1. $H_C$ and the pre-Lie product and the cubical chain complex

So consider the pair $(G, F)$ of a pre-Cutkosky graph with compatible forest $F$ with ordered edges. Assume there are graphs $G_1, G_2$ and forests $F_1, F_2$ such that

$$(G, F) = (G_1, F_1) \ast (G_2, F_2).$$

Here, $\ast$ is the pre-Lie product which is induced by the co-product $\Delta_{GF}$ by the Milnor–Moore theorem.
**Theorem 1.1** (K&Yeats). We can reduce the computation of the homology of the cubical chain complex for large graphs to computations for smaller graphs by a Leibniz rule:

\[ d ((G_1, F_1) \ast (G_2, F_2)) = (d(G_1, F_1)) \ast (G_2, F_2) + (-1)^{|E_1|} (G_1, F_1) \ast (d(G_2, F_2)). \]

Here, \( d = d_0 + d_1 \) is the boundary operator which either shrinks edges or cuts a graph.

\[ d \circ d = 0 \]
2. Flags

The idea: linearize!

\[ \text{Diagram}

\begin{align*}
\mathcal{G} & \xrightarrow{-1} \mathcal{O}_1 \boxtimes \mathcal{O}_2 \boxtimes \cdots \boxtimes \mathcal{O}_\mathcal{I} \\
& \overset{\text{linear}}{=} \mathcal{O}_\mathcal{I} \end{align*}
2.1. Flags of necklaces. Here we use it based on the core Hopf algebra introduced above. We introduce Sweedler’s notation for the reduced co-product in $H_{\text{core}}$:

$$\tilde{\Delta}_{\text{core}}(G) := \Delta_{\text{core}}(G) - I \otimes G - G \otimes I =: \sum G' \otimes G''.$$  

We define a flag $f \in \text{Aug}_{\text{core}}^k$ of length $k$ to be an element of the form

$$f = \gamma_1 \otimes \cdots \otimes \gamma_k,$$

where the $\gamma_i \in \text{Aug}_{\text{core}} \cap \langle H_{\text{core}} \rangle$ fulfill $\tilde{\Delta}_{\text{core}}(\gamma_i) = 0$, $|\gamma_i| = 1$. 

\[
\begin{align*}
\Delta (\alpha) & = 0 \\
(id \otimes \Delta) & = 0
\end{align*}
\]
Here, \( \langle H_{\text{core}} \rangle = \{ G \in H_{\text{core}} | |H^0(G)| = 1 \} \) is the linear \( \mathbb{Q} \)-span of bridge-free connected graphs as generators.

Note that for elements \( G \in \langle H_{\text{core}} \rangle \), we have \( \check{\Delta}^{-1}_{\text{core}}(G) \neq 0 \).

We have \( \Delta_{\text{core}} := (P \otimes P)\Delta_{\text{core}} \) for \( P : H_{\text{core}} \rightarrow \text{Aug}_{\text{core}} \) the projection into the augmentation ideal \( \text{Aug}_{\text{core}} \).

\[
\begin{align*}
\Delta^2 & \quad \Rightarrow \quad \Delta \otimes \Delta + \Delta \otimes \Delta \\
& \quad \Rightarrow \quad \eta \\
& \quad \Rightarrow \quad 2
\end{align*}
\]
Define the flag associated to a graph \( G \in \langle H_{\text{core}} \rangle \) to be a sum of flags of length \( |G| \) where in each flag each element \( \gamma_i \) has unit degree, \( |\gamma_i| = 1 \):

\[
Fl_G := \Delta_{\text{core}}^{[G]-1}(G) \in \text{Aug}_{\text{core}}^{\otimes [G]} \equiv \sum_i \gamma_i^1 \otimes \cdots \otimes \gamma_i^{[G]}.
\]
Similarly, for a pair \((G, F)\) we can define
\[
Fl_{G,F} := \tilde{\Delta}_{GF}^{-1}((G, F)) \in \text{Aug}_{GF}^{|G|},
\]
as a sum of flags
\[
Fl_{G,F} = \sum_i (\gamma_i, f_i)^i \otimes \cdots (\gamma_{|G|}, f_{|G|})^i,
\]
\(\tilde{\Delta}_{GF}((\gamma_l, f_l)^i) = 0, \forall i, l, 1 \leq l \leq |G|\).
3. Partial Fractions and Spanning Trees

The idea: linearize in each loop energy variable but one quadric.

\[ G \rightarrow \prod_{e \in f} \frac{1}{Q_e} \]

\(Q_e\) is quadratic in momenta \(0 < \varepsilon < 1\)

\[ (k + \delta e)^2 - m_e^2 + i \delta \]

\(Q - \delta \phi\) is linear in loop momenta

\[ 2lx \cdot (\varepsilon - \frac{\delta f}{g}) - m_e^2 + \frac{m_f^2}{(\varepsilon - \frac{\delta f}{g})^2} \]
3.1. Partial fractions. To make use of flags, let us consider as an example a one-loop graph $\gamma$. It can be regarded up to cyclic permutations of its vertices and hence regarded as belonging to an equivalence class forming necklace $p$ as a primitive element $\gamma \in H_{\text{core}}$.

