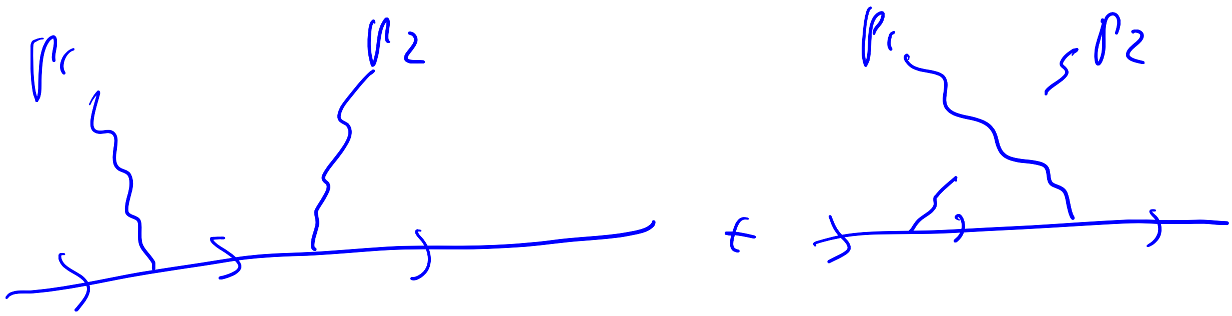
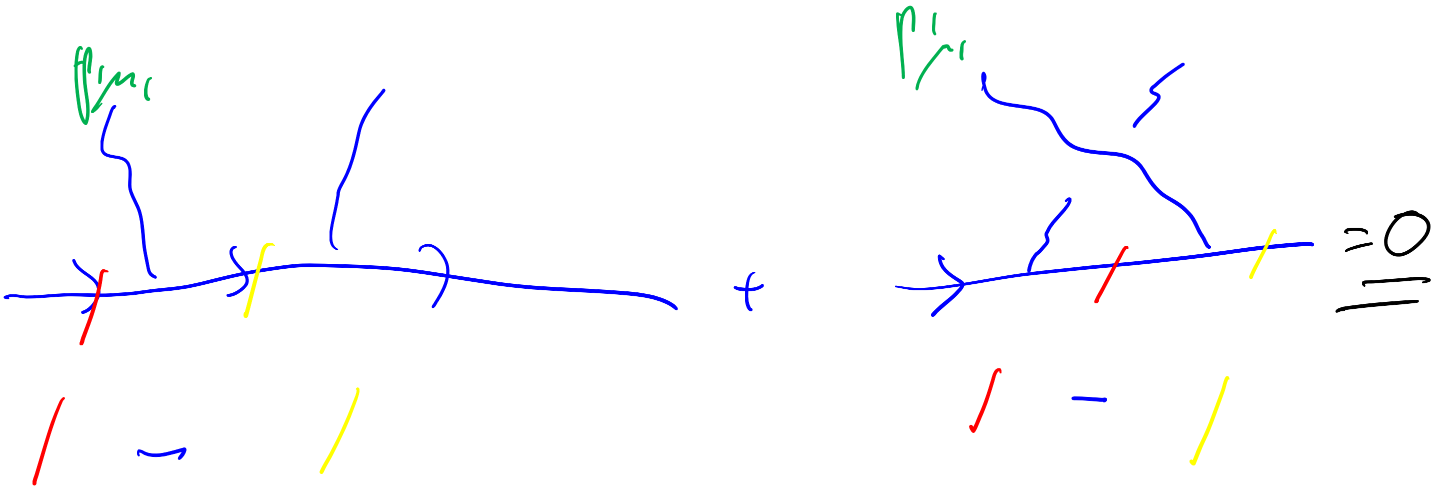
Dyson-Schwinger Eq's and Quantization of gauge theories (Summer '21)

