FEYNMAN DIAGRAMS AND THE S-MATRIX, AND OUTER SPACE (SUMMER 2020)

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1. Flags

The notion of flags of Feynman graphs was for example already used in [**?**, **?**].

Here we use it based on the core Hopf algebra introduced above.

We have $\tilde{\Delta}_{core} := (P \otimes P)\Delta_{core}$ for $P : H_{core} \to \mathrm{Aug}_{core}$ the projection into the augmentation ideal Aug_{core}.

We introduce Sweedler's notation for the reduced co-product in H_{core}:

$$\tilde{\Delta}_{core}(G) = \Delta_{core}(G) - \mathbb{I} \otimes G - G \otimes \mathbb{I} =: \tilde{\sum} G' \otimes G''.$$

Consider a graph G. We define an expanded flag associated to G a sequence of graphs

$$G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_{|G|} = G,$$

where $|G_1| = 1$ and $|G_i/G_{i-1}| = 1$ for all $i \geq 2$. We set $\gamma_i = G_i/G_{i-1}$ and $\gamma_1 = G_1$.

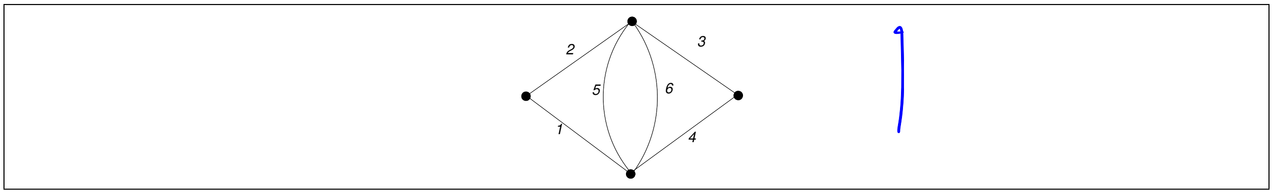
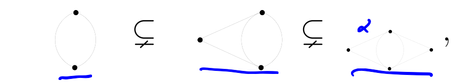
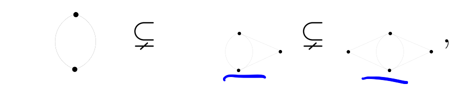
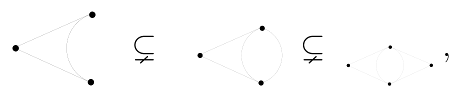


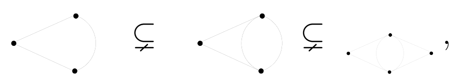
Figure 1: The graph generating the expanded flags of the example.

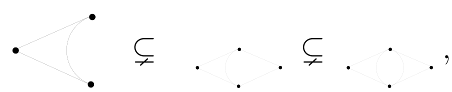
Write $F(G)$ for the collection of all expanded flags of G. Let us consider an example.

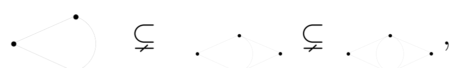
(1.1) 

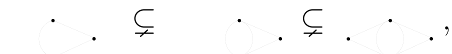
(1.2) 

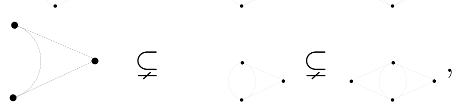
(1.3) 

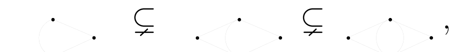
(1.4) 

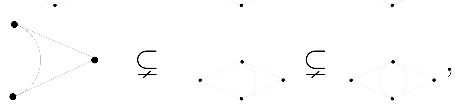
(1.5) 

(1.6) 

(1.7) 

(1.8) 

(1.9) 

(1.10) 

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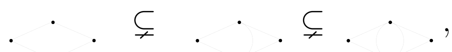
(1.12) 

are the twelve expanded flags for the graph given in Fig.(1). We omitted the edge labels in the above flags.

We define a flag $f \in \mathrm{Aug}_{core}^{\otimes k}$ of length k to be an element of the form

$$f = \gamma_1 \otimes \cdots \otimes \gamma_k,$$

where the $\gamma_i \in \mathrm{Aug}_{core} \cap \langle H_{core} \rangle$ fulfill $\tilde{\Delta}_{core}(\gamma_i) = 0$, $|\gamma_i| = 1$ which arises from an expanded flag.

Here, $\langle H_{core} \rangle = \{G \in H_{core} | \, h_0(G) = 1\}$ is the linear $\mathbb{Q}$-span of bridge-free connected graphs as generators.

Note that for elements $G \in (\langle H_{core} \rangle \cap \mathrm{Aug}_{core})$, we have $\tilde{\Delta}_{core}^{|G|-1}(G) \neq 0$.

Define the flag associated to a graph $G \in \langle H_{core} \rangle$ to be a sum of flags of length $|G|$ arising from all expanded flags so that in each flag each element γ_i has unit degree, $|\gamma_i| = 1$:

$$Fl_G := \tilde{\Delta}_{core}^{|G|-1}(G) \in \mathrm{Aug}_{core}^{\otimes |G|}.$$

With $\xi_G = |F(G)|$ the number of expanded flags a graph G has we can write

$$Fl_G = \sum_{i=1}^{\xi_G} \gamma_1^{(i)} \otimes \cdots \otimes \gamma_{|G|}^{(i)},$$

where for any of the orderings of the cycles l_j of G we have

$$\gamma_1 = l_1, \gamma_2 = l_2/E_{l_1 \cap l_2}, \ldots, \gamma_{|G|} = l_G/E_{l_1 \cap \cdots \cap l_{|G|-1}}. \tag{1.13}$$

Similarly, for a pair (G, F) we can define

$$Fl_{G,F} := \tilde{\Delta}_{GF}^{|G|-1}((G, F)) \in \mathrm{Aug}_{GF}^{\otimes |G|},$$

as a sum of flags

$$Fl_{G,F} = \sum_i (\gamma_1, f_1)^i \otimes \cdots (\gamma_{|G|}, f_{|G|})^i,$$

$\tilde{\Delta}_{GF}((\gamma_l, f_l)^i) = 0$, $\forall i, l$, $1 \leq l \leq |G|$.

2. Partial Fractions and Spanning Trees

2.1. **Divided differences.** Residue integrals can be expressed using divided differences. To this end consider a product L_γ of v_γ quadrics which constitute a one-loop graph γ.

