FEYNMAN DIAGRAMS AND THE S-MATRIX, AND OUTER SPACE (SUMMER 2020)

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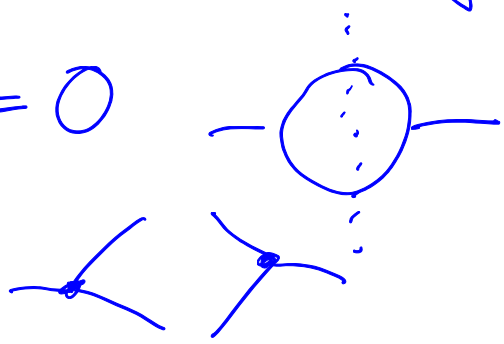
1. Cutkosky graphs

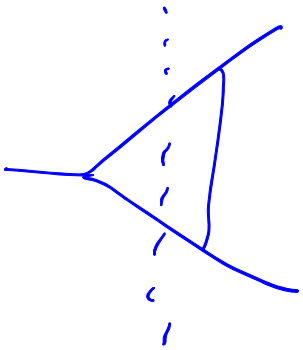
We first consider graphs in $H_C^{(0)}$.

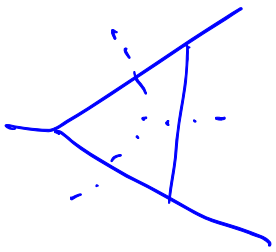
$$H_C = \bigoplus H_C^{(j)}$$

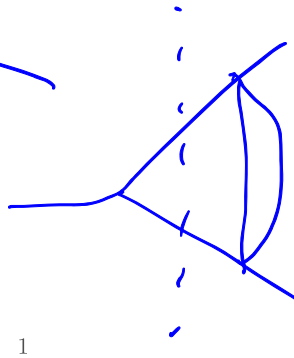
$$G \in H_C^{(j)} \iff \|G\| = j$$

$$H_C^{(0)} \iff \|G\| = 0$$



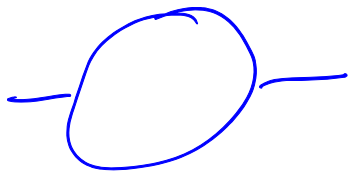






$\in H_C^{(7)}$

1.1. **An example.** Now let us consider the case that we have a Cutkosky cut. Let us again first consider the bubble on two edges. The integrand is $1/(Q_1 Q_2)$ as before. But now we put both edges on-shell, meaning we are integrating over the locus $Q_1 = Q_2 = 0$ which is the intersection of the two loci $l_1 : Q_1 = 0$ and $l_2 : Q_2 = 0$.



$$\Longleftrightarrow \frac{1}{Q_1 Q_2}$$

$$\int \delta_+(Q_1)\,\delta_+(Q_2)\, d^4 k$$

no partial fraction

$$Q_1 = Q_2 \quad (= 0)$$

$$\frac{1}{Q_1 Q_2} = \frac{1}{Q_1}\,\frac{1}{Q_2 - Q_1} + \cdots$$

We can not use a partial fraction as $Q_1 = Q_2$, but we can do the integral $\Phi(b)_C$ for the cut bubble in s straightforward manner where the distributions δ_+ restrict the domain of integration to the desired loci:

$$\Phi(b)_C := \int d^4k \delta_+(Q_1)\delta_+(Q_2) = 4\pi \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} \sqrt{s}\delta_+(k_0^2 - s - m_1^2)\delta_+((k_0+q_0)^2 - s - m_2^2)ds.$$

$$s = \vec{k} \cdot \vec{k} \qquad q^2 = q_0^2$$

$$\delta_+(k_0^2 - s - m_1^2)$$

$$k_0 = +\sqrt{s + m_1^2} \qquad \theta(k_0) \quad k_0 \geq 0$$

$$\delta(f(x)) = \sum_{f(x_i)=0} \frac{1}{|f'|} \quad)$$

Doing the k_0 integral we find

$$\Phi(b)_C = \int_0^\infty \frac{\sqrt{s}}{\sqrt{s+m_1^2}} \delta\left(\left(\sqrt{s+m_1^2}+q_0\right)^2 - s - m_2^2\right) ds.$$

This determines

$$s = \frac{\lambda(q^2, m_1^2, m_2^2)}{4q^2},$$

$$\lambda = q^2 + m_1^2 + m_2^2 - 2(q^2 m_1^2 + \dots)$$

hence

$$\Phi(b)_C = \frac{\sqrt{\lambda(q^2, m_1^2, m_2^2)}}{2q^2} \Theta(q^2 - (m_1+m_2)^2).$$

