FEYNMAN DIAGRAMS AND THE S-MATRIX, AND OUTER SPACE (SUMMER 2020)

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1. The Hopf algebra of rooted trees

We follow Loïc Foissy *An introduction to Hopf algebras of trees* (see link on course homepage).

2. General remarks on Hopf algebras and co-actions

We collect some material on Hopf algebras and co-actions. For simplicity, we only discuss vector spaces instead of modules, and we consider all vector spaces to be defined over $\mathbb{Q}$.

A coalgebra is a vector space H together with a coproduct

$$\Delta : H \to H \otimes H,$$

that is coassociative,

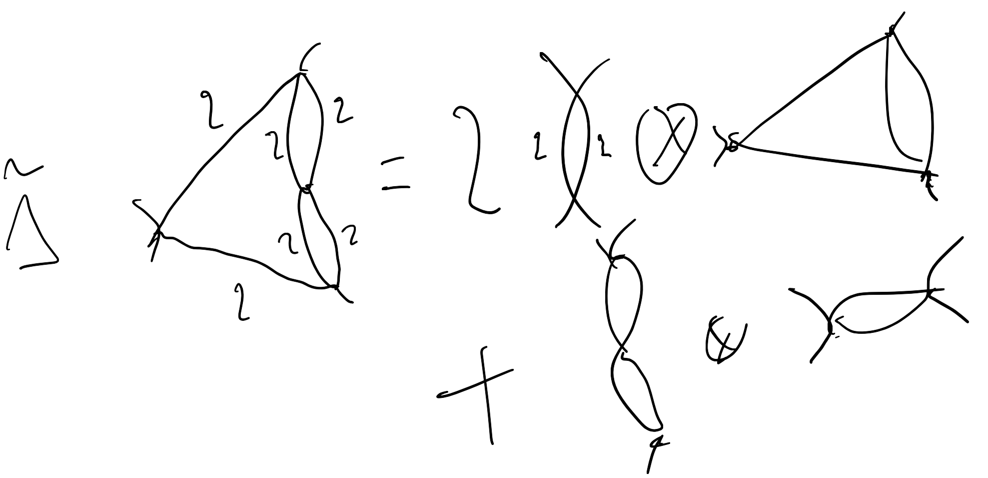
$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta.$$

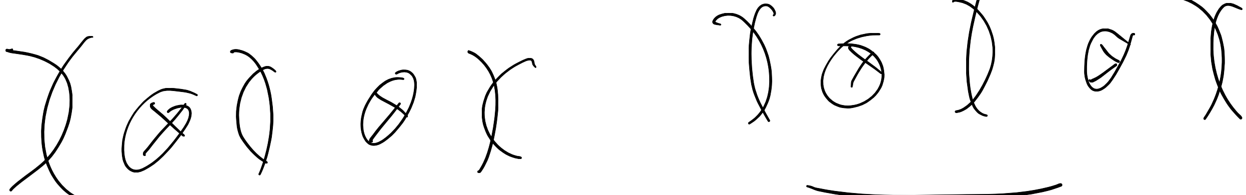
and is equipped with a counit, i.e., a map $\hat{\mathbb{I}} : H \to \mathbb{Q}$ such that

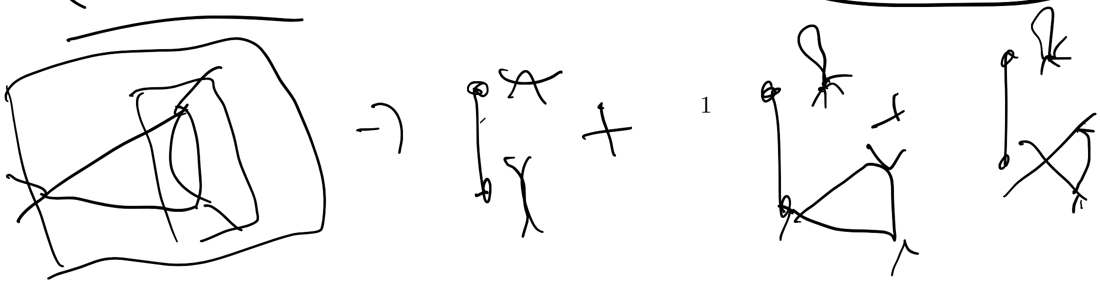
$$(\hat{\mathbb{I}} \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \hat{\mathbb{I}})\Delta = \mathrm{id}.$$