Without loss of generality we can asumme that each quadric has the form

$$Q_e = (k + r_e)^2 - m_e^2 + i\eta,$$

for some loop momentum k, four-vector r_e, mass m_e and $0 \lneq \eta \ll 1$. We write

$$L_\gamma := \prod_{e=1}^{v_\gamma} \frac{1}{Q_e}.$$

The divided difference with regard to the function $f : x \to x^{-1}$ delivers the partial fraction decomposition

$$L_\gamma = \sum_{e=1}^{v_\gamma} f(Q_e) \underbrace{\prod_{f \neq e} \frac{1}{Q_f - Q_e}}_{=:\mathbf{pf}_e^\gamma}. \tag{2.1}$$

Note that the coefficients of any $1/(Q_f - Q_e) \sim f(Q_e) - f(Q_f)$.

$$\frac{1}{Q_1 Q_2} = \frac{1}{Q_1}\frac{1}{Q_2 - Q_1} + \frac{1}{Q_2}\frac{1}{Q_1 - Q_2}$$

$$= \frac{1}{Q_2 - Q_1}\left[\frac{1}{Q_1} - \frac{1}{Q_2}\right]$$

2.2. **Partial fractions.** To make use of flags, let us consider L above. It can be regarded up to cyclic permutations of its vertices and hence regarded as a belonging to an equivalence class forming necklace p as a primitive element $\gamma \in H_{core}$.

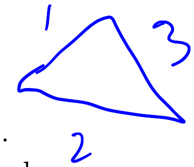
As an example for the bubble b we find :

$$L_b = \frac{1}{Q_1 Q_2} = \frac{1}{Q_1}\frac{1}{Q_2 - Q_1} + \frac{1}{Q_2}\frac{1}{Q_1 - Q_2}.$$

2.3. **pf and spanning trees.** Note that the edges $e_f \in E_\gamma$ in $\mathbf{pf}^\gamma_j$, for $f \neq e$, for any chosen edge $e \in E_\gamma$, form a spanning tree of γ.

We hence can write

$$L_\gamma = \sum_{T \in \mathcal{T}(\gamma)} \mathbf{pf}(T) \frac{1}{Q_{\acute{T}}},$$



$\frac{1}{Q_2}$

where $\acute{T}$ denotes the edge of γ which is not in T so that $\mathbf{pf}(T) = \mathbf{pf}^\gamma_{\acute{T}}$.

Note that $\mathbf{pf}(T)^{-1} = (\mathbf{pf}^\gamma_{\acute{T}})^{-1}$ is linear in the four-vector k and is real.

$$\frac{1}{Q_1 - Q_2} \quad \frac{1}{Q_3 - Q_2} \to \frac{}{(Q_1 - Q_2)(Q_3 - Q_2)}$$

liken ilk

$\cancel{x^2} - \ldots$

$$\sim \left(\frac{1}{Q_i} - \frac{1}{Q_j}\right)\Bigg|_{i=j} = 0$$

The divided difference structure gives

Proposition 2.1. *L_γ vanishes at any zero of any* $\mathbf{pf}(T)^{-1}$.

Proof. For $\mathbf{pf}(T)^{-1}$ to vanish, we need to have $\acute{T}, f$ such that $Q_f = Q_{\acute{T}}$. By the divided difference structure the coefficient of this zero is $1/Q_f - 1/Q_{\acute{T}}$ which vanishes. $\square$

All residues from propagators $\frac{1}{Q}$ only;

As a result the poles of L_γ in the variable k_0 are determined by the two zeroes of the quadric $Q_{\acute{T}}$ which are located in the upper and lower complex k_0-plane.

Indeed,

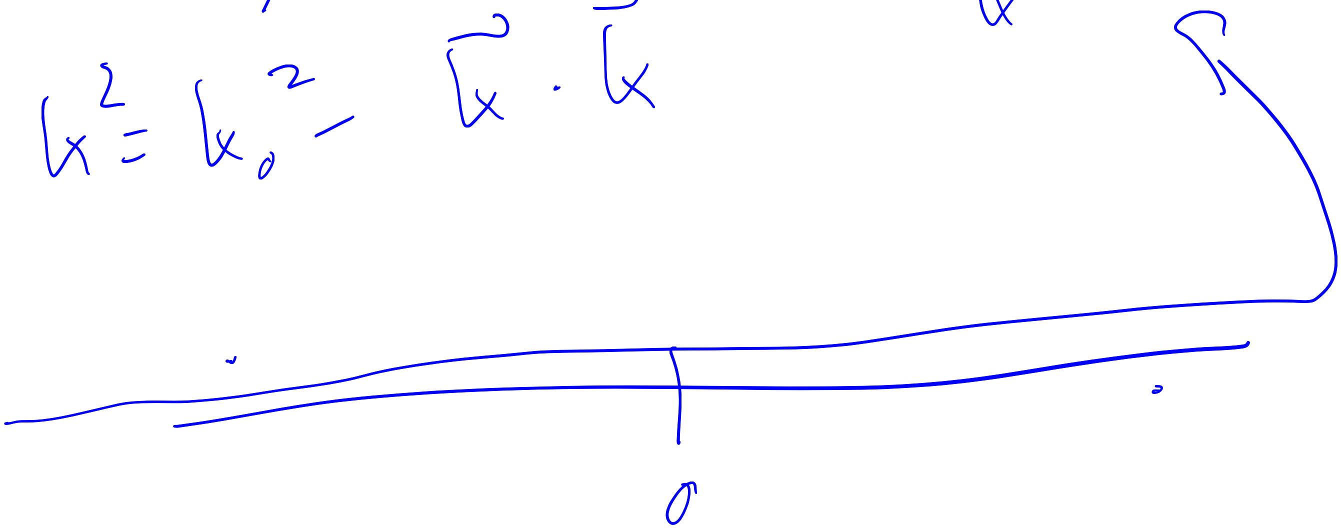
$$Q_{\acute{T}} = (k_0 + r_{\acute{T},0})^2 - (\vec{k} + \vec{r}_{\acute{T}})^2 - m_{\acute{T}}^2 + i\eta,$$

so that the zeroes are at

$$k_{0\,\pm}^{\acute{T}} = -r_{\acute{T},0} \pm \sqrt{(\vec{k} + \vec{r}_{\acute{T}})^2 + m_{\acute{T}}^2 - i\eta}.$$

$$k \equiv k_\mu = (k_0, \underbrace{k_1, k_2, k_3 \cdots}_{\vec{k}})$$

$$k^2 = k_0^2 - \vec{k} \cdot \vec{k}$$



2.4. **Shifts.** L_γ above has to be integrated:

$$\Phi(\gamma) := \int_{-\infty}^{\infty} dk_0 \int d^{D-1}\vec{k} L_\gamma.$$

Proposition 2.2. *For each term in the partial fraction decomposition the integral*

$$\Phi(\gamma, \acute{T}) := \int_{-\infty}^{\infty} dk_0 \int d^{D-1}\vec{k} f(Q_{\acute{T}})\mathbf{pf}(T),$$

exists as a unique Laurent-Taylor series with a pole of first order in $\epsilon = D/2 - 2$ and is invariant under the shifts $k_0 \to k_0 - r_{\acute{T},0}$ and $\vec{k} \to \vec{k} - \vec{r}_{\acute{T}}$.

