1. The Hopf algebra of rooted trees

We follow Loïc Foissy *An introduction to Hopf algebras of trees* (see link on course homepage).

2. General remarks on Hopf algebras and co-actions

We collect some material on Hopf algebras and co-actions. For simplicity, we only discuss vector spaces instead of modules, and we consider all vector spaces to be defined over $\mathbb{Q}$.

A coalgebra is a vector space $H$ together with a coproduct

$$\Delta : H \rightarrow H \otimes H,$$

that is coassociative,

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta \cdot$$

and is equipped with a counit, i.e., a map $\hat{1} : H \rightarrow \mathbb{Q}$ such that

$$(\hat{1} \otimes \text{id})\Delta = (\text{id} \otimes \hat{1})\Delta = \text{id}.$$
A commutative Hopf algebra is a commutative algebra (with product \( \cdot \)) that is at the same time a coalgebra (not necessarily co-commutative) such that the product and coproduct are compatible,

\[ \Delta(a \cdot b) = \Delta(a) \cdot \Delta(b), \]

and it is equipped with an antipode

\[ S : H \to H \]

such that

\[ S(a \cdot b) = S(b) \cdot S(a) = S(a) \cdot S(b), \]

and

\[ m(S \otimes \text{id})\Delta = m(\text{id} \otimes S)\Delta = \hat{I} \circ I, \]

where \( m \) denotes the multiplication in \( H \) and \( \hat{I} : \mathbb{Q} \to H \) is the unit (map), \( \hat{I}(1) = I \) is the unit in \( H \).

\[ S \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) = \]

\[ - \left( a \cdot l \right) + l \cdot a + 2 \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right) \]
A (left-)comodule over a coalgebra $H$ is a vector space $M$ together with a map (co-action) $$\rho : M \rightarrow H \otimes M$$ such that $$(\text{id} \otimes \rho)\rho = (\Delta \otimes \text{id})\rho, M \rightarrow H \otimes H \otimes M,$$ and $$(\mathbb{f} \otimes \text{id})\rho = \text{id}.$$ Our Hopf algebras are commutative and graded, $H = \bigoplus_{j=0}^{\infty} H^{(j)}$ and connected $H^{(0)} \sim \mathbb{Q}$, the $H^{(j)}$ are finite-dimensional $\mathbb{Q}$-vectorspaces.

The vectorspace $H_C$ of Cutkosky graphs forms a left comodule over the core Hopf algebra $H_{\text{core}}$. 

\[ \rho = \Delta \] 
\[ \tilde{\rho} \left( \begin{array}{c} \vdots \\ \ast \\ \vdots \end{array} \right) = \begin{array}{c} \vdots \\ \ast \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \ast \\ \vdots \end{array} 
\]
\[ \tilde{\rho} \left( \begin{array}{c} \vdots \\ \ast \\ \vdots \end{array} \right) = 0 \] 
\[ \tilde{\rho} \left( \begin{array}{c} \vdots \\ \ast \\ \vdots \end{array} \right) = \mathbb{I} \otimes \begin{array}{c} \vdots \\ \ast \\ \vdots \end{array} \]
3. The vector space $H_C$

Consider a Cutkosky graph $G$ with a corresponding $v_G$-refinement $P$ of its set of external edges $L_G$. It is a maximal refinement of $V_G$.

The core Hopf algebra co-acts on the vector-space of Cutkosky graphs $H_C$.

\[ \Delta_{\text{core}} : H_C \rightarrow H_{\text{core}} \otimes H_C. \]

We say $G \in H_C^{(n)} \iff |G| = n$ and define $\text{Aut}_C = \bigoplus_{i=1}^{\infty} H_C^{(i)}$. 

\[ S = \Delta_{\text{core}} \]

\[ \bigotimes_{G \in H_{\text{core}}} (\alpha, \beta, \gamma, \delta) \]
Note that the sub-vectorspace $H_C^{(0)}$ is rather large: it contains all graphs $G = ((H_G, V_G, E_G), (H_H, V_H, E_H))$ such that $\|G\| = 0$. These are the graphs where the cuts leave no loop intact.

For any $G \in H_C$ there exists a largest integer $\text{cor}_C(G) \geq 0$ such that \[
\tilde{\Delta}_{\text{core}}^\text{cor}_C(G) \neq 0, \quad \tilde{\Delta}_{\text{core}}^\text{cor}_C(G) : H_C \rightarrow H_{\text{core}}^{\text{cor}_C(G)} \otimes H_G^{(0)},
\] whilst $\tilde{\Delta}_{\text{core}}^{\text{cor}_C(G)+1}(G) = 0$. 

\[
\Delta_{\text{core}}((i \cdot \alpha \otimes i \cdot \beta) \otimes \xi) = \xi \otimes \Delta_{\text{core}}((i \cdot \alpha \otimes i \cdot \beta)) = \xi \otimes (\Delta_{\text{core}}((i \cdot \alpha \otimes i \cdot \beta)) \otimes (P(\Pi) = 0 \quad \tilde{F} = (P \otimes \text{id}) S
\]

$P_A \cup c = A \cup c$
Proposition 3.1.

\[ \text{cor}_C(G) = ||G||. \]

Proof. The primitives of \( H_{\text{core}} \) are one-loop graphs.

