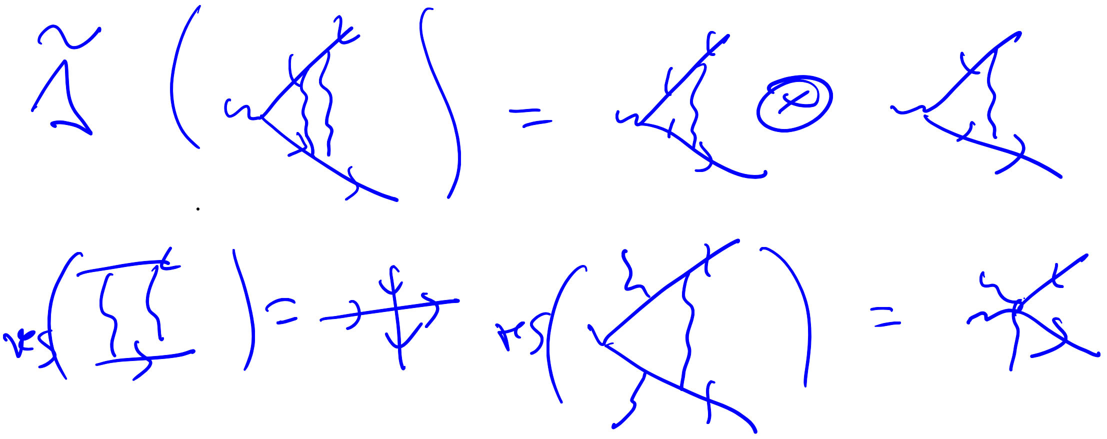
$\mathrm{res}(\gamma_i) \in$ 

where:

$\mathrm{res}(\Gamma)$: $\Gamma\big|_{\text{all internal edges have length zero.}}$

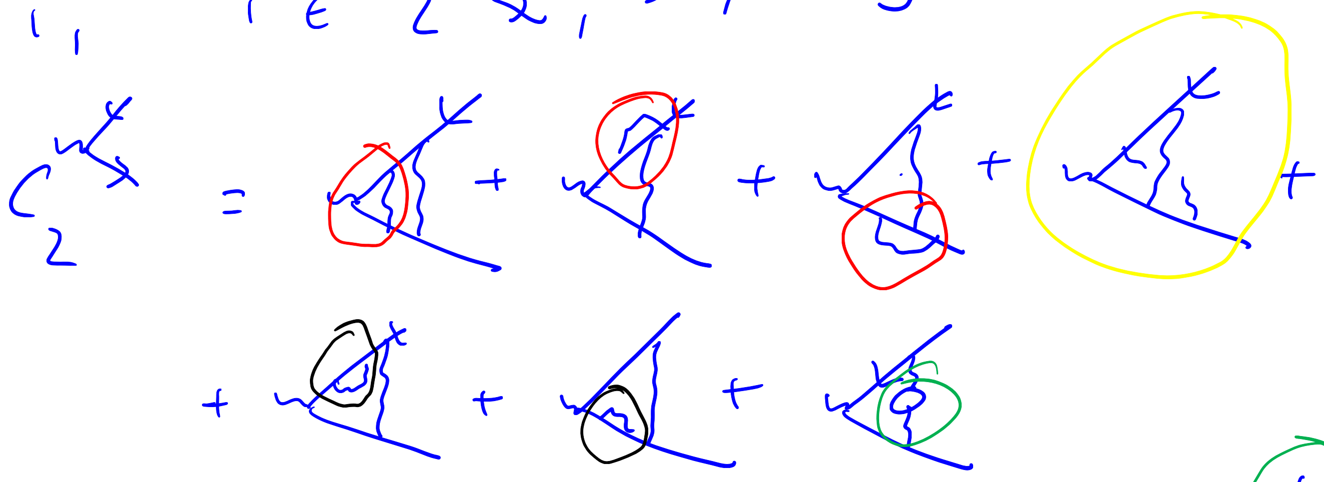
So:

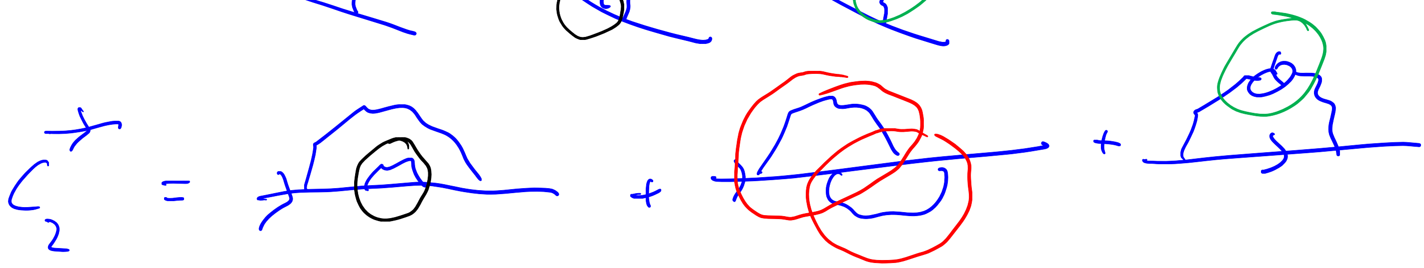




So $H_{QED}(m, I, \hat{I}, \Delta, S)$ as usual. Now where is the co-ideal?