Proof. Elementary properties of dimensional regularisation, [**?**]. $\square$

Collins

$$\int_{\mathbb{R}^D} d^D k = \int dk_0 \underbrace{\int d^{D-1} \vec{k}}$$ "Renormalization"

$$\mathrm{Vol}(S_{D-1}) \sim \frac{f(D,k)}{\pi} \Big/ \Gamma(\ldots) \qquad \underbrace{\int_0^\infty d|\vec{k}|\, |\vec{k}|^{D-2} \int \Omega_{\vec{k}}}$$

Assume from now on that for each $\Phi(\gamma, \acute{T})$ the indicated shift has been performed so that $Q_{\acute{T}} = k^2 - m_{\acute{T}} + i\eta$, let

$$\bar{\mathbf{pf}}(T) = \mathbf{pf}(T)_{k_0 \to k_0 - r_{\acute{T},0}, \vec{k} \to \vec{k} - \vec{r}_{\acute{T}}}.$$

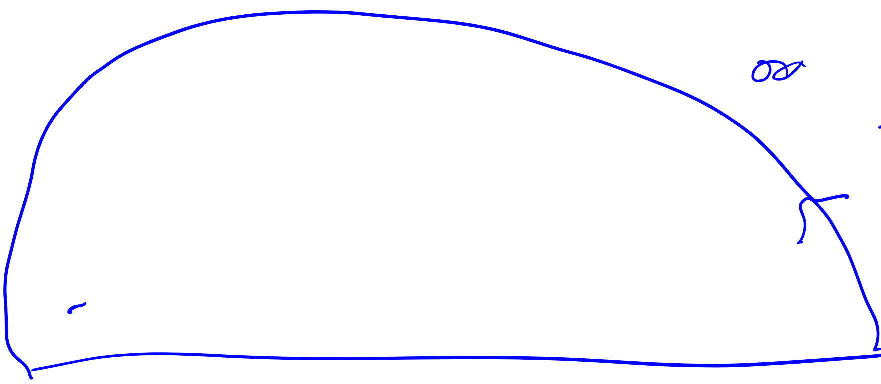
We get

$$\Phi(\gamma) = \sum_{T \in \mathcal{T}(\gamma)} \Phi(\gamma, \acute{T}) = \int_{-\infty}^{\infty} dk_0 \int d^{D-1}\vec{k} \sum_{T \in \mathcal{T}(\gamma)} \bar{\mathbf{pf}}(T) \frac{1}{k_0^2 - \vec{k}^2 - m_{\acute{T}}^2 + i\eta}.$$

Doing the k_0-integral by a contour integration closing in the upper complex half-plane we find

$$\Phi(\gamma) = \int d^{D-1}\vec{k} \sum_{T \in \mathcal{T}(\gamma)} \bar{\mathbf{pf}}(T)_{|k_0 = +\sqrt{\vec{k}^2 - m_{\acute{T}}^2 + i\eta}}.$$

$\forall \gamma$, the result is Li_2 'dilog'



$\gamma_1 \otimes \cdots \otimes \gamma_{|G|}$

2.5. **Partial Fractions for generic graphs.** A generalization to generic graphs is then straightforward.

We define

$$L(Fl_G) := \sum_{i}^{\xi_G} \prod_{j=1}^{|G|} L_{\gamma_j^{(i)}}.$$

This is a homogeneous polynomial of degree $|G|$ in inverse quadrics $1/Q_e$. The $\gamma_j^{(i)}$ are determined as above in Eq.(1.13).

For the integral $\Phi(G)$ we have

$$\Phi(G) := \sum_{i=1}^{\xi_G} \left(\prod_{j=1}^{|G|} \int_{\infty}^{\infty} dk_0(j) \int d^{D-1}\vec{k}(j) \right) \times \left(\prod_{j=1}^{|G|} L_{\gamma_j^{(i)}} \right).$$

Carrying out all $k_0(j)$-integrals we find

$$\Phi(G) := \sum_{i=1}^{\xi_G} \left(\prod_{j=1}^{|G|} \int d^{D-1}\vec{k}(j) \right) \times \prod_{j=1}^{|G|} \sum_{T \in \mathcal{T}(\gamma_j^{(i)})} \bar{\mathbf{pf}}(T)_{k_0(j)=+\sqrt{\vec{k}(j)^2 - m_T^2 + i\eta}}.$$

Note that for each of the $|G|!$ terms in the above sum, the spannng trees T of the graphs $\gamma_j^{(i)}$ combine to a spanning tree $U \in \mathcal{T}(G)$. Furthermore each term in the summand indicates one of the $|G|!$ possible orders of the $|G|$ independent cycles of the graph.

2.6. **General structure.** It is then useful to count the number of spanning trees of a graph to control its computation as well as for example the number of Hodge matrices describing the analytic structure of an evaluated Feynman graph.

So we let $spt : H_{core} \to \mathbb{N}$, $G \to spt(G)$ be the number of spanning trees of G and define

$$\mathbf{spt} : H_{core} \to \mathbb{N}, \mathbf{spt}(G) := spt(G)|G|!.$$

We have

Proposition 2.3.

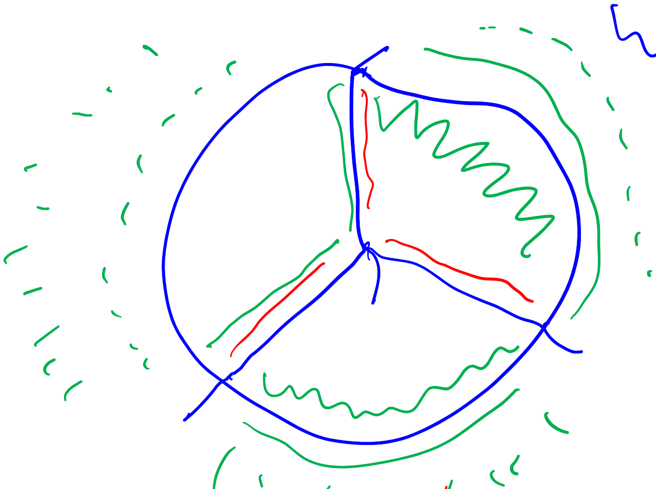
$$\mathbf{spt}(G) = \tilde{\sum} \mathbf{spt}(G')\mathbf{spt}(G''),$$

using the reduced coproduct and

$$\mathbf{spt}(G) = spt^{|G|}\tilde{\Delta}_{core}^{|G|-1}(G).$$

Proof. Immediate pairing off edges in the spanning trees. □

In fact $\mathbf{spt}(G)$ counts the nubers of residues when integrating the energy components in each loop momentum as we have seen above.