$n4 - \sum_c 2 = \omega$

$\omega \doteq 0$







A commutative Hopf algebra is a commutative algebra (with product $\cdot$) that is at the same time a coalgebra (not necessarily co-commutative) such that the product and coproduct are compatible,

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b),$$

$(a \otimes b) \cdot (c \otimes d)$

$= (a \cdot c) \otimes (b \cdot d)$

and it is equipped with an antipode

$$S : H \to H$$

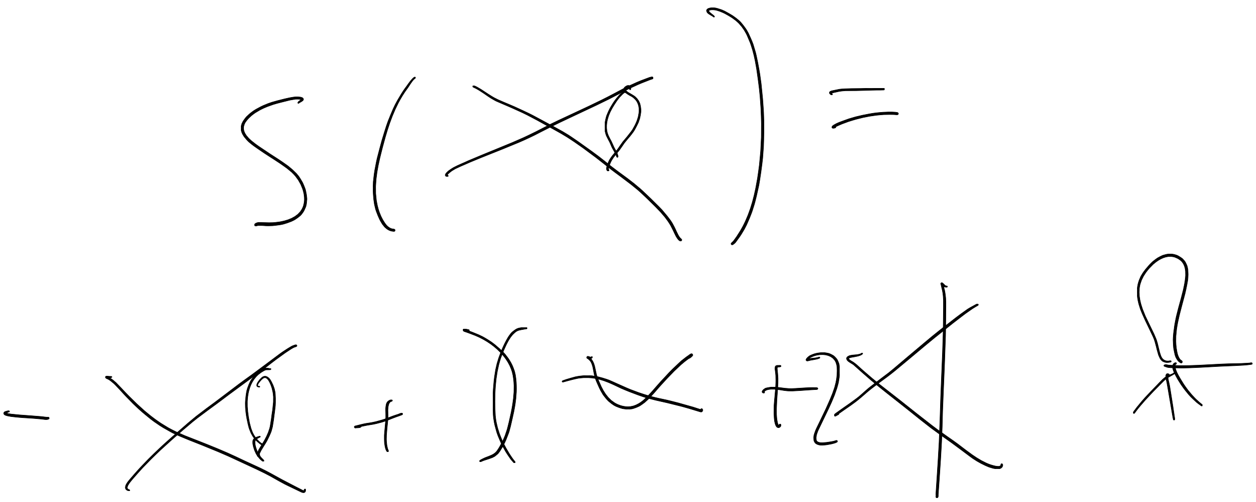
such that

$$S(a \cdot b) = S(b) \cdot S(a) = S(a) \cdot S(b),$$

and

$$m(S \otimes \mathrm{id})\Delta = m(\mathrm{id} \otimes S)\Delta = \hat{\mathbb{I}} \circ \mathbb{I},$$

where m denotes the multiplication in H and $\mathbb{I} : \mathbb{Q} \to H$ is the unit (map), $\mathbb{I}(1) = \mathbb{I}$ is the unit in H.



A (left-)comodule over a coalgebra H is a vector space M together with a map (co-action)

