

DYSON-SCHWINGER EQUATIONS AND QUANTIZATION OF GAUGE THEORIES (SUMMER '21)

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1. CO-IDEALS AND GAUGE THEORY

Reference: <https://arxiv.org/abs/hep-th/0509135> . ←

Slavnov-Taylor id's and Ward id's can be captured via co-ideals.

" Anatomy of a gauge theory identifies those S-T id's

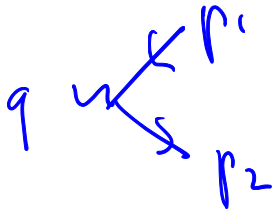
$$\frac{X^{\leftarrow}}{X^{\rightarrow}} = \frac{X^{\leftarrow}}{X^{\leftarrow}} = \frac{X^{\leftarrow}}{X^{\rightarrow}} = \frac{X^{n:2'}}{X^{-\rightarrow}}$$

Fa today: QED

1.1. QED: the Ward Takahashi identity.

The Ward - Takahashi identity:

$$q = p_2 - p_1$$



$$G_{\mu}^{\alpha\beta}(p_1, p_2) = \alpha_j \frac{p_1^j}{m} \beta_j \frac{p_2^j}{m^2}$$

$\Sigma(p, m, p^2, d)$ fermion self-energy

$p = m + \Sigma$ is the inverse propagator.

G_{μ} at tree level: $-ie \gamma_{\mu}$
 $\sim \gamma_{\mu}$ Clifford matrix

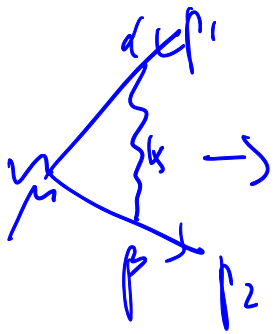
$$g^{\mu\nu} \gamma_{\mu} = \cancel{g} = \cancel{p_2} - p_1$$

$$= (p_2 - m) - (p_1 - m)$$

WT at tree level:

$$g^{\mu\nu} G_{\mu} = (p_2 - m + \Sigma(p_2, \dots)) - (p_1 - m + \Sigma(p_1, \dots)).$$

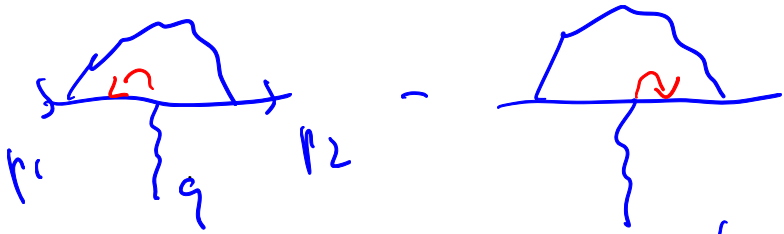
Quick reminder: how do you move it to all loop orders?



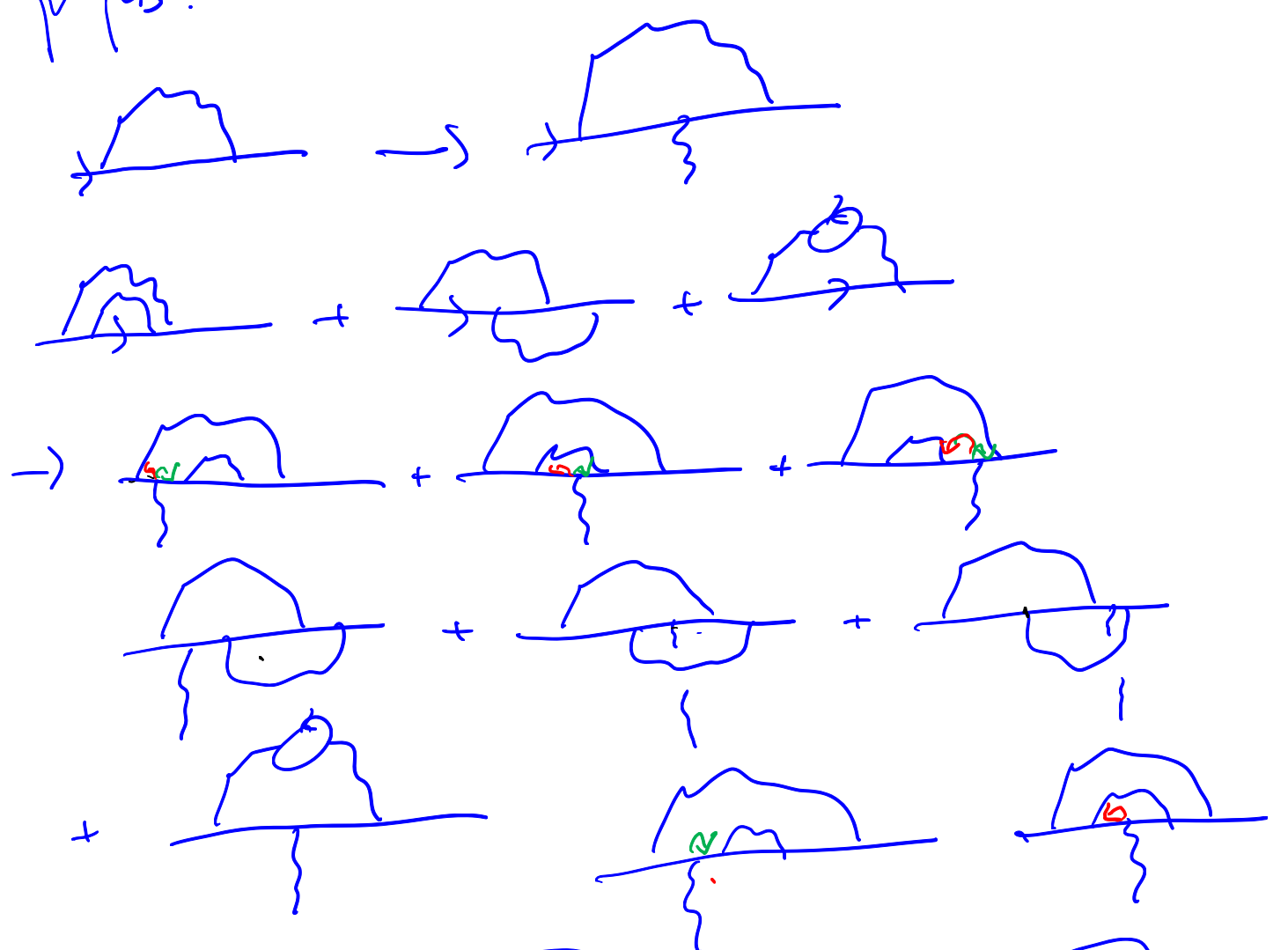
$$\int d^4 k \gamma_{\alpha} \frac{1}{p_1 + k - m} \gamma_{\mu} \frac{1}{p_2 + k - m} \gamma_{\beta} D_{\alpha\beta}(k^2)$$