W_3

$|W_3| = 3$

$\boxed{16 \cdot |W_3|!} = 96$

red edges:

sp. trees with x red edges

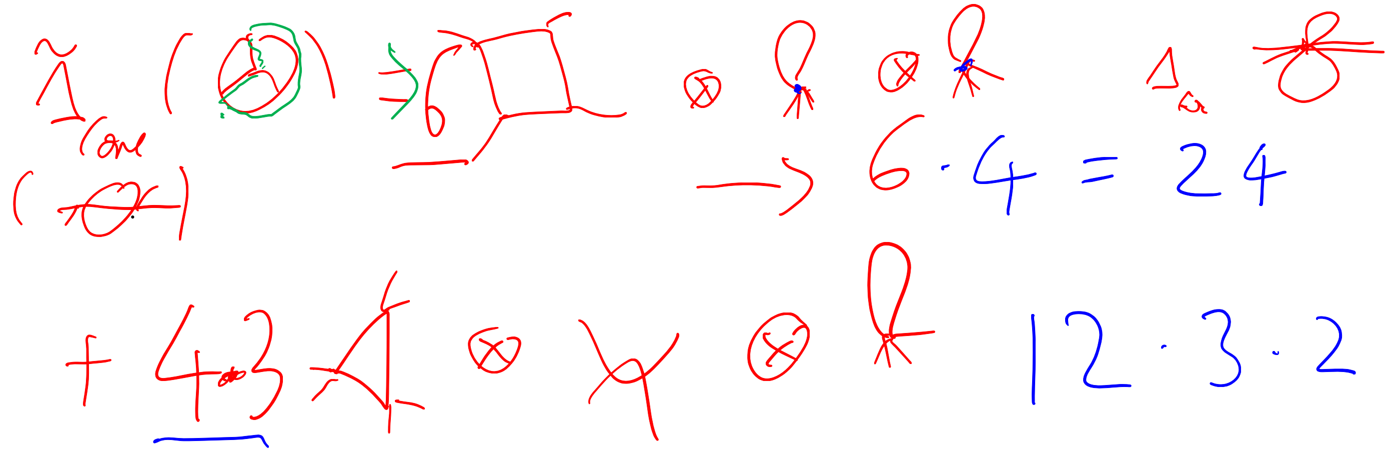
x = 3 → 1

x = 0 → 0

x = 1 → 3 × 3 = 9

x = 2 → 3 × 2 = 6

16



$\to 6 \cdot 4 = 24$

$+\ 4 \cdot 3 \quad 12 \cdot 3 \cdot 2$

$72 + 24 = 96\ !$

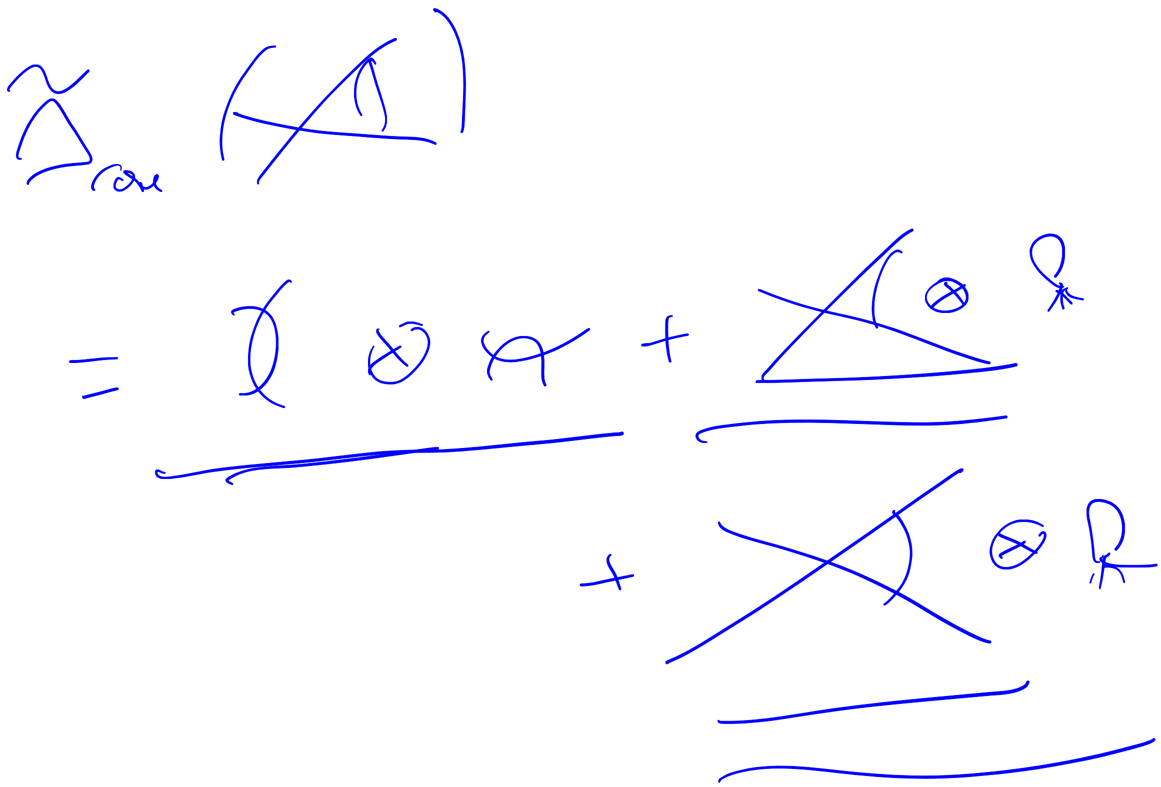
Proposition 2.4.

$$L(Fl_G) = \xi_G \frac{1}{\prod_{e \in E_G} Q_e},$$

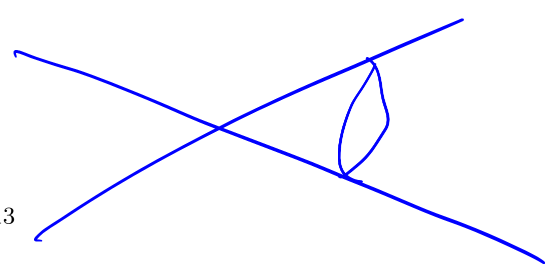
As before ξ_G is the number of distinct flags in Fl_G.

