FEYNMAN DIAGRAMS AND THE S-MATRIX, AND OUTER SPACE (SUMMER 2020)

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1. Outer Space

1.

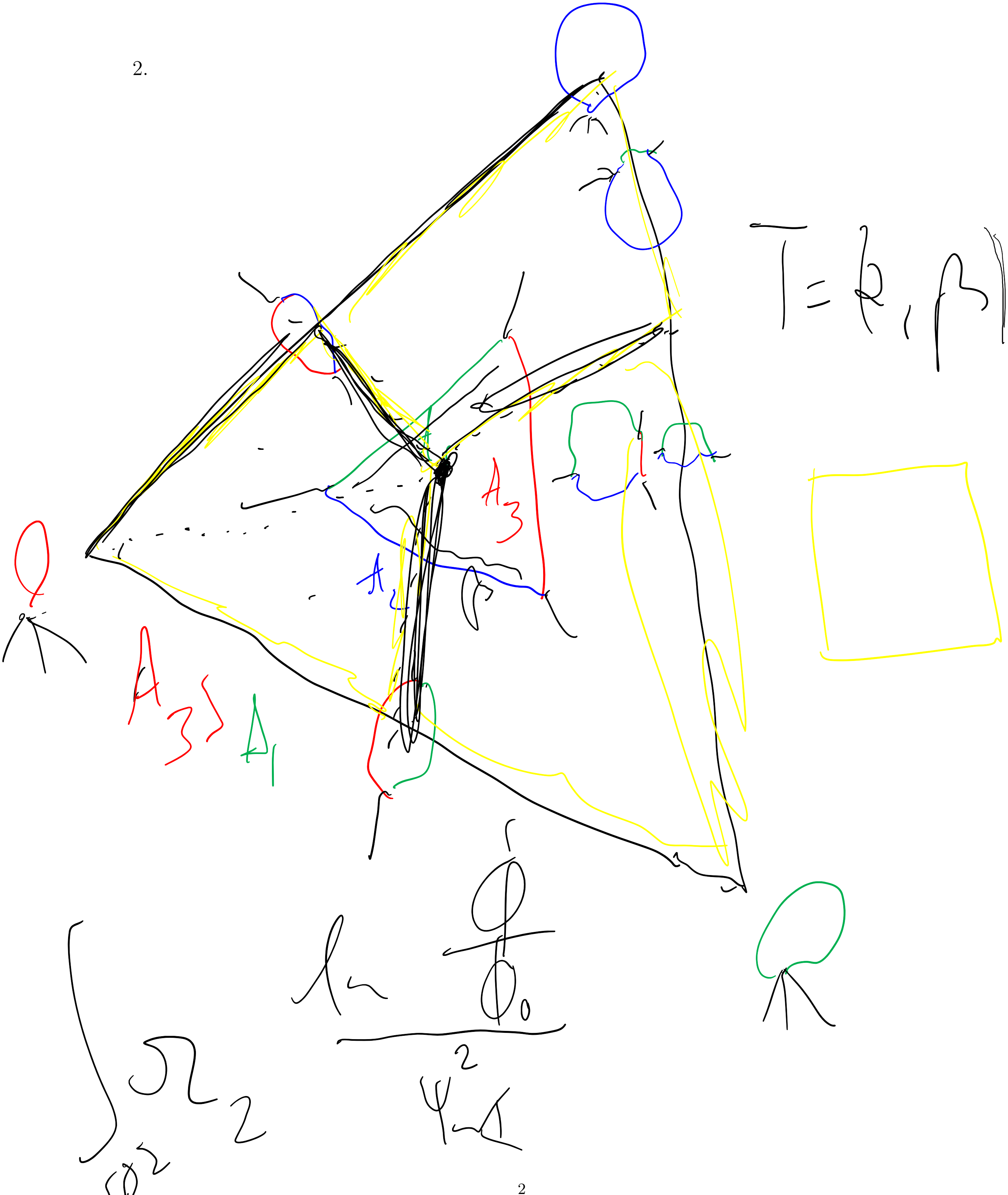
$$\int_0^{\infty} \int d^4\alpha \prod_e dA_e \, e^{-A_1 Q_1 - A_2 Q_2 \ldots}$$