Let c_n^i be the sum of all 1PI n-loop graphs with residue i, $i \in$ .

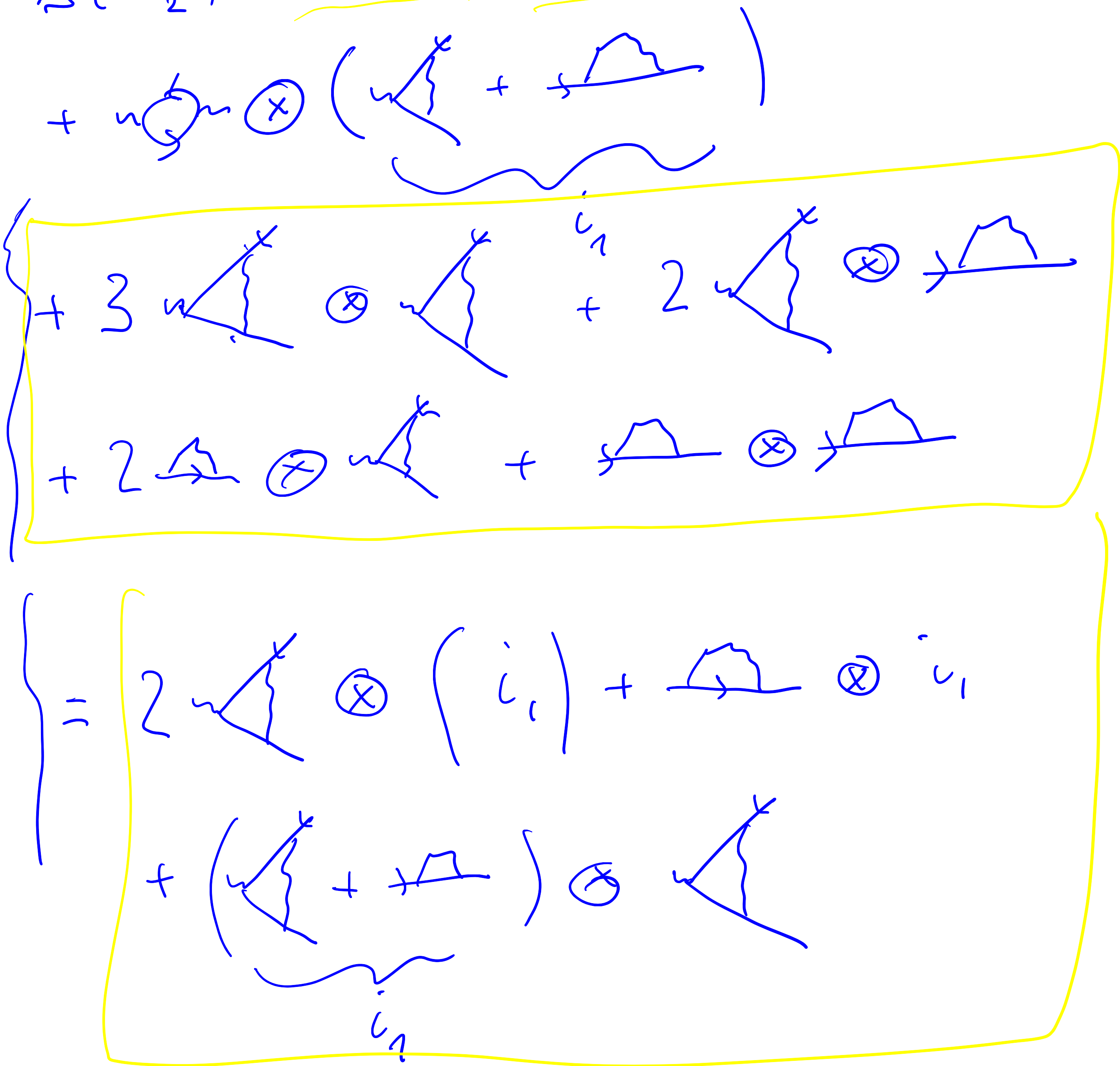




Thm: $\Delta(\mathrm{id}_n) \subseteq H_{QED} \otimes I + I \otimes H_{QED}$,

where $I = \mathrm{span}(\mathrm{id}_n)_n$

$\Delta(i_2) = i_2 \otimes \mathbb{I} + \mathbb{I} \otimes i_2$



↳ $\Delta(i_2) \subseteq H \otimes I + I \otimes H$

By counting # of insertion places

one moves it to all orders.