It has an associated product $Q(\gamma)$ of say $k$ quadrics

$$Q(\gamma) := \prod_{j=1}^{v_\gamma} \frac{1}{Q_j}.$$  

We have an obvious partial fraction

$$Q(\gamma) = \sum_{j=1}^{v_\gamma} \frac{1}{Q_j} \prod_{i \neq j} \frac{1}{Q_i - Q_j}.$$

\begin{align*}
1 & \quad \rightarrow \quad \frac{1}{Q_1} \cdot \frac{1}{Q_2} \cdot \frac{1}{Q_3} \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
3 & \quad \rightarrow \quad Q_1 Q_2 Q_3
\end{align*}

\[= \frac{1}{Q_1} \left( \frac{1}{Q_2 - Q_1} - \frac{1}{Q_3 - Q_1} \right) \quad \frac{1}{Q_2} \left( \frac{1}{Q_1 - Q_2} - \frac{1}{Q_3 - Q_2} \right) \quad \frac{1}{Q_3} \left( \frac{1}{Q_1 - Q_3} - \frac{1}{Q_2 - Q_3} \right)\]
As an example for the bubble $b$ we find:

$$Q(b) = \frac{1}{Q_1 Q_2} = \frac{1}{Q_1} \frac{1}{Q_2 - Q_1} + \frac{1}{Q_2} \frac{1}{Q_1 - Q_2}.$$

Here we define $p f_j$ so that

$$Q(\gamma) = \sum_{j=1}^{v_s} \frac{1}{Q_j} p f_j.$$
Before we come to terms with the parametric approach, let us discuss first a simple example in greater detail.

Let us go back to the one-loop bubble $b$, $|b| = 1$, on two edges with integrand

$$I_b(q^2, m_1^2, m_2^2) := \frac{1}{Q_1 Q_2} = \frac{1}{Q_1} \frac{1}{Q_2 - Q_1} + \frac{1}{Q_2} \frac{1}{Q_1 - Q_2}.$$

The two quadrics are

$$Q_1 := k_0^2 - s - m_1^2, \quad Q_2 := (k_0 + q_0)^2 - s - m_2^2.$$

Here, the four-vector $k = (k_0, k) \tau$ defines $s = k \cdot k$. We can work in a frame where the external time-like four-vector $q$ fulfills $q = (q_0, 0)^T$, $q^2 \equiv q_0^2 > 0$.

The renormalized integral is in $D$ dimensions (spherical coordinates in $(D-1)$-space)

$$\Phi_R(b)(q^2, m_1^2, m_2^2) = \Omega_D \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} ds \sqrt{s}^{(D-3)} \left( I_b(q^2, m_1^2, m_2^2) - I_b(\mu^2, m_1^2, m_2^2) \right),$$

with $\Omega_D = 4\pi$. 

$$x = \begin{pmatrix} k_0 \cr k \end{pmatrix}$$

$$\int \sqrt{s} \left( I_b(q^2, m_1^2, m_2^2) - 2 \right)$$
Using the partial fractions above we have

\[
I_5(q^2, m_1^2, m_2^2) = \frac{1}{k_0^2 - s - m_1^2} \left( \frac{1}{(k_0 + q_0)^2 - s - m_2^2} \right) + \frac{1}{(q_0^2 + 2q_0k_0 - m_2^2 + m_1^2)}
\]

We can now integrate terms \(I, II\) in \(D\) dimensions separately as both terms are regularized and give Laurent series in \(D - 4\) with poles of finite (actually, first) order.

\[
\Phi_e = (\mathbf{k} - \mathbf{e})^2 - m_e^2 + i\varepsilon
\]

\[
\mathbf{k} \to \mathbf{k} + \varepsilon \Rightarrow \Phi_e = \mathbf{k}^2 - m_e^2 + i\varepsilon
\]
\[ I_b^R := I_b(q^2, m_1^2, m_2^2) - I_b(\mu^2, m_1^2, m_2^2) \] then gives four such terms such that the limit \( D \to 4 \) exists in their sum.