$$Q_i(\{x_e\})$$

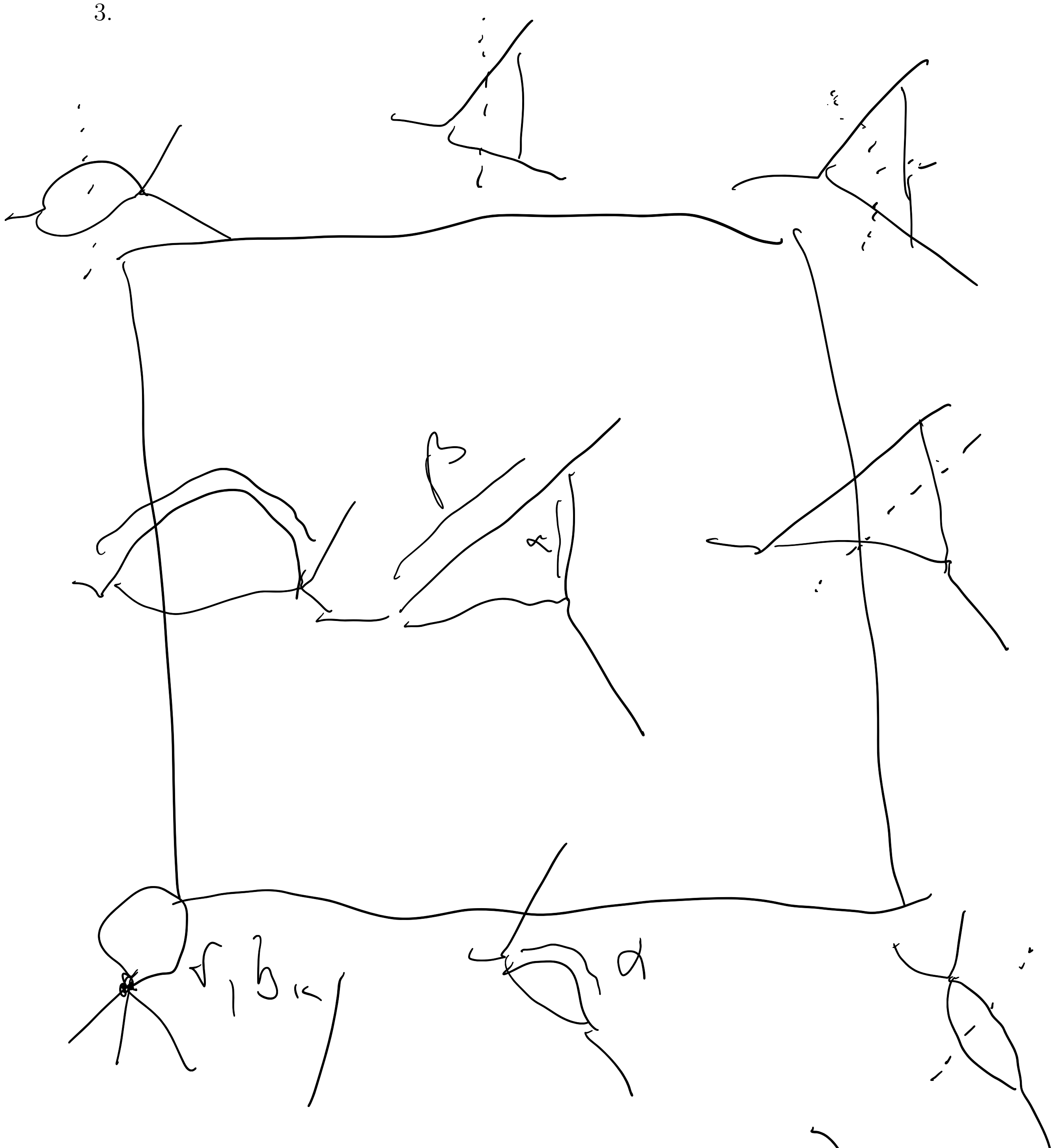
$$\hookrightarrow \int \prod_e dA_e \int d^4k \, e^{f(A_i)k^2 + g(A_i)k \cdot q + h(A_i)\{m^2, q^2\}}$$

$$\frac{\ldots}{\varphi^2}$$

2.



