FEYNMAN DIAGRAMS AND THE S-MATRIX, AND OUTER SPACE
(SUMMER 2020)

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1. Cutkosky graphs

We first consider graphs in $H_C^{(0)}$.

\[ H_C = \bigoplus H_C^{(i)} \]

$G \in H_C^{(i)}$ \( \Rightarrow \) \( \| G \| = i \)

$H_C^{(i)}$ \( \Rightarrow \) \( \| 0 \| = 0 \)

\[ \begin{array}{c}
\includegraphics[scale=0.5]{diagram1.png} \\
\includegraphics[scale=0.5]{diagram2.png} \\
\includegraphics[scale=0.5]{diagram3.png} \\
\includegraphics[scale=0.5]{diagram4.png}
\end{array} \]
1.1. An example. Now let us consider the case that we have a Cutkosky cut. Let us again first consider the bubble on two edges. The integrand is $1/(Q_1 Q_2)$ as before. But now we put both edges on-shell, meaning we are integrating over the locus $Q_1 = Q_2 = 0$ which is the intersection of the two loci $l_1 : Q_1 = 0$ and $l_2 : Q_2 = 0$.

\[ \left( \delta_+ (Q_1) \delta_+ (Q_2) \right) d^4 x \]

no partial fraction

$Q_1 = Q_2 \ ( = 0)$

\[ \frac{1}{Q_1 Q_2} = \frac{1}{Q_1} \frac{1}{Q_2 - Q_1} + \ldots \]
We cannot use a partial fraction as \( Q_1 = Q_2 \), but we can do the integral \( \Phi(b)_C \) for the cut bubble in a straightforward manner where the distributions \( \delta_+ \) restrict the domain of integration to the desired loci:

\[
\Phi(b)_C := \int d^4k \delta_+(Q_1) \delta_+(Q_2) = 4\pi \int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} \sqrt{s} \delta_+(k_0^2 - s - m_1^2) \delta_+((k_0 + q_0)^2 - s - m_2^2) ds.
\]

\[
S = x \cdot \bar{x}
\]

\[
\Omega = \frac{q^2}{6}
\]

\[
\delta_+ (k_0^2 - s - m_1^2)
\]

\[
k_0 = + \sqrt{s + m_1^2}
\]

\[
\Theta (k_r) \quad b_0 \geq 0
\]
To proceed we have to define integrands of Cutkosky graphs formally.

1.2. The general formula for $H_C^{(0)}$. Consider a Cutkosky graph $G \in H_C^{(0)}$ generated by a unique forest $F$ and associated set of edges $E_{on}$ with $e \in E_{on} \iff e \notin E_F$ so that $E_{on} \cup E_F = E_G$. 

\[ 2 \rightarrow 2 \]
\[ \epsilon + \epsilon \]
\[ \epsilon \in E_F \]
\[ \epsilon \in E_{on} \]
Then, 

\[ \Phi(G) = \int \prod_{j=1}^{[G]} d^{D}k_j \left( \frac{1}{\prod_{c \in B_c} Q_c} \right) \cap_{j \in B_{\text{on}}(Q_j=0)} \]

It remains to describe the threshold divisor preserved by \( \cap_{j \in B_{\text{on}}(Q_j=0)} \).
We first note that $|E_{on}| \geq |G|$. Next, consider $G/E_P$. This graph is a Cutkosky graph which has all its edges cut.

Any chosen refinement of $L_G$ which includes $F$ defines a refinement of $V_{G/E_P}$ and therefore a set of variables $k_{i,0}$ and $s_i$ which are determined by the set $E_{on}$. As $|E_{on}| \geq |G|$, all $k_{i,0}$ are fixed and so is $t$, where we set $s_i = t\tilde{s}_i$ for all $i$ and integrate $t$ over the positive real half-axis, whilst the $\tilde{s}_i$ are integrated in each ordering over the corresponding simplex $\Delta_{\sigma} \subset f_{G/E_P}$.