Proof. By definition of Fl_G we can write $Fl_G = \sum_{j=1}^{\xi_G} \gamma_1^{(j)} \otimes \cdots \otimes \gamma_{|G|}^{(j)}$. Each $L(\gamma_k^{(j)}) = \prod_{e \in E_{\gamma_k^{(j)}}} \frac{1}{Q_e}$ and we use $L(u \otimes v) = L(u)L(v)$ where we extend L as a map $Q : H_{core} \to \mathbb{C}$. $\square$

$Q = L$



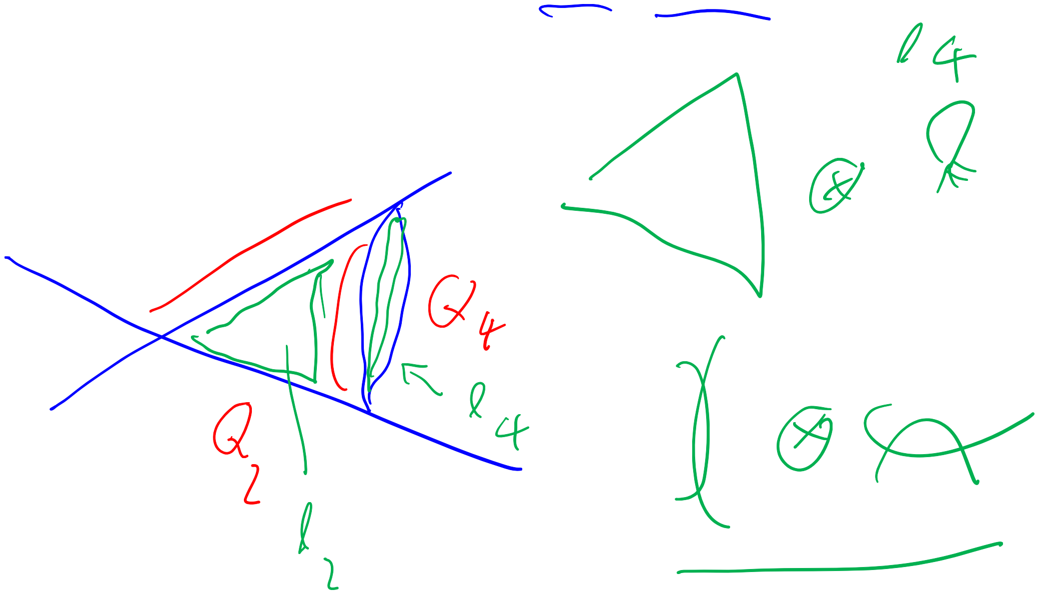
$\chi_G = 3$



2.7. **Example.** For the Dunce's cap graph dc we find the following residues in accordance with the core coproduct, see Figs.(2,3):

$$
\begin{aligned}
Q(Fl_{dc}) = {} & \frac{1}{Q_4}\left(\frac{1}{Q_1}\frac{1}{(Q_2-Q_1)(Q_3-Q_1)} + \frac{1}{Q_2}\frac{1}{(Q_3-Q_2)(Q_1-Q_2)} + \frac{1}{Q_3}\frac{1}{(Q_1-Q_3)(Q_2-Q_3)}\right)\\
+ {} & \frac{1}{Q_3}\left(\frac{1}{Q_1}\frac{1}{(Q_2-Q_1)(Q_4-Q_1)} + \frac{1}{Q_2}\frac{1}{(Q_4-Q_2)(Q_1-Q_2)} + \frac{1}{Q_4}\frac{1}{(Q_1-Q_4)(Q_2-Q_4)}\right)\\
+ {} & \left(\frac{1}{Q_1}\frac{1}{Q_2-Q_1} + \frac{1}{Q_2}\frac{1}{Q_1-Q_2}\right)\left(\frac{1}{Q_3}\frac{1}{Q_4-Q_3} + \frac{1}{Q_4}\frac{1}{Q_3-Q_4}\right).
\end{aligned} \tag{2.2}
$$

We have $\xi_{dc} = 3$, corresponding to the three parts in $10 = (3 \times 1) + (3 \times 1) + (2 \times 2)$. The ten terms correspond to the five spanning trees, with each spanning tree defining two loops l_1, l_2, which contribute to the core coproduct as either $l_1 \otimes l_2$ or $l_2 \otimes l_1$.



For example choosing the spanning tree on edges e_2, e_3 in the Dunce's cap, the two loops are $l_1 = \{e_1, e_2, e_3\}, l_2 = \{e_3, e_4\}$. The term $l_1 \otimes l_2$ has l_1 as the sub-graph and $dc/l_1 = e_4$ as the co-graph. The corresponding term in $Q(Fl_{dc})$ is

$$\frac{1}{Q_4}\left(\frac{1}{Q_1}\frac{1}{(Q_2-Q_1)(Q_3-Q_1)}\right). \tag{2.3}$$

The term $l_2 \otimes l_1$ has l_2 as the sub-graph and $dc/l_2 = \{e_1, e_2\}$ as the co-graph. The corresponding term in $Q(Fl_{dc})$ is

$$\left(\frac{1}{Q_1}\frac{1}{Q_2-Q_1}\right)\left(\frac{1}{Q_4}\frac{1}{(Q_3-Q_4)}\right). \tag{2.4}$$

For both Eqs.(2.3,2.4) the conditions $Q_1 = 0$ and $Q_4 = 0$ determine $k_0 = \sqrt{s+m_1^2}$ and $l_0 = \sqrt{t+m_4^2}$. Both equations have by construction the same residue as

$$(Q_3 - Q_4)_{0=Q_1=Q_4} = (Q_3 - Q_1)_{0=Q_1=Q_4},$$

in accordance with the divided difference structure of contour integrals.