$$\rho : M \to H \otimes M$$

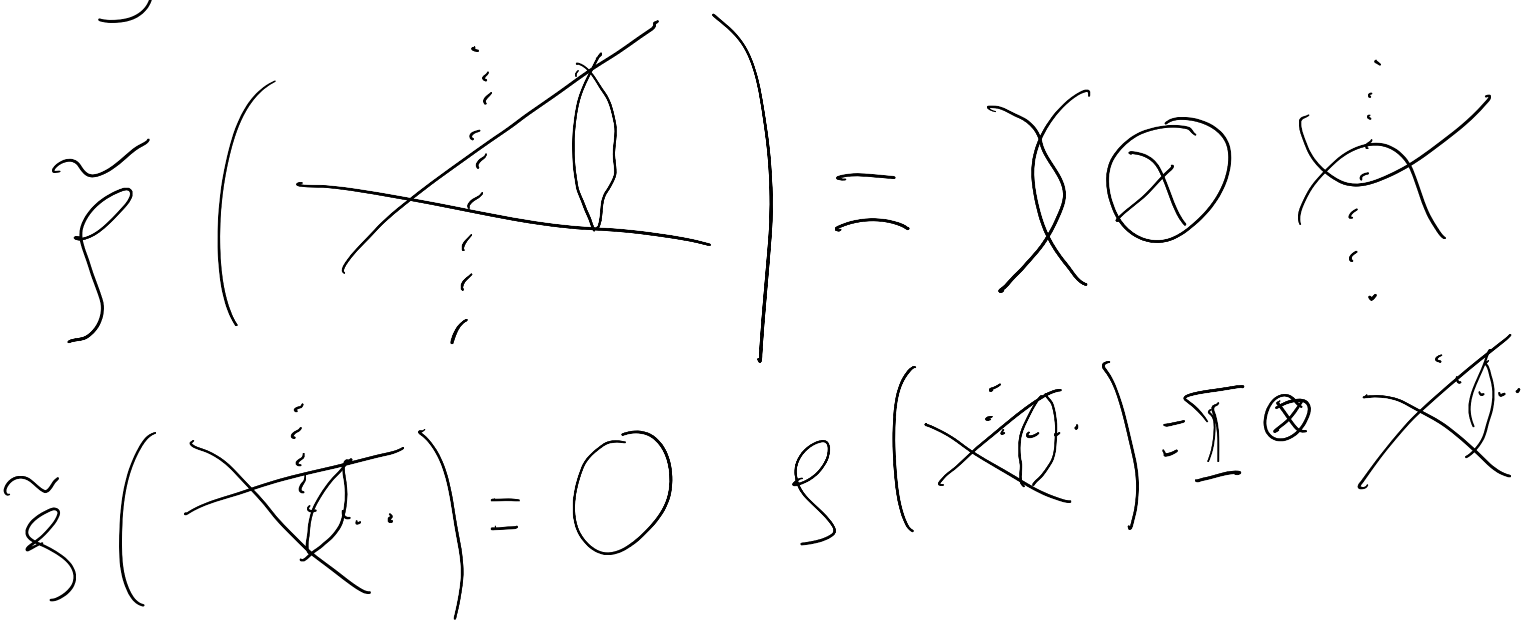
such that

$$(\mathrm{id} \otimes \rho)\rho = (\Delta \otimes \mathrm{id})\rho, M \to H \otimes H \otimes M,$$

and $(\hat{\mathbb{I}} \otimes \mathrm{id})\rho = \mathrm{id}$. Our Hopf algebras are commutative and graded, $H = \oplus_{j=0}^{\infty} H^{(j)}$ and connected $H^{(0)} \sim \mathbb{Q}\mathbb{I}$, the $H^{(j)}$ are finite-dimensional $\mathbb{Q}$-vectorspaces.

The vectorspace H_C of Cutkosky graphs forms a left comodule over the core Hopf algebra H_{core}.