As a result, the $|E_{on}|$ constraints make sure that the remaining integrals are over a compactum $C_{G/E_P}$ and give the volume of $C_{G/E_P}$.

\[ G/E_P \rightarrow \]

\[ H(0) \]

\[ \tilde{s}_i : = t \tilde{s}_i \]

\[ \int \cdots \cdots \int \]

\[ k_{i,1,0} = \sqrt{s_i + \omega^2} \]

\[ \int \limits_{-\infty}^{\infty} dk_{i,0} \]
Now consider $G$ itself. The side-constraints are unchanged. The compactum $C_G$ fulfills
\[ \dim(C_G) = \dim(C_{G/E_F}) - e_F. \]

**Theorem 1.1.** For $G \in H_C^{(0)}$ with $h_0(F) \geq 2$, $\Phi(G)$ exists and determines a threshold $s_F(G)$ in the variable $s_0$ defined by the 2-refinement of $L_G$. 
2. THE TRIANGLE

Consider the one-loop triangle with vertices \( \{A, B, C\} \) and edges \( \{(A, B), (B, C), (C, A)\} \), and quadrics:

\[
\begin{align*}
P_{AB} &= k_0^2 - k_1^2 - k_2^2 - k_3^2 - M_1, \\
P_{BC} &= (k_0 + q_0)^2 - k_1^2 - k_2^2 - k_3^2 - M_2, \\
P_{CA} &= (k_0 - p_0)^2 - (k_1)^2 - (k_2)^2 - (k_3 - p_3)^2 - M_3.
\end{align*}
\]

Here, we Lorentz transformed into the rest frame of the external Lorentz 4-vector \( q = (q_0, 0, 0, 0)^T \), and oriented the space like part of \( p = (p_0, \vec{p})^T \) in the 3-direction: \( \vec{p} = (0, 0, p_3)^T \).

Using \( q_0 = \sqrt{q^2}, q_0 p_0 = q_\mu p^\mu \equiv q.p, \ \vec{p} \cdot \vec{p} = \frac{q^2 - q_\mu p^\mu}{q^2} \), we can express everything in covariant form whenever we want to.
We consider first the two quadrics $P_{AB}, P_{BC}$ which intersect in $\mathbb{C}^4$.

The real locus we want to integrate is $\mathbb{R}^4$, and we split this as $\mathbb{R} \times \mathbb{R}^3$, and the latter three dimensional real space we consider in spherical variables as $\mathbb{R} \times S^1 \times [-1,1]$, by going to coordinates $k_1 = \sqrt{s} \sin \phi \sin \theta, k_2 = \sqrt{s} \cos \phi \sin \theta, k_3 = \sqrt{s} \cos \theta, s = k_1^2 + k_2^2 + k_3^2, z = \cos \theta$.

$$\int \left( (k_1^2 + s - m_1^2 + i\gamma) \right)$$

$$\int \left( ((k_1 + s_1)^2 - s - m_2^2 + i\gamma) \right)$$
We have

\[ P_{AB} = k_0^2 - s - M_1, \]
\[ P_{BC} = (k_0 + q_0)^2 - s - M_2. \]

So we learn say \( s = k_0^2 - M_1 \) from the first and

\[ k_0 = k_r := \frac{M_2 - M_1 - q_0^2}{2q_0} \]

from the second, so we set

\[ s_r = \frac{M_2^2 + M_1^2 + (q_0^2)^2 - 2(M_1M_2 + q_0^2M_1 + q_0^2M_2)}{4q_0^2}. \]
The integral over the real locus transforms to

$$\int_{\mathbb{R}^4} d^4 k \to \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}^1} \sqrt{s} \delta_+(p_{AB}) \delta_+(p_{BC}) dk_0 ds \times \int_{-1}^{1} \int_{-1}^{1} d\phi \delta_+(p_{CA}) dz.$$ 

We consider $k_0, s$ to be base space coordinates, while $p_{CA}$ also depends on the fibre coordinate $z = \cos \theta$. Nothing depends on $\phi$ (for the one-loop box it would).