$|\vec{k}| < |\vec{l}|$

$|\vec{l}| < |\vec{k}|$

$$\left(\prod_{e\in T}\frac{1}{Q_e}\right)\Bigg|_{Q_f=0\ \forall f\notin T}$$

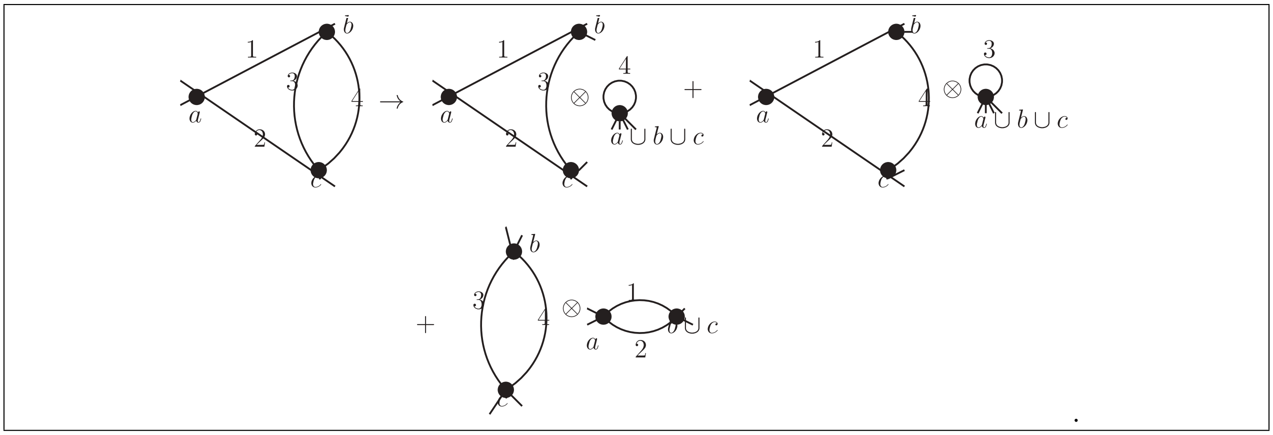


Figure 2: The flag decomposition of the Dunce's cap. Note that the Dunce's cap has five spanning trees and two loops. This gives ten terms which appear on the rhs by counting spanning trees in the sub- and co-graphs: $2\times 5 = 10 = 3\times 1 + 3\times 1 + 2\times 2.$ Indeed, the five spanning trees of the Dunce's cap are the five pairs of edges $\{(1,2),(1,3),(1,4),(2,3),(2,4)\}$. Each spanning tree defines a basis for the loops in the Dunce's cap by connecting the two endpoints of one of the edges not in the spanning tree by a path through that tree. With two edges not in the spanning tree and five spanning trees this gives a five element set. Looking at the flag decomposition on the right, the triangle on edges $1,2,3$ has three spanning trees given by the pairs $\{(1,2),(2,3),(3,1)\}$ while the tadpole has a single spanning tree given by the vertex $a\cup b\cup c$. The triangle on edges $1,2,4$ has three spanning trees given by the pairs $\{(1,2),(2,4),(4,1)\}$ while the tadpole again has a single spanning tree given by the vertex $a\cup b\cup c$. Each of the two edges $3,4$ of the bubble forms a spanning tree and similar each of the two edges $1,2$ of the other bubble.

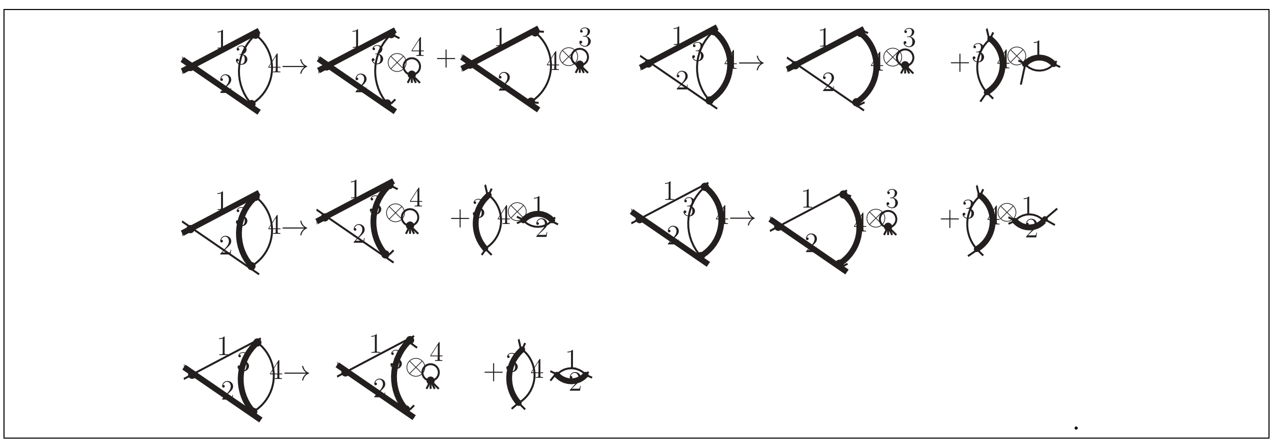


Figure 3: $\tilde{\Delta}_{GF}((G,T)) = Fl_{G,T}$ for G the Dunce' s cap and for its five spanning trees T.

What remains are $|G|$ integrations over variables s_i, to be integrated over the positive real half-axis $\mathbb{R}_+$. They span a $|G|$-dimensional hypercube $\mathbb{R}_+^{|G|}$. We dissect it into $|G|!$ sectors $s_i < s_j$, one for each possible ordering.

We perform the integrals over the energy variables as contour integrals. This is possible as the quadrics Q_{e_i} are quadratic, so the single presence of a factor $1/Q_{e_i}$ ensures convergence when closing the contour in the upper half-plane.

$|\tilde{K}.|$

~~Let $spt(G) \equiv |\mathcal{T}_G|$ be the number of spanning trees of G.~~

Lemma 2.5. *There are $|G|! \times spt(G) =: \mathbf{spt}(G)$ contributing residues.*

Proof. Consider a given spanning tree $T \in \mathcal{T}(G)$. The locus $\cap_{e \notin T} Q_e = 0$, defines $|G|!$ residues through the $|G|!$ possible orders of evaluation of $\prod_{e \in T} 1/Q_e$ corresponding to the $|G|!$ sectors in the above hypercube.

Consider

$$\mathrm{Res}_G(T) := \prod_{e \in E_T} \frac{1}{Q_e}.$$

For any chosen order and fixed chosen T, the contour integrals above deliver

$$\mathrm{Res}_G(T) = \left(\prod_{e \in E_T} \frac{1}{Q_e} \right)_{|l_{0,i} = +\sqrt{s_i + m_i^2}}.$$

Next, let us consider the set of residues in the energy integrals which can contribute. Come back to the cycle space of G. A choice of a spanning tree determines a basis for this space.