Each of the terms is separately invariant under shifts \( k \to k + r \) of the loop momentum. In particular we can shift in term \( II \)

\[ I_b(q^2, m_1^2, m_2^2) = \frac{1}{k^2_0 - s - m_1^2 q_0^2 + 2q_0 k_0 - m_2^2 + m_1^2} + \frac{1}{k^2_0 - s - m_2^2 q_0^2 - 2q_0 k_0 + m_2^2 - m_1^2}, \]

and similarly for \( I_b(\mu^2, m_1^2, m_2^2) \).

\[ \ell (T) ! \]
We can exchange the order of the $k_0$ and $s$ integral. The $k_0$ integral exists in all four terms of $I^R_k$ separately and can be done as a contour integral in the complex $k_0$-plane upon closing in the upper half-plane, using that we assign an infinitesimal imaginary part $m_i^2 - i\eta$, $0 < \eta \ll 1$ to each mass square $m_i^2$.

Each of the four terms has the form $1/(Q_j(Q_i - Q_j))$. We note that the pole at $Q_i = Q_j$ does not contribute as its residue is

$$0 = \left( \frac{1}{Q_1} - \frac{1}{Q_2} \right)_{Q_1 = Q_2 \neq 0}.$$

$$k_0^2 - s - m_i^2 + i\eta \rightarrow k_0 = \pm \sqrt{s - m_i^2 + i\eta}$$

$$k_0^2 = s - m_i^2 - i\eta$$

$$\frac{1}{Q_0}$$

not pole

on-shell the so are $k_0 \geq 0$ the on-shell condition

$$\int_{-\infty}^{\infty} dk_0$$
So the \( k_0 \)-contour integral is precisely picking up the four terms with poles in the upper \( k_0 \)-halfplane for \( Q_1 = 0 \) and the remaining \( s \)-integral has an integrand as a function of \( s \) (for timelike \( q = (q_0, 0)^T \))

\[
\int dk_0 I^R_6(q^2, \mu^2, m_1^2, m_2^2)(s) = \frac{1}{q_0^2 + 2q_0 \sqrt{s + m_1^2 - m_2^2 + m_1^2}} + \frac{1}{q_0^2 - 2q_0 \sqrt{s + m_2^2 + m_2^2 - m_1^2}} - \left( \frac{1}{\mu_0^2 + 2\mu_0 \sqrt{s + m_1^2 - m_2^2 + m_1^2}} + \frac{1}{\mu_0^2 - 2\mu_0 \sqrt{s + m_2^2 + m_2^2 - m_1^2}} \right),
\]

which allows for the limit \( D \to 4 \) so that the result is

\[
\Phi_R(b)(q^2, m_1^2, m_2^2) = 4\pi \int_0^\infty I^R_6(q^2, m_1^2, m_2^2)(s) ds.
\]
The integral is elementary and confirms the known
\[
\Phi_R(b)(q^2, m_1^2, m_2^2) =
\left( \frac{\sqrt{\lambda(q^2, m_1^2, m_2^2)}}{2q^2} \ln \frac{m_1^2 + m_2^2 - q^2 - \sqrt{\lambda(q^2, m_1^2, m_2^2)}}{m_1^2 + m_2^2 - q^2 + \sqrt{\lambda(q^2, m_1^2, m_2^2)}} - \frac{m_1^2 - m_2^2}{2q^2} \ln \frac{m_1^2}{m_2^2} \right)
- \left( \{q^2 \rightarrow \mu^2\} \right),
\]
where \(\lambda(a, b, c) := a^2 + b^2 + c^2 - 2(ab + bc + ca)\) is the usual Källen function.
3.2. **pf and spanning trees.** Note that the edges $e_i \in E_{\gamma}$ in $pf_j$, for $i \neq j$, for any chosen edge $e_j \in E_{\gamma}$, form a spanning tree of $\gamma$.

We hence can write

$$Q(\gamma) = \sum_{T \in T(\gamma)} pf(T) \frac{1}{Q_T},$$

where $T$ denotes the edge of $\gamma$ which is not in $T$ so that $pf(T) = pf_T$.

A generalization to generic graphs is the straightforward.
We define
\[ Q(\text{Fl}_G) := \sum_{i} \prod_{j=1}^{[G]} Q(\gamma_j^{(i)}). \]
This is a homogeneous polynomial of degree $|G|$ in inverse quadrics $1/Q_e$.

**Proposition 3.1.**

\[ Q(\text{Fl}_G) = \xi_G \frac{1}{\prod_{e \in E_G} Q_e}. \]

Here, $\xi_G$ is the number of distinct flags in $\text{Fl}_G$.