3.



Cube (2-dim.) in the cubical chain complex.

4. Spine: all possible paths from G to the nodes. $= G/T$

$$d(G, F) \quad \text{for } d = 0$$

$$= \sum_{j=1}^{|E_F|} (-1)^{j} \Big((G/e_j,\ F/e_j) - (G,\ F - e_j) \Big)$$

5.

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1.1. **One-loop graphs.** Consider the one-loop triangle with vertices $\{A, B, C\}$ and edges $\{(A,B),(B,C),(C,A)\}$, and quadrics:

$$P_{AB} = k_0^2 - k_1^2 - k_2^2 - k_3^2 - M_1,$$
$$P_{BC} = (k_0 + q_0)^2 - k_1^2 - k_2^2 - k_3^2 - M_2,$$
$$P_{CA} = (k_0 - p_0)^2 - (k_1)^2 - (k_2)^2 - (k_3 - p_3)^2 - M_3.$$

Here, we Lorentz transformed into the rest frame of the external Lorentz 4-vector $q = (q_0, 0, 0, 0)^T$, and oriented the space like part of $p = (p_0, \vec{p})^T$ in the 3-direction: $\vec{p} = (0, 0, p_3)^T$.

Using $q_0 = \sqrt{q^2}$, $q_0 p_0 = q_\mu p^\mu \equiv q.p$, $\vec{p}\cdot\vec{p} = \frac{q.p^2 - p.pq.q}{q^2}$, we can express everything in covariant form whenever we want to.

We consider first the two quadrics P_{AB}, P_{BC} which intersect in $\mathbb{C}^4$.

The real locus we want to integrate is $\mathbb{R}^4$, and we split this as $\mathbb{R}\times\mathbb{R}^3$, and the latter three dimensional real space we consider in spherical variables as $\mathbb{R}_+ \times S^1 \times [-1,1]$, by going to coordinates $k_1 = \sqrt{s}\sin\phi\sin\theta$,$k_2 = \sqrt{s}\cos\phi\sin\theta$, $k_3 = \sqrt{s}\cos\theta$, $s = k_1^2 + k_2^2 + k_3^2$, $z = \cos\theta$.

We have

$$P_{AB} = k_0^2 - s - M_1,$$
$$P_{BC} = (k_0 + q_0)^2 - s - M_2.$$

So we learn say $s = k_0^2 - M_1$ from the first and

$$k_0 = k_r := \frac{M_2 - M_1 - q_0^2}{2q_0}$$

from the second, so we set

$$s_r := \frac{M_2^2 + M_1^2 + (q_0^2)^2 - 2(M_1 M_2 + q_0^2 M_1 + q_0^2 M_2)}{4q_0^2}.$$

The integral over the real locus transforms to

$$\int_{\mathbb{R}^4} d^4k \to \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} \sqrt{s}\delta_+(P_{AB})\delta_+(P_{BC})dk_0 ds \times \int_0^{2\pi}\int_{-1}^{1} d\phi\delta_+(P_{CA})dz.$$

We consider k_0, s to be base space coordinates, while P_{CA} also depends on the fibre coordinate $z = \cos\theta$. Nothing depends on ϕ (for the one-loop box it would).

Integrating in the base and integrating also ϕ trivially in the fibre gives

$$\frac{1}{2}\frac{\sqrt{s_r}}{2q_0}2\pi\int_{-1}^{1}\delta_+(P_{CA}(s = s_r, k_0 = k_r))dz.$$

For P_{CA} we have

$$P_{CA} = (k_r - p_0)^2 - s_r - \vec{p}\cdot\vec{p} - 2|\vec{p}|\sqrt{s_r}z - M_3 =: \alpha + \beta z. \tag{1.1}$$

Integrating the fibre gives a very simple expression (the Jacobian of the δ-function is $1/(2\sqrt{s_r}|\vec{p}|)$, and we are left with the Omnès factor

$$\frac{\pi}{4|\vec{p}|q_0}. \tag{1.2}$$

This contributes as long as the fibre variable

$$z = \frac{(k_r - p_0)^2 - s_r - \vec{p}\cdot\vec{p} - M_3}{2|\vec{p}|\sqrt{s_r}}$$

lies in the range $(-1, 1)$. This is just the condition that the three quadrics intersect.