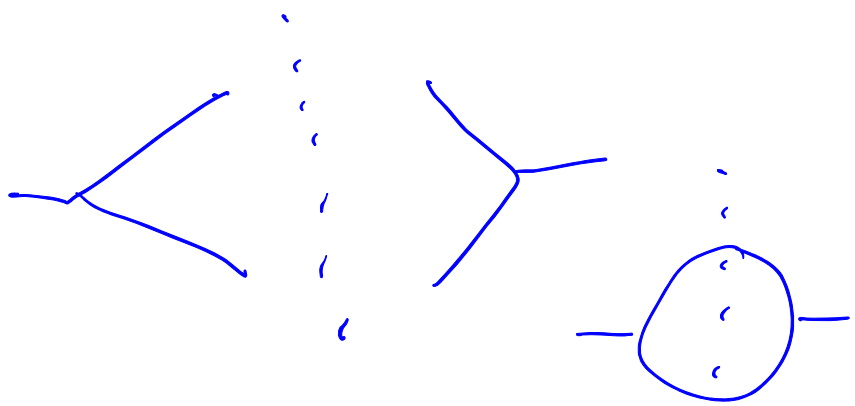
on the principal sheet.

$$\lambda(q^2, m_1^2, m_2^2) \geq 0 \qquad q^2 \geq (m_1 \pm m_2)^2$$

$$\lambda(q^2, m_1^2, m_2^2) \leq 0 \qquad (m_1 - m_2)^2 < q^2 < (m_1 + m_2)^2$$

$$S = \mathbb{1} + iT$$

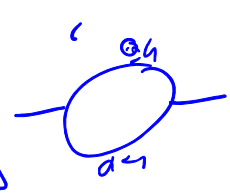
$$\mathrm{Im}\, T = \sum T T^\dagger$$



$$\frac{1}{k^2 - m^2}$$

$$CP \frac{1}{k^2 - m^2} + i\pi\delta(k^2 - m^2)$$

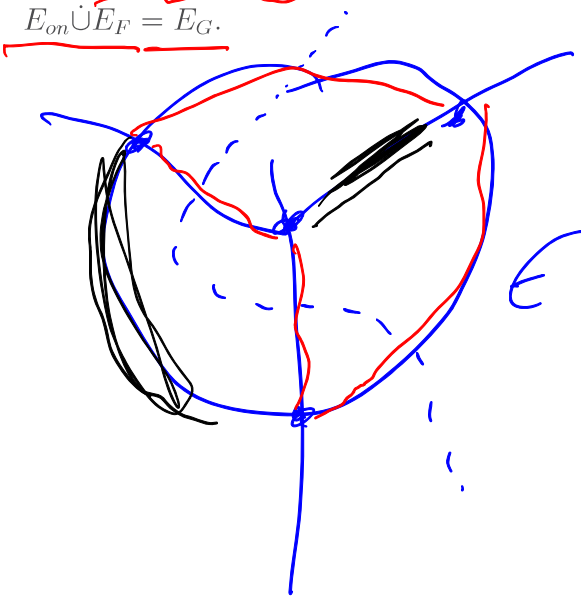
$$= \int$$



LHS

To proceed we have to define integrands of Cutkosky graphs formally.

1.2. **The general formula for $H_C^{(0)}$.** Consider a Cutkosky graph $G \in H_C^{(0)}$ generated by a unique forest F and associated set of edges E_{on} with $e \in E_{on} \Leftrightarrow e \notin E_F$ so that $E_{on} \dot\cup E_F = E_G$.