Dirk Kreimer Lect. May 18

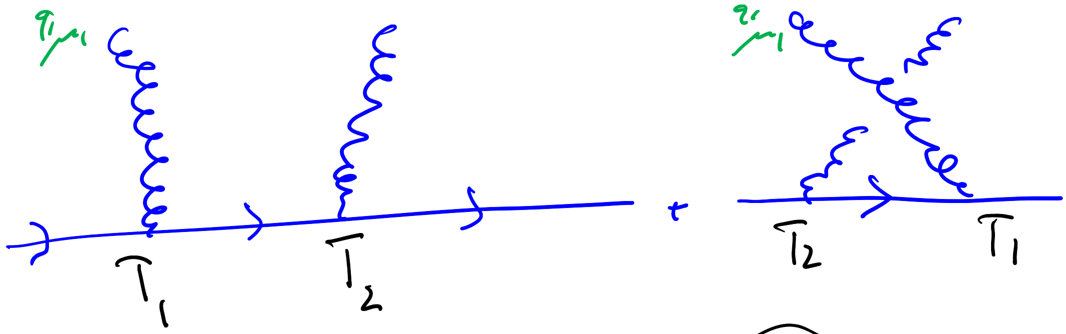
How to go from QED (abelian gauge theory) to QCD (non-abelian ...) ?



Contract a photon

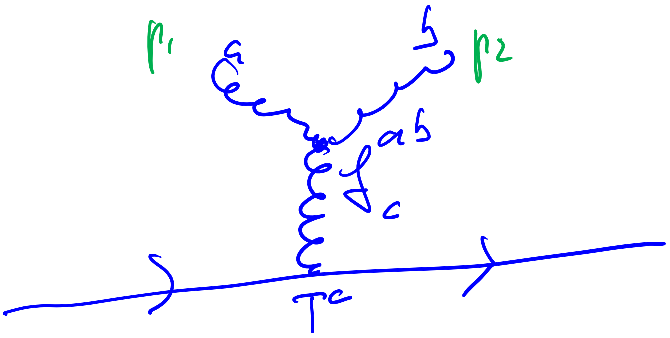


In QCD



$\neq 0$

$$\Rightarrow \quad \sim [T_1, T_2] = f^{ab}_{\ c}\, T^c$$



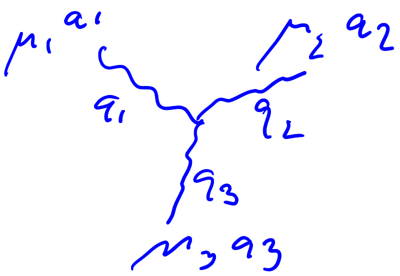
repairs transversality.

So we need a 3 gluon vertex.

Locality again gives you 3-gluon vertex.

3-gluon vertex: $\quad q_1 + q_2 + q_3 = 0$

$$X^{\mu_1 \mu_2 \mu_3}_{a_1 a_2 a_3}(q_1, q_2, q_3) =$$



$$\left\{(q_1-q_2)_{\mu_3}\, g_{\mu_1\mu_2} + (q_2-q_3)_{\mu_1}\, g_{\mu_2\mu_3} + (q_3-q_1)_{\mu_2}\, g_{\mu_3\mu_1}\right\} f^{a_1a_2a_3}$$

Bose symmetric expression

$$q_3 = -q_1 - q_2$$

$$q_3^{\mu_3}\, X^{\mu_1\mu_2\mu_3}_{a_1a_2a_3}\,(\dots) =$$

$$\Big\{ \underline{-(q_1+q_2)\cdot(q_1-q_2)\, g_{\mu_1\mu_2}}$$

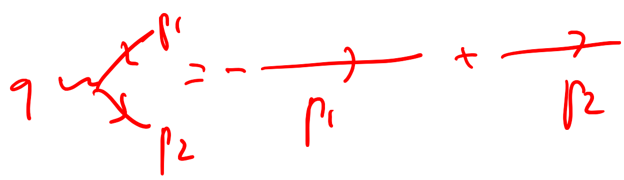
$$-(q_2-q_3)_{\mu_1}(q_1+q_2)_{\mu_2} - (q_3-q_1)_{\mu_2}(q_1+q_2)_{\mu_1}\Big\} f^{a_1a_2a_3}$$

$$= \Big\{(q_2^2-q_1^2)\, g_{\mu_1\mu_2} - (2q_2+q_1)_{\mu_1}(q_1+q_2)_{\mu_2}$$

$$+ (2q_1+q_2)_{\mu_2}(q_1+q_2)_{\mu_1}\Big\} f^{a_1a_2a_3}$$

$$= \Big\{ \overbrace{\{q_2^2\, g_{\mu_1\mu_2} - q_{2\mu_1} q_{2\mu_2}\}}^{g^T_{\mu_1\mu_2}(q_2)} - \overbrace{\{q_1^2\, g_{\mu_1\mu_2} - q_{1\mu_1} q_{1\mu_2}\}}^{g^T_{\mu_1\mu_2}(q_1)} \Big\}$$

$$\times f^{a_1a_2a_3}$$

QED: 

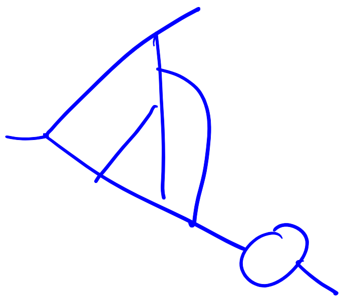
i) graph homology

ii) cycle homology

Start with a theory of scalar 3-regular graphs.