$$g^{\mu\nu} \dots \rightarrow \int d^4 k \gamma_{\alpha} \frac{1}{\dots} \frac{(p_2 + k - m) - (p_1 + k - m)}{p_2 + k - m} \frac{1}{p_2 + k - m} \gamma_{\beta}$$

$$= \Sigma^1(p_2) - \Sigma^1(p_1)$$



All $1PI$ vertex graphs can be obtained from all self energy graphs:



$$\int d^4l d^4k \quad \text{[Diagram 1]} \rightarrow \text{[Diagram 2]}$$

$p_1 \quad p_1+k \quad p_1+k+l \quad p_1+l \quad p$

$p_1 \quad p_1+k \quad p_1+k+l \quad p_1+l \quad p_2$

Proof by "telescoping"

1.1.1. the WT identity as an ideal. Reference: Walter van Suijlekom, *The Hopf Algebra of Feynman Graphs in Quantum Electrodynamics* Letters in Mathematical Physics (2006) 77:265-281.

$$\frac{\partial}{\partial q_\mu} \Sigma(q) = \text{Ward id.} \quad \text{Diagram: } \text{circle with } q \text{ and } \mu \text{ labels}$$

There is a wonderful co-ideal in the Hopf alg. of QED graphs.

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{1} + \mathbb{1} \otimes \Gamma + \sum_{\gamma \neq \Gamma} \gamma \otimes \Gamma/\gamma$$

$\gamma = \cup \gamma_i$
 $\text{res}(\gamma_i) \in \{\text{cut, loop}\}$

where:

$\text{res}(\Gamma)$: Γ | all internal edges have length zero.

So:

$$\text{res}(\text{cut}) = \text{cut} \quad \text{res}(\text{loop}) = \text{loop}$$

$$\text{res}(\text{cut}) = \text{cut} \quad \text{res}(\text{loop}) = \text{loop}$$

$$\hat{\Delta}(\text{triangle}) = \text{triangle} \oplus \text{triangle}$$

$$\text{res}(\text{triangle}) = \text{triangle} \quad \text{res}(\text{triangle}) = \text{triangle}$$

So $H_{\text{QED}}(m, \mathbb{I}, \hat{\mathbb{I}}, \Delta, S)$

as usual. Now where is the co-ideal?

let c_n^i be the sum of all $1PI$ n -loop graphs with residue i , $i \in \{\text{triangle}, \rightarrow, \text{mushroom}\}$.

$$c_2^{\text{triangle}} = \text{triangle} + \text{triangle} + \text{triangle} + \text{triangle} + \text{triangle} + \text{triangle} + \text{triangle}$$

$$c_2^{\rightarrow} = \text{triangle} + \text{triangle} + \text{triangle}$$

$$id_n = \underbrace{c_n^{\text{triangle}} + c_n^{\rightarrow}}_{} + c_n^{\text{mushroom}}$$