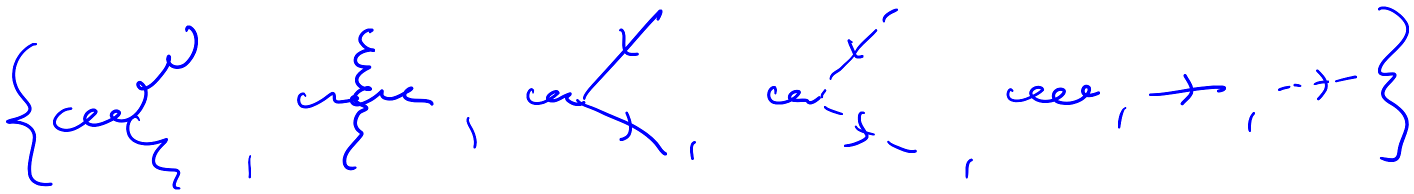
This makes for a nice interpretation of WT identities:

Define
$$\phi_R(i_n) = q^\mu \phi_R(c_n)$$
$$- \phi_R(c_n)(p_2)$$
$$- \phi_R(c_n)(p_1)$$

$$\Rightarrow \quad \phi_R(I) \equiv 0.$$

$$\left(\phi_R(i_m\, i_n) = \phi_R(i_m)\, \phi_R(i_n) \right.$$

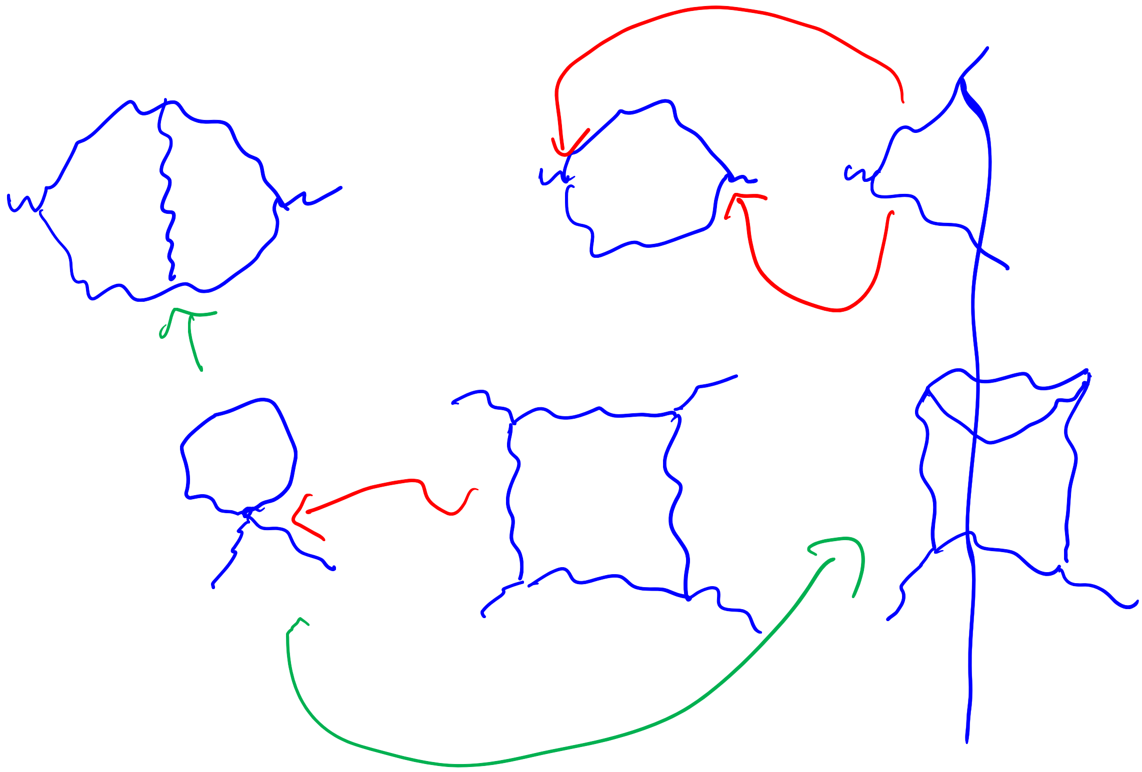
Why is this much more complicated in QCD?



7 residues, and four instead

of one ideal.

Have to study: # insertion places, # bijections between ext. leg. structure and internal edges and vertices, filtration by topology of a graph.



S-T derivable without using gauge invariance of Lagrangean.

$\longrightarrow$ See Weinberg "Feynman rules for any spin"