Choose an ordering of the cycles. This defines a sequence corresponding to a flag

$$l_1, l_2/l_1, \ldots, l_{|G|}/l_{|G|-1}/\cdots/l_1.$$

Now any choice of an ordering of the cycles, or equivalently of the edges $e \notin T$, defines the Feynman integral as an iterated integral, and therefore a sequence $s_1 > s_2 > \cdots > s_{|G|} > 0$. We get $\mathbf{spt}(G) = spt(G) \times |G|!$ such iterated integrals. □

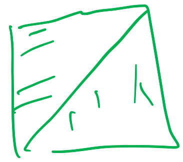
2.8. **The energy integral.** Summarising, we have

Theorem 2.6. *The energy integral I_G^0 is given as*

$$I_G^0 = \int_{-\infty}^{\infty} \prod_{i=1}^{|G|} dk_{i,0} \frac{1}{\prod_{e \in E_G} Q_e} = \sum_{i=1}^{\xi_G} \left(\prod_{j=1}^{|G|} \int d^{D-1}\vec{k}(j) \right) \times \prod_{j=1}^{|G|} \sum_{T \in \mathcal{T}(\gamma_j^{(i)})} \bar{\mathbf{pf}}(T)_{k_0(j)=+\sqrt{\vec{k}(j)^2 - m_{\hat{T}}^2 + i\eta}}.$$

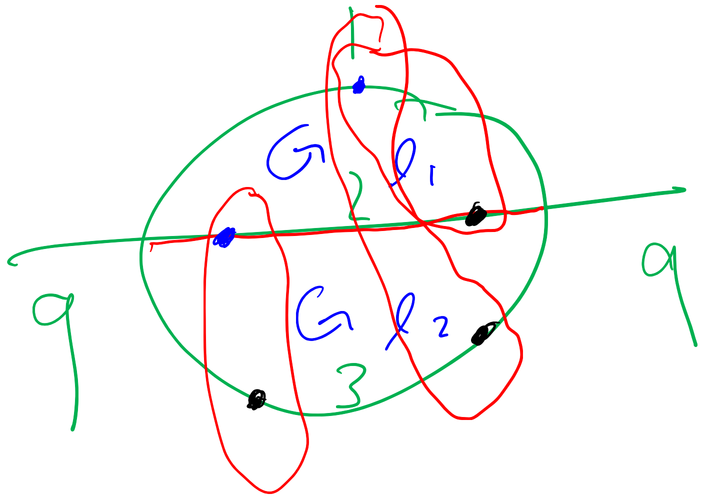
This can be written as a sum over all spanning trees of G together with a sum of all orderings of the space like integrations and for the full integral we find

$$\Phi_R^G = \sum_{T \in \mathcal{T}(G)} \sum_{\sigma \in S_{|G|}} \int_{0<\sigma(1)<\cdots<\sigma(|G|)} \left(\prod_{e \in E_T} \frac{1}{Q_e} \right)_{|k(j)_0^2 = s_j + m_j^2,\, j \notin E_T} \prod_{j \notin E_T} ds(j)$$



$\sigma_1 < \sigma_2 \simeq s(1) < s(2)$

2.9. **The three-edge banana.** Here, we see a new phenomenon. We have a two-loop graph, and the first question is for the number of contributing residues. This has a straight combinatorial answer which we will first exhibit on the example of the three-edge banana b_3.



3 sp trees $\Rightarrow 3 \cdot 2 = 6$ residue configurations

$$\frac{1}{Q_1 Q_2 Q_3}$$

Let us consider the integrand:

$$I_{b_3} = \frac{1}{Q_1 Q_2 Q_3} = \frac{1}{\underbrace{(k_0^2 - s - m_1^2)}_{1}\underbrace{(l_0^2 - t - m_2^2)}_{2}\underbrace{((k_0 - l_0 + q_0)^2 - s - t - 2\sqrt{s}\sqrt{t}z - m_3^2)}}.$$

This has to be integrated against $d^4k d^4l$, concretely

$$\int_0^\infty ds dt \int_1^1 dz \int_{-\infty}^\infty dk_0 dl_0.$$

$$\frac{\vec{k}\cdot\vec{l}}{|\vec{k}||\vec{l}|} = z$$

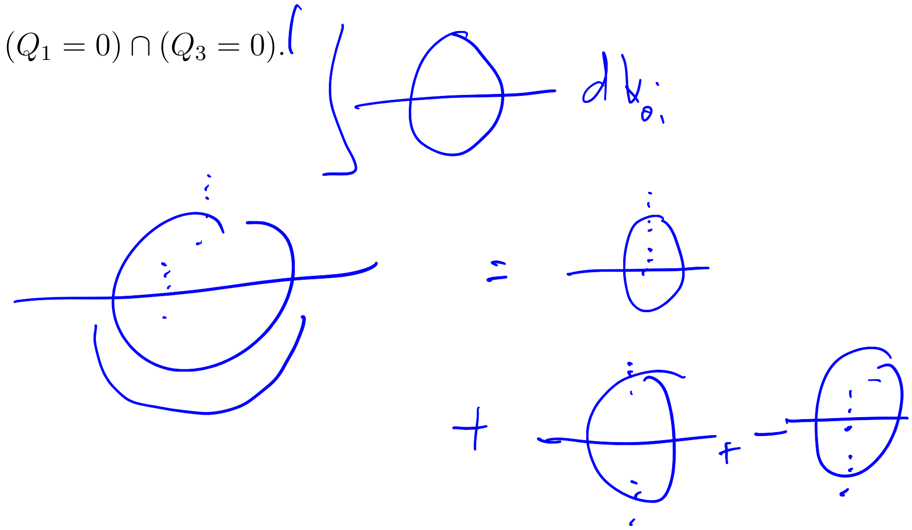
We do the k_0, l_0 integrals as contour integrals. Assume we integrate k_0 first and then l_0.

Q_1 and Q_3 have poles in the upper k_0 half-plane. Let us start with Q_1. Evaluating the product $1/(Q_2 Q_3)$ at the residue defined by $Q_1 = 0$ is a product which has two poles in the upper l_0 half-plane, at $Q_2 = 0$ or at $Q_3 = 0$. This gives two loci

$$(Q_1 = 0) \cap (Q_2 = 0),$$

and

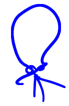
$$(Q_1 = 0) \cap (Q_3 = 0).$$



Next, the pole in the upper k_0 half-plane at $Q_3 = 0$ evaluates the residue of the product $1/(Q_1 Q_2)$ as a term with a single pole in the upper l_0 half-plane at $Q_2 = 0$ providing the locus

$$(Q_3 = 0) \cap (Q_2 = 0).$$

Altogether we find three loci. Precisely the same three loci are obtained when we interchange the order of the k_0, l_0 contour integrals.