$\rho \equiv \Delta_{core}$



3. THE VECTORSPACE H_C

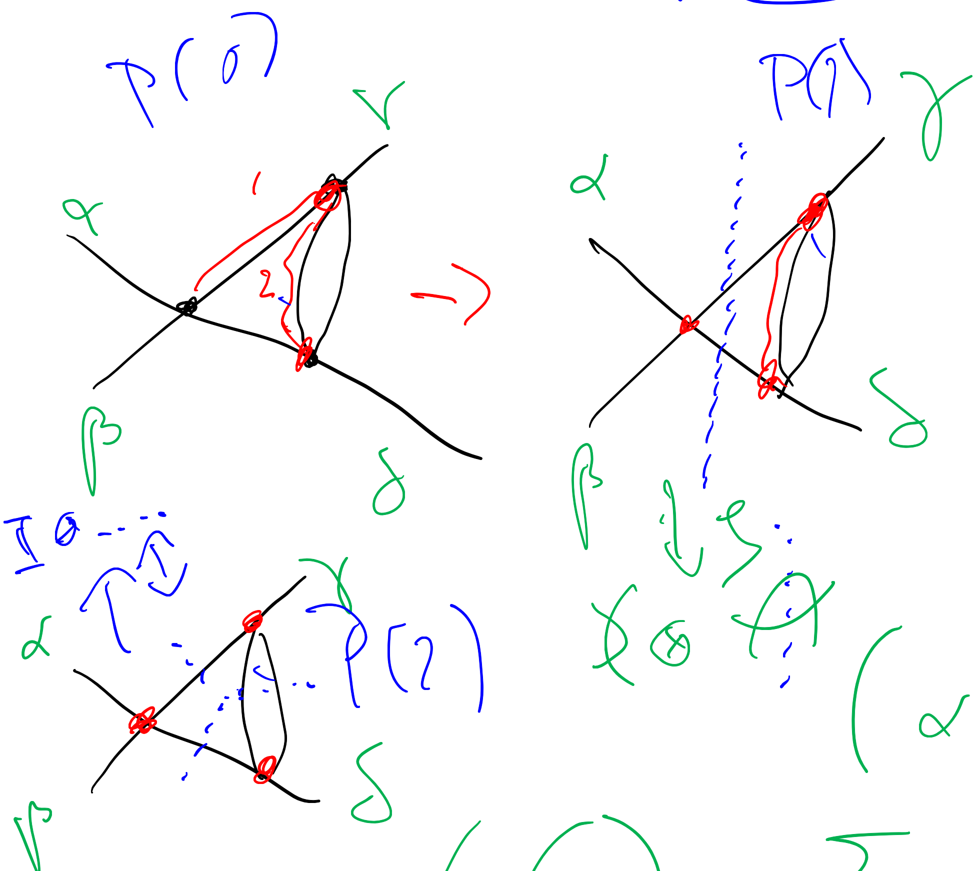
Consider a Cutkosky graph G with a corresponding v_G-refinement P of its set of external edges L_G. It is a maximal refinement of V_G.

The core Hopf algebra co-acts on the vector-space of Cutkosky graphs H_C.

$$\Delta_{core} : H_C \to H_{core} \otimes H_C. \tag{3.1}$$

We say $G \in H_C^{(n)} \Leftrightarrow |G| = n$ and define $\mathrm{Aut}_C = \oplus_{i=1}^{\infty} H_C^{(i)}$.

$H_C = H_C^{ra} \oplus \mathrm{Aut}_C$



$(\alpha, \beta, \gamma, \delta)$

$\downarrow$

$(\alpha, \beta), (\gamma, \delta)$

$\downarrow$

$(\alpha\beta), (\gamma), (\delta)$

$S = \Delta_{core}$

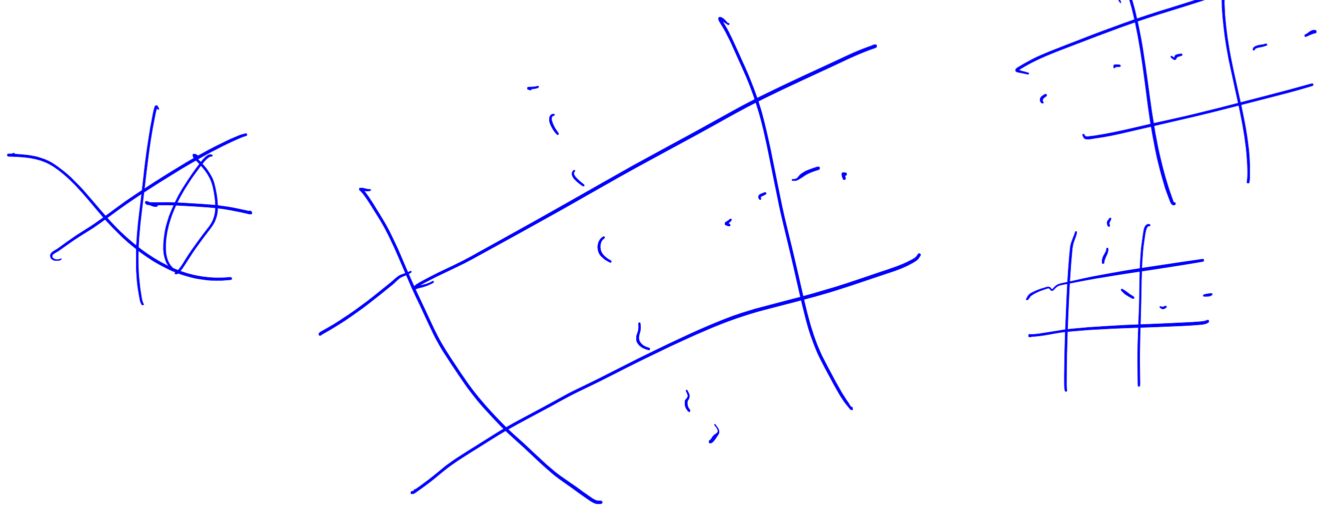
$\Delta(G) = \sum_{\substack{\gamma \subsetneq G \\ \in H_{core}}} \gamma \otimes G/\gamma$