An anomalous threshold below the normal theshold appears when $(m_1 - m_2)^2 < q^2 < (m_1 + m_2)^2$. In that range, s_r is negative, hence its square root imaginary. It follows that z can be real only for $z = 0$, and this delivers

$$s_r = (k_r - p_0)^2 - \vec{p} \cdot \vec{p} - M_3,$$

which is negative for sufficiently large M_3, as expected.

On the other hand, when we leave the propagator P_{CA} uncut, we have the integral

$$\frac{1}{2} \frac{\sqrt{s_r}}{2q_0} 2\pi \int_{-1}^{1} \frac{1}{P_{CA\,(s=s_r, k_0=k_r)}} dz.$$

This delivers a result as foreseen by S-Matrix theory [?].

The two δ_+-functions constrain the k_0- and t-variables, so that the remaining integrals are over the compact domain S^2.

As the integrand does not depend on ϕ, this gives a result of the form

$$2\pi C \underbrace{\int_{-1}^{1} \frac{1}{\alpha+\beta z} dz}_{:=J_{CA}} = 2\pi \frac{C}{\beta} \ln \frac{\alpha+\beta}{\alpha-\beta} = \frac{1}{2} \mathrm{Var}(\Phi_R(b_2)) \times J_{CA}, \tag{1.3}$$

where $C = \sqrt{s_r}/2q_0$ is intimitaly related to $\mathrm{Var}(\Phi_R(b_2))$ for b_2 the reduced triangle graph (the bubble), and the factor $1/2$ here is $\mathrm{Vol}(S^1)/\mathrm{Vol}(S^2)$.

Here, α and β are given through (see Eq.(1.1)) $l_1 \equiv \vec{p}^2 = \lambda(q^2, p^2, (p+q)^2)/4q^2$ and $l_2 := s_r = \lambda(q^2, M_1, M_2)/4q^2$ as

$$\alpha = (k_r - p_0)^2 - l_2 - l_1 - M_3,\ \beta = 2\sqrt{l_1 l_2}.$$

Note that

$$\frac{C}{\beta} = \frac{1}{\sqrt{\lambda(q^2, p^2, (q+p)^2)}} = \frac{1}{2q_0|\vec{p}|},$$

in Eq.(1.3) is proportional to the Omnès factor Eq.(1.2).

In summary, there is a Landau singularity in the reduced graph in which we shrink P_{CA}. It is located at

$$q_0^2 = s_{normal} = (\sqrt{M_1} + \sqrt{M_2})^2.$$

It corresponds to the threshold divisor defined by the intersection $(P_{AB} = 0) \cap (P_{BC} = 0)$.

This is not a Landau singularity when we unshrink P_{CA} though. A (leading) Landau singularity appears in the triangle when we also intersect the previous divisor with the locus $(P_{CA} = 0)$.

It has a location which can be computed from the parametric approach. One finds

$$\begin{aligned} q_0^2 &= s_{anom} = (\sqrt{M_1} + \sqrt{M_2})^2 + \\ &+ \frac{4M_3(\sqrt{\lambda_2}\sqrt{M_1} - \sqrt{\lambda_1}\sqrt{M_2})^2 - \left(\sqrt{\lambda_1}(p^2 - M_2 - M_3) + \sqrt{\lambda_2}((p+q)^2 - M_1 - M_3)\right)^2}{4M_3\sqrt{\lambda_1}\sqrt{\lambda_2}}, \end{aligned}$$

with $\lambda_1 = \lambda(p^2, M_2, M_3)$ and $\lambda_2 = \lambda((p+q)^2, M_1, M_3)$.

Humboldt U. Berlin