Then: $\Delta(\text{id}_n) \subseteq \underbrace{H \otimes \mathbb{I} + \mathbb{I} \otimes H}_{\text{QED}}$

where $\mathbb{I} = \text{span}(\text{id}_n)$

$$\Delta(i_2) = \underbrace{i_2 \otimes \mathbb{I}} + \underbrace{\mathbb{I} \otimes i_2} + n \circledast \left(\underbrace{\text{diagram}_1 + \text{diagram}_2} \right)$$

$$+ 3 \text{diagram}_1 \otimes \text{diagram}_1 + 2 \text{diagram}_1 \otimes \text{diagram}_2 + 2 \text{diagram}_2 \otimes \text{diagram}_1 + \text{diagram}_2 \otimes \text{diagram}_2$$

$$= 2 \text{diagram}_1 \otimes (i_1) + \text{diagram}_2 \otimes i_1 + (\underbrace{\text{diagram}_1 + \text{diagram}_2}_{i_1}) \otimes \text{diagram}_1$$

$\hookrightarrow \Delta(i_2) \subseteq H \otimes \mathbb{I} + \mathbb{I} \otimes H$

By counting # of insertion places

one power it to all sides.

This makes for a nice
in representation of WT identities:

$$\text{Define } \phi_R(i_n) \equiv \phi_R(c_n^{\rightarrow}) - \phi_R(c_n^{\leftarrow}) - \phi_R(c_n^{\rightarrow}) (p_2) - \phi_R(c_n^{\leftarrow}) (p_1)$$

$$\Rightarrow \phi_R(\mathbb{I}) \equiv 0.$$

$$(\phi_R(i_n i_n) = \phi_R(i_n) \phi_R(i_n).$$

Why is this much more complicated
in QCD?

$$\left\{ ccc, ccc, ccc, ccc, ccc, \rightarrow, \rightarrow \right\}$$

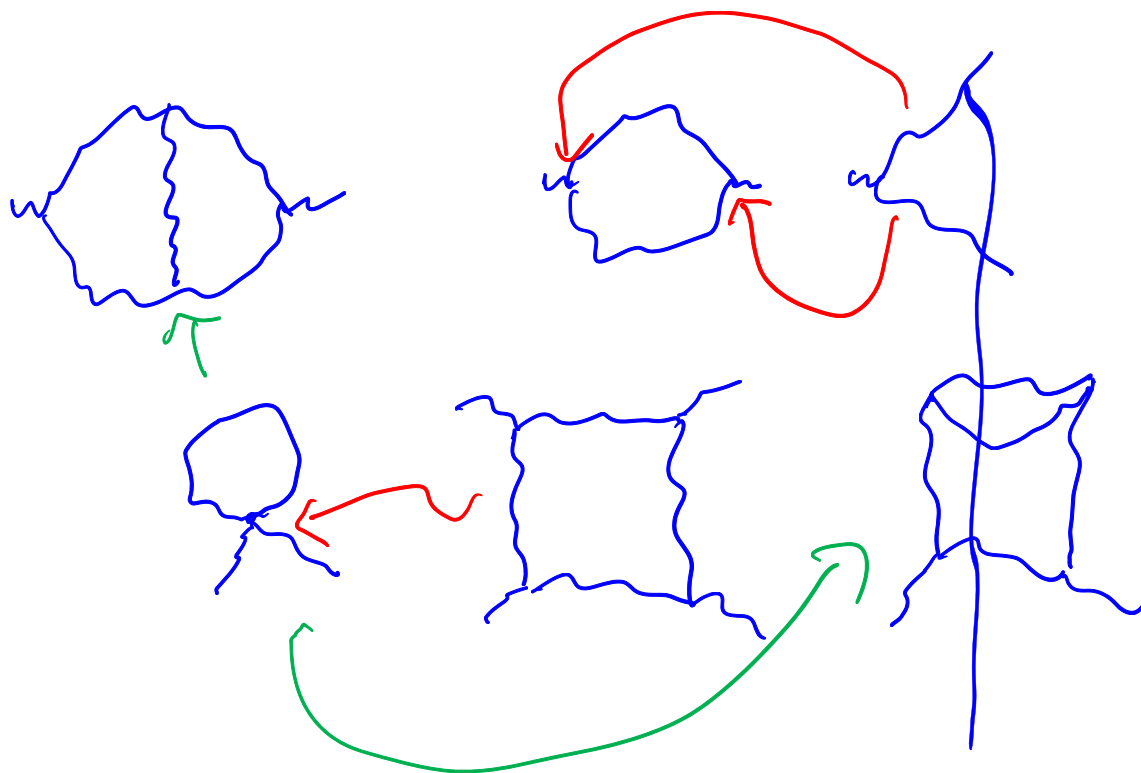
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7 residues, and four instead

of our ideal.

HUMBOLDT U. BERLIN

Have to study: $\#$ insertion
places, $\#$ bijections between
ext. leg. structure and internal
edges and vertices, filtration by
topology of a graph.



S-T derivable without using
gauge invariance of Lagrangian.
See Weiberg ⁹ Feynman rules,
for any sp_d