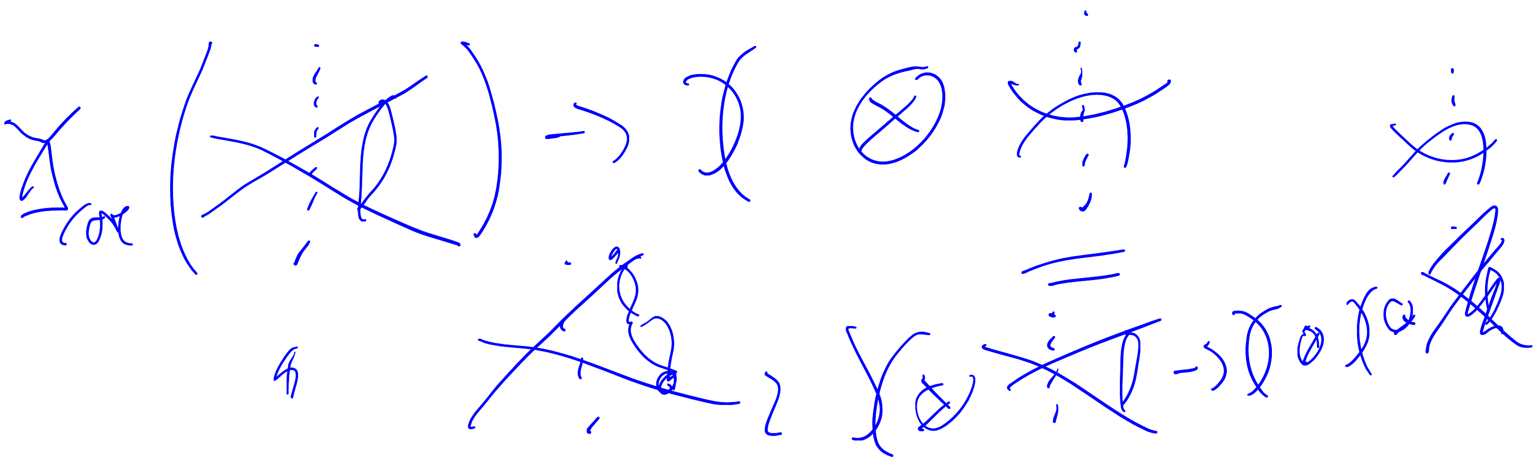
Note that the sub-vectorspace $H_C^{(0)}$ is rather large: it contains all graphs $G = ((H_G, \mathcal{V}_G, \mathcal{E}_G), (H_G, \mathcal{V}_G, \mathcal{E}_H)) \in H_C$ such that $||G|| = 0$. These are the graphs where the cuts leave no loop intact.

For any $G \in H_C$ there exists a largest integer $\mathrm{cor}_C(G) \geq 0$ such that

$$\tilde{\Delta}_{core}^{\mathrm{cor}_C(G)}(G) \neq 0,\ \tilde{\Delta}_{core}^{\mathrm{cor}_C(G)}(G) : H_C \to H_{core}^{\otimes corc_C(G)} \otimes H_C^{(0)},$$

whilst $\tilde{\Delta}_{core}^{\mathrm{cor}_C(G)+1}(G) = 0$.