**Proof.** By definition of $\text{Fl}_G$ we can write $\text{Fl}_G = \sum_{j=1}^{[G]} \gamma_1^{(j)} \otimes \cdots \otimes \gamma_{[G]}^{(j)}$. Each $Q(\gamma_k^{(j)}) = \prod_{e \in E, j^{(k)}} \frac{1}{Q_e}$, and we use $Q(u \otimes v) = Q(u)Q(v)$ where we extend $Q$ as a map $Q : H_{\text{core}} \to \mathbb{C}$. \qed
3.3. Example. For the Dunce's cap graph $dc$ we find the following residues in accordance with the core coproduct, see Figs.(1,2):

$$Q(F^c_{dc}) = \frac{1}{Q_4} \left( \frac{1}{Q_1 (Q_2 - Q_1)(Q_3 - Q_1)} + \frac{1}{Q_2 (Q_3 - Q_2)(Q_1 - Q_2)} + \frac{1}{Q_3 (Q_1 - Q_3)(Q_2 - Q_3)} \right)$$

$$+ \frac{1}{Q_3} \left( \frac{1}{Q_1 (Q_2 - Q_1)(Q_4 - Q_1)} + \frac{1}{Q_2 (Q_4 - Q_2)(Q_1 - Q_2)} + \frac{1}{Q_4 (Q_1 - Q_4)(Q_2 - Q_4)} \right)$$

$$+ \left( \frac{1}{Q_1 (Q_2 - Q_1) + \frac{1}{Q_2 (Q_1 - Q_2)}} \right) \left( \frac{1}{Q_3 (Q_4 - Q_3) + \frac{1}{Q_4 (Q_3 - Q_4)}} \right).$$

We have $\xi_{dc} = 3$, corresponding to the three parts in $10 = (3 \times 1) + (3 \times 1) + (2 \times 2)$. The ten terms correspond to the five spanning trees, with each spanning tree defining two loops $l_1, l_2$, which contribute to the core coproduct as either $l_1 \otimes l_2$ or $l_2 \otimes l_1$. 

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For example choosing the spanning tree on edges $e_2, e_3$ in the Dunce's cap, the two loops
are $l_1 = \{e_1, e_2, e_3\}, l_2 = \{e_3, e_4\}$. The term $l_1 \otimes l_2$ has $l_1$ as the sub-graph and $dc/l_1 = e_4$ as
the co-graph. The corresponding term in $Q(Fl_{dc})$ is

\[
(3.2) \quad \frac{1}{Q_4} \left( \frac{1}{Q_1 (Q_2 - Q_1)} \frac{1}{Q_3 - Q_1} \right).
\]

The term $l_2 \otimes l_1$ has $l_2$ as the sub-graph and $dc/l_2 = \{e_1, e_2\}$ as the co-graph. The corre-
sponding term in $Q(Fl_{dc})$ is

\[
(3.3) \quad \left( \frac{1}{Q_1 Q_2 - Q_1} \right) \left( \frac{1}{Q_4 (Q_3 - Q_4)} \right).
\]

For both Eqs.(3.2,3.3) the conditions $Q_1 = 0$ and $Q_4 = 0$ determine $k_0 = \sqrt{s + m_1^2}$ and
$l_0 = \sqrt{s + m_4^2}$. Both equations have by construction the same residue as

\[
(Q_3 - Q_4)_{0=Q_1=Q_2} = (Q_3 - Q_1)_{0=Q_1=Q_4},
\]

in accordance with the divided difference structure of contour integrals.
Figure 1: The flag decomposition of the Dunce's cap. Note that the Dunce's cap has five spanning trees and two loops. This gives ten terms which appear on the rhs by counting spanning trees in the sub- and co-graphs: $2 \times 5 = 10 = 3 \times 1 + 3 \times 1 + 2 \times 2$. Indeed, the five spanning trees of the Dunce's cap are the five pairs of edges $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}$. Each spanning tree defines a basis for the loops in the Dunce's cap by connecting the two endpoints of one of the edges not in the spanning tree by a path through that tree. With two edges not in the spanning tree and five spanning trees this gives a five element set. Looking at the flag decomposition on the right, the triangle on edges 1, 2, 3 has three spanning trees given by the pairs $\{(1, 2), (2, 3), (3, 1)\}$ while the tadpole has a single spanning tree given by the vertex $a \cup b \cup c$. The triangle on edges 1, 2, 4 has three spanning trees given by the pairs $\{(1, 2), (2, 4), (4, 1)\}$ while the tadpole again has a single spanning tree given by the vertex $a \cup b \cup c$. Each of the two edges 3, 4 of the bubble forms a spanning tree and similar each of the two edges 1, 2 of the other bubble.
Figure 2: \( \Delta_{GF}(G, T) = F_{G,T} \) for \( G \) the Dunce' s cap and for its five spanning trees \( T \).