\[ C \]

\[ (k_0 - \rho_0)^2 - \bar{k} \cdot \bar{k} - F \cdot \bar{F} \]

\[ \frac{1}{2} - 2 \frac{\bar{p}}{\sqrt{s}} \frac{z}{2} - \frac{m^2}{2} \]

\[ \delta_4(p_{AB}) \delta_4(p_{BC}) \delta_4(p_{CA}) \]
Integrating in the base and integrating also \( \phi \) trivially in the fibre gives

\[
\frac{1}{2} \pi \int_{-1}^{1} \frac{\sqrt{s_r}}{q_0} 2 \delta_+(P_{CA}(s = s_r, k_0 = k_r)) dz.
\]

For \( P_{CA} \) we have

\[
P_{CA} = (k_r - p_0)^2 - s_r - \vec{p} \cdot \vec{p} - 2|\vec{p}| \sqrt{s_r} z - M_3.
\]

Integrating the fibre gives a very simple expression (the Jacobian of the \( \delta \)-function is \( 1/2\sqrt{s_r}|\vec{p}| \)), and we are left with

\[
\frac{\pi}{4|\vec{p}| q_0^2}.
\]

This is the famous Omnès factor.
This contributes as long as the fibre variable

\[ z = \frac{(k_r - p_0)^2 - s_r - \vec{p} \cdot \vec{p} - M_3}{2|p|\sqrt{s_r}} \]

lies in the range \((-1, 1)\). This is just the condition that the three quadrics intersect.

An anomalous threshold below the normal threshold appears when \((m_1 - m_2)^2 < q^2 < (m_1 + m_2)^2\). In that range, \(s_r\) is negative, hence its square root imaginary. It follows that \(z\) can be real only for \(z = 0\), and this delivers

\[ s_r = (k_r - p_0)^2 - \vec{p} \cdot \vec{p} - M_3, \]

which is negative for sufficiently large \(M_3\), as expected.
As the integrand does not depend on $\phi$, this gives a result of the form

$$2\pi C \int_{-1}^{1} \frac{1}{\alpha + \beta z} \, dz = 2\pi C \frac{\ln \frac{\alpha + \beta}{\alpha - \beta}}{\beta},$$

where $C$ is intuitively related to $\text{Var}(\Phi_R(b_2)) = 2C$, and the factor $2$ here is $\text{Vol}(S^2)/\text{Vol}(S^1)$.

$$c \sim \sqrt{\lambda (\xi_1^2 + \xi_2^2)}.$$
Here, $\alpha$ and $\beta$ are given through $l_1 := \lambda(s, p^2, p^2)$ and $l_2 := \lambda(s, m_\alpha^2, m_\alpha^2)$ as
\[
\alpha := (m_y^2 - m_t^2 - s - p_a, p_c)^2 - l_1 - l_2, \quad \beta := 2\sqrt{l_1 l_2}
\]
Accordingly, we find a single generator for the monodromy in the complement of the threshold divisors: either for the normal threshold at $s_0 = (m_r + m_y)^2$ or for the anomalous threshold at $s_1$, with $l_r = p^2 - m_r^2$, $l_y = (p + q)^2 - m_y^2$, $l_1 = \lambda(p^2, m_r^2, m_y^2)$, $\lambda_1 = \lambda((p + q)^2, m_\alpha^2, m_\beta^2)$ it is [?] given as,
\[
\tag{2.1}
\frac{4m_y^2(\sqrt{\lambda_2}m_r - \sqrt{\lambda_1}m_y)^2 - (\sqrt{\lambda_1}l_y + \sqrt{\lambda_2}l_r)^2}{4m_y^2\sqrt{\lambda_1}\sqrt{\lambda_2}}.
\]
We will derive it later using the parametric representation.

Today: $H_c$

\[H_c \quad \text{parametric reps.}\]

Quadric reps (Fornlum '98)

Good for $H_c$

$H_c \rightarrow$ parametrically

$H_c$ hybrid method

Have interpretations in orbifold space
3. Generic graphs: Using the co-action