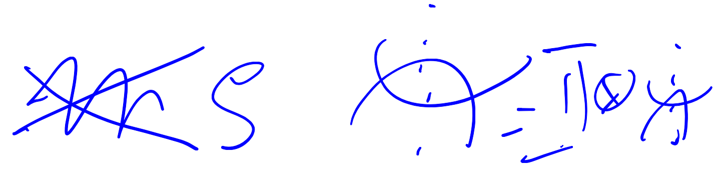


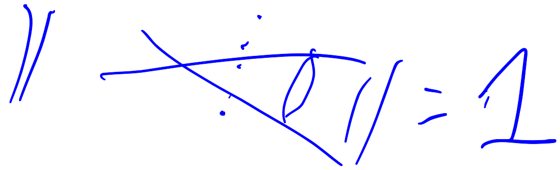
$(\mathrm{id} \otimes \mathrm{id} \otimes \tilde{S})\, S \quad (\mathrm{id} \otimes \tilde{S})\, S \quad S$

$P(\mathbb{I}) = 0$

$\tilde{S} = (P \otimes \mathrm{id}) S$

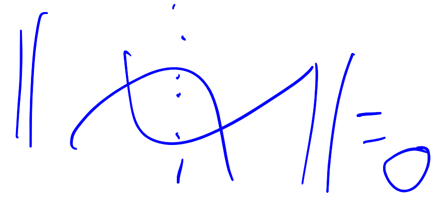
$P\, \mathrm{Aug}_C = \mathrm{Aug}_C$





Proposition 3.1.

$$\mathrm{cor}_C(G) = ||G||.$$



Proof. The primitives of H_{core} are one-loop graphs. □

In particular there is a unique element $g \otimes G/g \in H_{core} \otimes H_C^{(0)}$:

$$\Delta_{core}(G) \cap \left(H_{core} \otimes H_C^{(0)}\right) = g \otimes G/g,$$

with $|g| = ||G||$.

For any graph G we let $\mathbf{G} = \sum_{T \in \mathcal{T}_G} (G, T)$. Here $\mathcal{T}_G$ is the set of all spanning trees of G and we set for $G = \dot{\cup}_i G_i$, $\mathcal{T}_G = \dot{\cup}_i \mathcal{T}_{G_i}$.

The maximal refinement P induces for each partition $P(i), 0 \le i \le v_G$ a unique spanning forest f_i of G/g. The set $\mathcal{F}_{G,P(i)}$ of spanning forests of G compatible with $P(i)$ is then determined by f_i and the spanning trees in $\mathcal{T}_g$.

Define $\mathbf{G}_i := \sum_{F \in \mathcal{F}_{G,P(i)}} (G, F)$.

$$\tilde{\Delta}^{||G||}_{G,F} \mathbf{G}_i = \sum_{i=1} \mathbf{G}_i^{(1)} \otimes \cdots \otimes \mathbf{G}_i^{(||G||+1)}. \tag{3.2}$$

Note that $|\mathbf{G}_i^k| = 1$, $\forall k \lneq (||G|| + 1)$ and $|\mathbf{G}_i^{||G||+1}| = 0$.

3.1. **The pre-Lie product and the cubical chain complex.** So consider the pair (G, F) of a pre-Cutkosky graph with compatible forest F with ordered edges. Assume there are graphs G_1, G_2 and forests F_1, F_2 such that

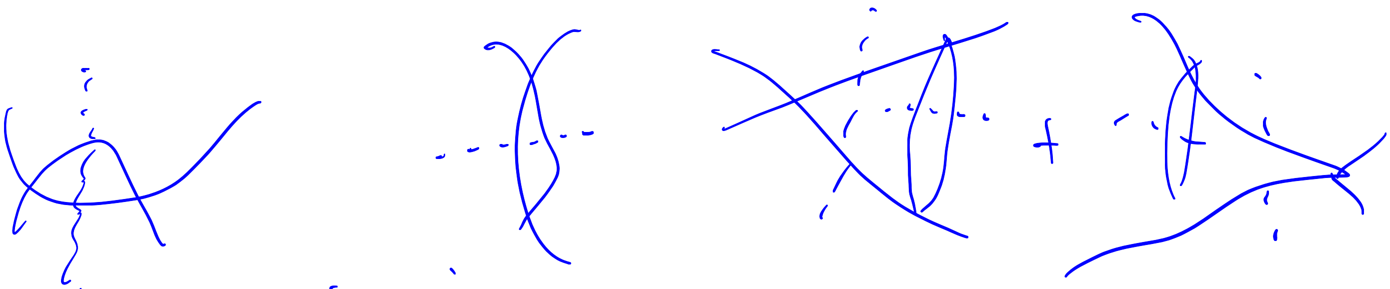
$$(G, F) = (G_1, F_1) \star (G_2, F_2).$$