The three loci correspond to

$$
\begin{aligned}
(Q_1 = 0) \cap (Q_2 = 0) \quad &\Leftrightarrow \quad \left(\frac{1}{Q_3 - Q_1} + \frac{1}{Q_3 - Q_2} \right) \frac{1}{Q_1 Q_2} \Leftrightarrow I_{l_1,e_3} \times I_{e_2} + I_{l_2,e_3} \times I_{e_1}, \\
(Q_2 = 0) \cap (Q_3 = 0) \quad &\Leftrightarrow \quad \left(\frac{1}{Q_1 - Q_2} + \frac{1}{Q_1 - Q_3} \right) \frac{1}{Q_2 Q_3} \Leftrightarrow I_{l_3,e_1} \times I_{e_3} + I_{l_1,e_1} \times I_{e_2}, \\
(Q_3 = 0) \cap (Q_1 = 0) \quad &\Leftrightarrow \quad \left(\frac{1}{Q_2 - Q_3} + \frac{1}{Q_2 - Q_1} \right) \frac{1}{Q_3 Q_1} \Leftrightarrow I_{l_2,e_2} \times I_{e_1} + I_{l_3,e_2} \times I_{e_3}.
\end{aligned}
$$

We have to discuss this in detail. Look at the first line. The locus $(Q_1 = 0)$ determines $k_0 = \sqrt{s + m_1^2}$ with $s + m_1^2 \in \mathbb{R}_+$ strictly positive. The locus $(Q_2 = 0)$ determines $l_0 = \sqrt{t + m_2^2}$ with again $t + m_2^2 \in \mathbb{R}_+$.

Intersecting the two loci evaluates

$$Q_3 = \left(\sqrt{s+m_1^2} - \sqrt{t+m_2^2} + q_0\right)^2 - s - t - 2\sqrt{s}\sqrt{t}z - m_3^2.$$

Now consider I_{b_3} with the three internal edges e_1, e_2, e_3. Assume we choose e_3 to define a spanning tree of the graph. This defines as a basis of the cycle space $\{l_1, l_2, l_3\}$ of b_3 the two loops generated by e_1, e_2 together with e_3:

$$l_1 = \{e_1, e_3\},$$

and

$$l_2 = \{e_2, e_3\}.$$

Also $l_3 = \{e_1, e_2\}$.

Furthermore we note that $b_3/l_1 = l_2/(l_1 \cap l_2) = \{e_2\}$ and $b_3/l_2 = l_1/(l_2 \cap l_1) = \{e_1\}$.

We have for the coproduct of the pair of the graph b_3 with its spanning tree e_3,

$$\tilde{\Delta}_{GF}((b_3, e_3)) = l_1 \otimes e_2 + l_2 \otimes e_1,$$

so that either l_1 or l_2 appear as subgraph.

The integrands for these sub- and co-graphs are

$$I_{l_1} = \frac{1}{Q_1 Q_3} = \frac{1}{Q_3 - Q_1}\left(\frac{1}{Q_1} - \frac{1}{Q_3}\right) =: I_{l_1,e_3} + I_{l_1,e_1},$$

$$I_{l_2} = \frac{1}{Q_2 Q_3} = \frac{1}{Q_3 - Q_2}\left(\frac{1}{Q_2} - \frac{1}{Q_3}\right) =: I_{l_2,e_3} + I_{l_2,e_2},$$

$$I_{e_1} = \frac{1}{Q_1},\ I_{e_2} = \frac{1}{Q_2}.$$

where we introduced the integrands for the pair of a graph with its spanning tree I_{l_j,e_k}.

For any choice of order of the contour integrals we do the $s = \vec{k}^2$ and $t = \vec{l}^2$ integrals in the same order as iterated integrals.

This gives us six integrals which converge in dimensional regularization so that we can treat the three loci separately.

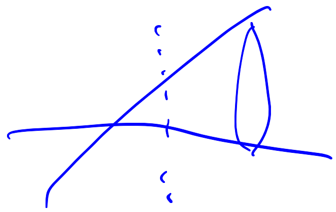
In particular in each of the three loci we can route the external momentum q through the remaining edge.

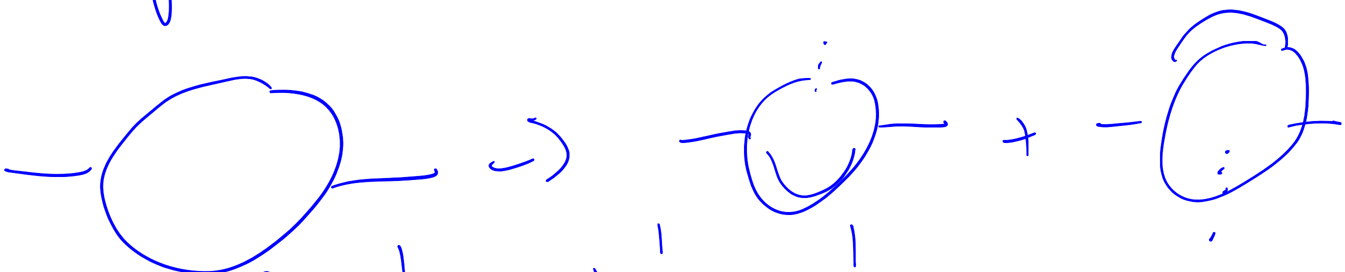
$$\Gamma \Rightarrow \sum_{T \in \mathcal{T}(\Gamma)} \left(G \Big|_{\text{cut } e \notin T} \right)$$

$$\Phi(G) = \int dk_{i_0} \int d\Pi k_j$$

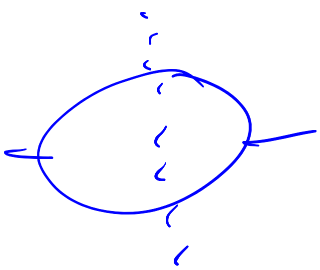
$$\operatorname{Res}^{\Pi}_{e \in T} \frac{1}{Q_e} \Bigg|_{Q_f = 0\ \forall f \notin T}$$

Standard integrals: with cut





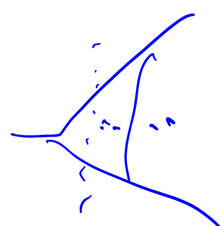
$$\frac{1}{Q_1 Q_2} = \underbrace{\frac{1}{Q_1}\frac{1}{Q_2 - Q_1}}_{} + \underbrace{\frac{1}{Q_2}\frac{1}{Q_1 - Q_2}}_{}$$



$$(Q_1 = 0) \cap (Q_2 = 0) \qquad Q_1 = Q_2$$

$$Q = 0 = Q_2$$

$$\int \delta_+(Q_1)\,\delta_+(Q_2)$$



$$2$$

$$k_0 = +\sqrt{\vec{k}^2 + m_1^2} \qquad \underline{\delta_+\left((k_0+q_0)^2 - \vec{k}^2 - m_2^2\right)}$$

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