Every vertex is 3-valent.

Take connected graphs.



Study graph homology.

Graph homology:

give an orientation to a graph.

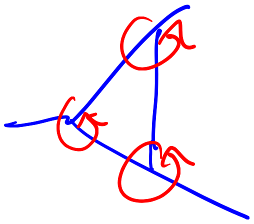
Equivalent to chosing an orientation for each corolla.

Find a Riemann surface of minimal genus on which you can draw your graph without self-intersections.

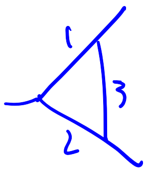
(planar graphs on S^1), else ...

Orient your Riemann surface

$\Rightarrow$ orientation of your graph.

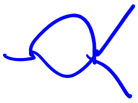
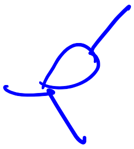
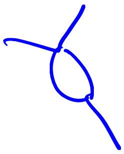


$\Rightarrow$ labeled oriented graph upon labeling the corollas.



$$S(\Gamma) = \sum_{e \in \text{internal edge}} (-1)^{e} (\Gamma / e)$$

shrink e

$-$  $+$  $-$ 

If you do everything right,

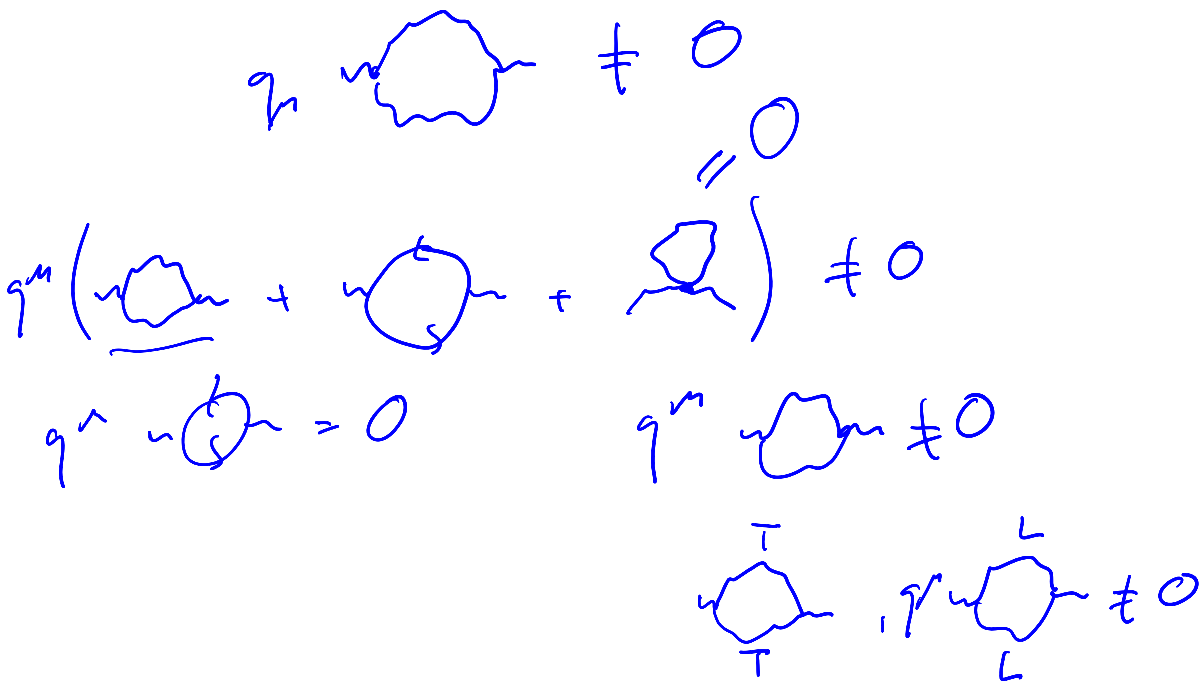
$$S \circ S = 0.$$

So S is a boundary operator for "graph homology"

Look at the Feynman rule for a 4-gluon vertex:

They come from applying edge contraction to 3-regular graphs.