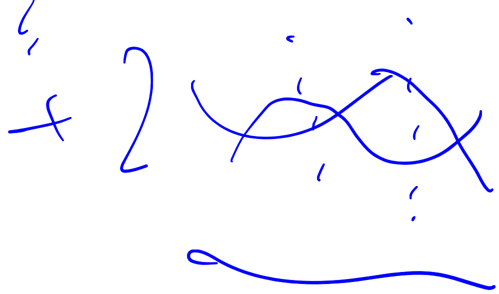
Here, $\star$ is the pre-Lie product which is induced by the co-product Δ_{GF} by the Milnor–Moore theorem.

Theorem 3.2. ~~[?]~~ *We can reduce the computation of the homology of the cubical chain complex for large graphs to computations for smaller graphs by a Leibniz rule:*

$$d\left((G_1, F_1) \star (G_2, F_2)\right) = (d(G_1, F_1)) \star (G_2, F_2) + (-1)^{|E_{F_1}|}(G_1, F_1) \star (d(G_2, F_2)).$$

Here, $d = d_0 + d_1$ is the boundary operator which either shrinks edges or cuts a graph.





$$d \circ d = 0$$

4. Flags

4.1. **Bamboo.** The notion of flags of Feynman graphs was for example already used in [**?**, **?**].

4.2. **Flags of necklaces.** Here we use it based on the core Hopf algebra introduced above. We introduce Sweedler's notation for the reduced co-product in H_{core}:

$$\tilde{\Delta}_{core}(G) := \Delta_{core}(G) - \mathbb{I} \otimes G - G \otimes \mathbb{I} =: \sum{}' G' \otimes G''.$$

We define a flag $f \in \mathrm{Aug}_{core}^{\otimes k}$ of length k to be an element of the form

$$f = \gamma_1 \otimes \cdots \otimes \gamma_k,$$

where the $\gamma_i \in \mathrm{Aug}_{core} \cap \langle H_{core} \rangle$ fulfill $\tilde{\Delta}_{core}(\gamma_i) = 0$, $|\gamma_i| = 1$.

Here, $\langle H_{core}\rangle = \{G \in H_{core} | \, |H^0(G)| = 1\}$ is the linear $\mathbb{Q}$-span of bridge-free connected graphs as generators.

Note that for elements $G \in \langle H_{core}\rangle$, we have $\tilde{\Delta}_{core}^{|G|-1}(G) \neq 0$.

We have $\tilde{\Delta}_{core} := (P \otimes P)\Delta_{core}$ for $P : H_{core} \to \mathrm{Aug}_{core}$ the projection into the augmentation ideal Aug_{core}.

Define the flag associated to a graph $G \in \langle H_{core} \rangle$ to be a sum of flags of length $|G|$ where in each flag each element γ_i has unit degree, $|\gamma_i| = 1$:

$$Fl_G := \tilde{\Delta}_{core}^{|G|-1}(G) \in \mathrm{Aug}_{core}^{\otimes |G|}.$$

Similarly, for a pair (G, F) we can define

$$Fl_{G,F} := \tilde{\Delta}_{GF}^{|G|-1}((G, F)) \in \mathrm{Aug}_{GF}^{\otimes |G|},$$

as a sum of flags

$$Fl_{G,F} = \sum_i (\gamma_1, f_1)^i \otimes \cdots (\gamma_{|G|}, f_{|G|})^i,$$

$\tilde{\Delta}_{GF}((\gamma_l, f_l)^i) = 0$, $\forall i, l$, $1 \leq l \leq |G|$.

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