In particular there is a unique element \( g \otimes G/g \in H_{\text{core}} \otimes H_C^{(0)} \):

\[ \Delta_{\text{core}}(G) \cap \left( H_{\text{core}} \otimes H_C^{(0)} \right) = g \otimes G/g, \]

with \( |g| = ||G|| \).

For any graph \( G \) we let \( G = \sum_{T \in \mathcal{T}(G)} (G, T) \). Here \( \mathcal{T}(G) \) is the set of all spanning trees of \( G \) and we set for \( G = \bigcup_i G_i \), \( \mathcal{T}(G) = \bigcup_i \mathcal{T}(G_i) \).

The maximal refinement \( P \) induces for each partition \( P(i), 0 \leq i \leq v_G \) a unique spanning forest \( f_i \) of \( G/g \). The set \( \mathcal{F}_{G,P(i)} \) of spanning forests of \( G \) compatible with \( P(i) \) is then determined by \( f_i \) and the spanning trees in \( \mathcal{T}_g \).

Define \( G_i := \sum_{F \in \mathcal{F}_{G,P(i)}} (G, F) \).

(3.2) \[ \tilde{\Delta}_{G,F} G_i = \sum_{i=1}^{||G||} G_i^{||G||+1} \otimes \cdots \otimes G_i^{1}. \]

Note that \( |G_i^k| = 1, \forall k \leq (||G|| + 1) \) and \( |G_i^{||G||+1}| = 0. \)
3.1. The pre-Lie product and the cubical chain complex. So consider the pair \((G, F)\) of a pre-Cutkosky graph with compatible forest \(F\) with ordered edges. Assume there are graphs \(G_1, G_2\) and forests \(F_1, F_2\) such that

\[
(G, F) = (G_1, F_1) \star (G_2, F_2).
\]

Here, \(\star\) is the pre-Lie product which is induced by the co-product \(\Delta_{GF}\) by the Milnor–Moore theorem.

**Theorem 3.2.** We can reduce the computation of the homology of the cubical chain complex for large graphs to computations for smaller graphs by a Leibniz rule:

\[
d((G_1, F_1) \star (G_2, F_2)) = (d(G_1, F_1)) \star (G_2, F_2) + (-1)^{|E_{F_1}|}(G_1, F_1) \star (d(G_2, F_2)).
\]

Here, \(d = d_0 + d_1\) is the boundary operator which either shrinks edges or cuts a graph.
4. Flags

4.1. Bamboo. The notion of flags of Feynman graphs was for example already used in [?, ?].

4.2. Flags of necklaces. Here we use it based on the core Hopf algebra introduced above. We introduce Sweedler’s notation for the reduced co-product in $H_{core}$:

$$\tilde{\Delta}_{core}(G) := \Delta_{core}(G) - \mathbb{I} \otimes G - G \otimes \mathbb{I} =: \sum' G' \otimes G''.$$ 

We define a flag $f \in \text{Aug}_{core} \otimes^{k}$ of length $k$ to be an element of the form

$$f = \gamma_{1} \otimes \cdots \otimes \gamma_{k},$$

where the $\gamma_{i} \in \text{Aug}_{core} \cap \langle H_{core} \rangle$ fulfill $\tilde{\Delta}_{core}(\gamma_{i}) = 0$, $|\gamma_{i}| = 1$. 
Here, $\langle H_{\text{core}} \rangle = \{ G \in H_{\text{core}} \mid |H^0(G)| = 1 \}$ is the linear $\mathbb{Q}$-span of bridge-free connected graphs as generators.

Note that for elements $G \in \langle H_{\text{core}} \rangle$, we have $\tilde{\Delta}_{\text{core}}^{[G]-1}(G) \neq 0$.

We have $\tilde{\Delta}_{\text{core}} := (P \otimes P)\Delta_{\text{core}}$ for $P : H_{\text{core}} \to \text{Aug}_{\text{core}}$ the projection into the augmentation ideal $\text{Aug}_{\text{core}}$. 
Define the flag associated to a graph $G \in \langle H_{\text{core}} \rangle$ to be a sum of flags of length $|G|$ where in each flag each element $\gamma_i$ has unit degree, $|\gamma_i| = 1$:

$$F \ell_G := \tilde{\Delta}_{\text{core}}^{-1}(G) \in \text{Aug}_{\text{core}}^{\otimes|G|}.$$
Similarly, for a pair \((G, F)\) we can define

\[
F l_{G,F} := \bar{\Delta}_{GF}^{-1} \left((G, F)\right) \in \text{Aug}_{G,F}^{\otimes |G|},
\]

as a sum of flags

\[
F l_{G,F} = \sum_i (\gamma_1, f_1)^i \otimes \cdots \otimes (\gamma_{|G|}, f_{|G|})^i,
\]

\[
\bar{\Delta}_{GF}((\gamma_l, f_l)^i) = 0, \forall i, l, 1 \leq l \leq |G|.
\]