But: longitudinal gluons can be in internal loops.



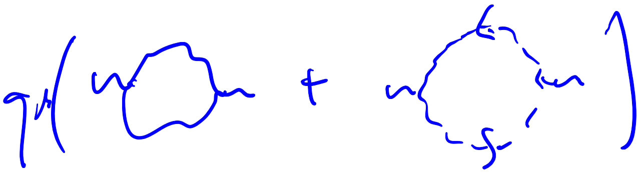
Need a new particle:

i) must be a boson because it must annihilate longitudinal gauge bosons.

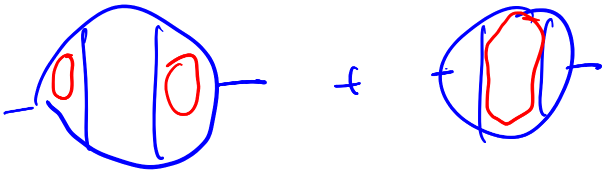
ii) must be a fermion, because we need (-1) for each closed long. boson loop.

iii) Wigner: any particle is either boson or fermion.

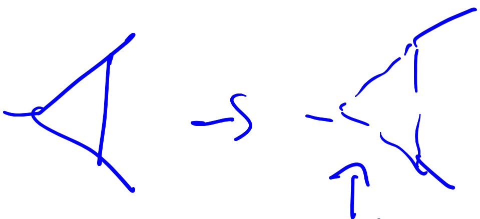
$\Rightarrow$ therefore this thing cannot exist in reality, so it is a ghost.

gr  $= 0$ Hooray!

Look at 3-regular graphs and all disjoint loops in them:



label your loops and define an operator

δ : 

ghost loop.

$\delta \circ \delta = 0$ and this gives cycle homology.

graph homology + cycle homology give you BRST homology: b, $\underline{b \circ b = 0}$

So gauge invariance:

$$b(\text{physical amplitude}) = 0$$

physical amplitude are BRST closed but not exact

is equivalent to
physical amplitudes are
closed but not exact under
$(s+\delta)$, with $s\circ s=0$, $\delta\circ\delta=0$

$\Leftarrow)$ $(s+\delta)\circ(s+\delta)=0$, $\{s,\delta\}=0$

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i) Start with 3-regular scalar
graphs.

ii) generate graphs with 3-gluon
vertex by a corolla polynomial

iii) generate all graphs from
an application of s, δ

and get the full YM or gauge-theory
amplitudes from this set-up.

Corolla polynomial:

Feynman rules come from
1st, 2nd Symanzik polynomial

$\exists$ a corolla polynomial C_G based on half edges such that

the transition from scalar field

th to gauge th comes from

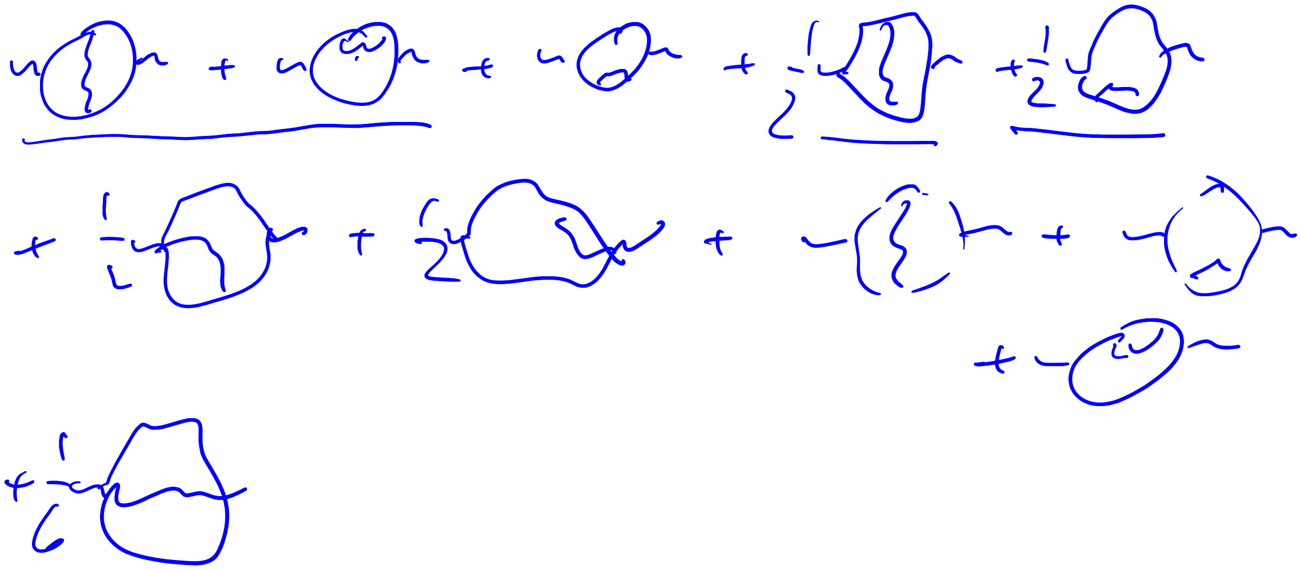
the application of $\hat{G}$ to scalar

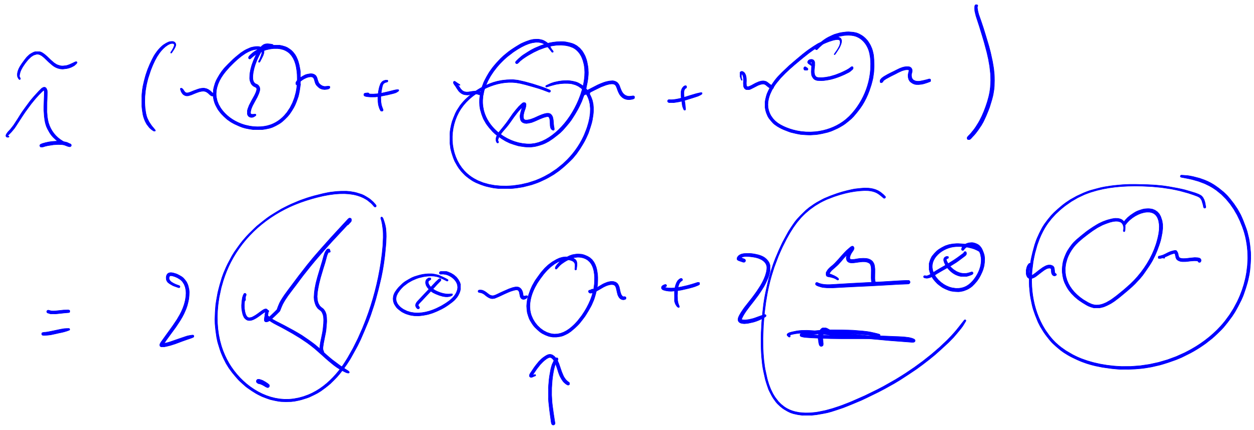
graphs.

gives complete results for

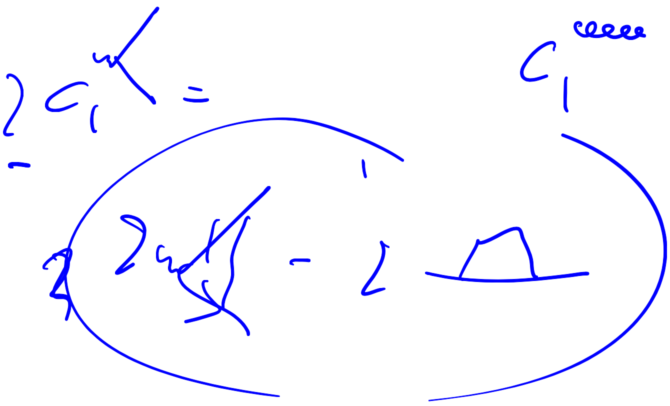
3-, 4- valent gauge vertices, for

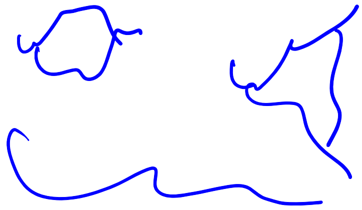
ghosts, . . .





$2 c_1 \,\text{—}\, =$ $\qquad c_1$





$2 c_1^{\text{coup}}$

$\|$