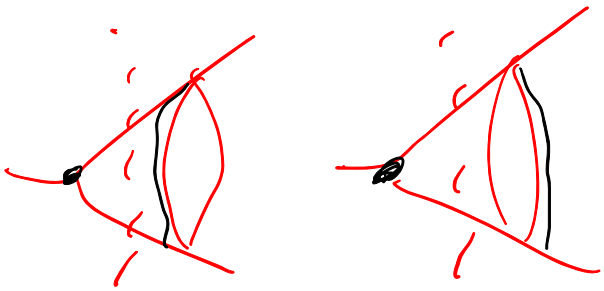


$2 \to 2$

$\in H_c^{(0)}$

$- \in E_F$

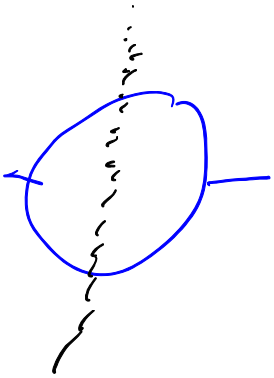
$- \in E_{on}$



Then,

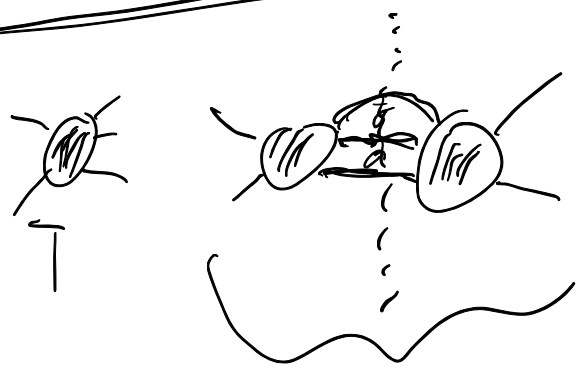
$$\Phi(G) = \int \prod_{j=1}^{|G|} d^D k_j \left(\frac{1}{\prod_{e \in E_F} Q_e} \right)_{\cap_{f \in E_{on}} (Q_f = 0)} .$$

It remains to describe the threshold divisor ~~prescibed~~ by $\cap_{f \in E_{on}} (Q_f = 0)$.

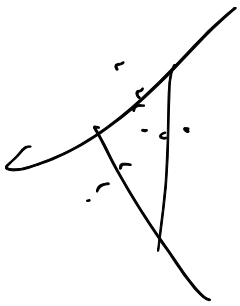


$S = \Pi + i T$

$\mathrm{Im}\, T = \sum T T^{+}$?



Cut kos kg



$k_0 = + \sqrt{\ }$

$Q_1 = 0$

$k_0^2 = \vec{k}^2 + m_1^2$

$Q_2 = (k_0 + p_0)^2 - \vec{k}^2 - m_2^2 = 0$

$\uparrow$ $\quad \vec{k}^2 = s$ $\quad \hookrightarrow 4\pi$

$\frac{1}{Q_3}$

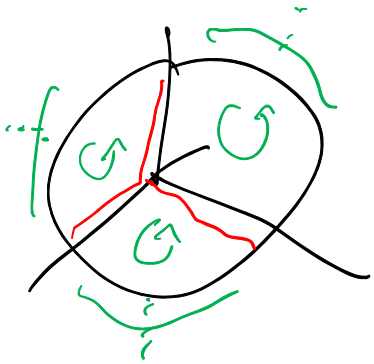
$k_0 = \sqrt{\ }$

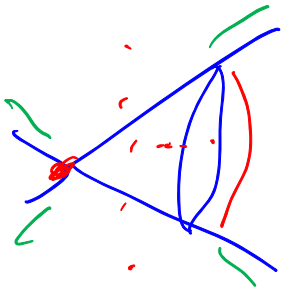
$s = \ldots \quad \sqrt{\frac{\lambda}{4 p^2}}$

We first note that $|E_{on}| \gneq |G|$. Next, consider G/E_F. This graph is a Cutkosky graph which has all its edges cut.

Any chosen refinement of L_G which includes F defines a refinement of V_{G/E_F} and therefore a set of variables $k_{i,0}$ and s_i which are determined by the set E_{on}. As $|E_{on}| \gneq ||G||$, all $k_{i,0}$ are fixed and so is t, where we set $s_i = t\tilde{s}_i$ for all i and integrate t over the positive real half-axis, whilst the $\tilde{s}_i$ are integrated in each ordering over the corresponding simplex $\Delta_\sigma \subset I^{|G/E_F|}$.

As a result, the $|E_{on}|$ constraints make sure that the remaining integrals are over a compactum C_{G/E_f} and give the volume of C_{G/E_F}.

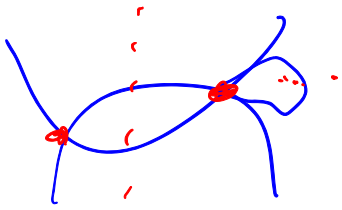




G/E_F

$\Rightarrow$

$H_L^{(0)}$



$s_i = t\tilde{s}_i$

$\int_{-\infty}^{+\infty} dk_{i0}$

$k_{i,0} = \sqrt{s_i + m_i^2}$

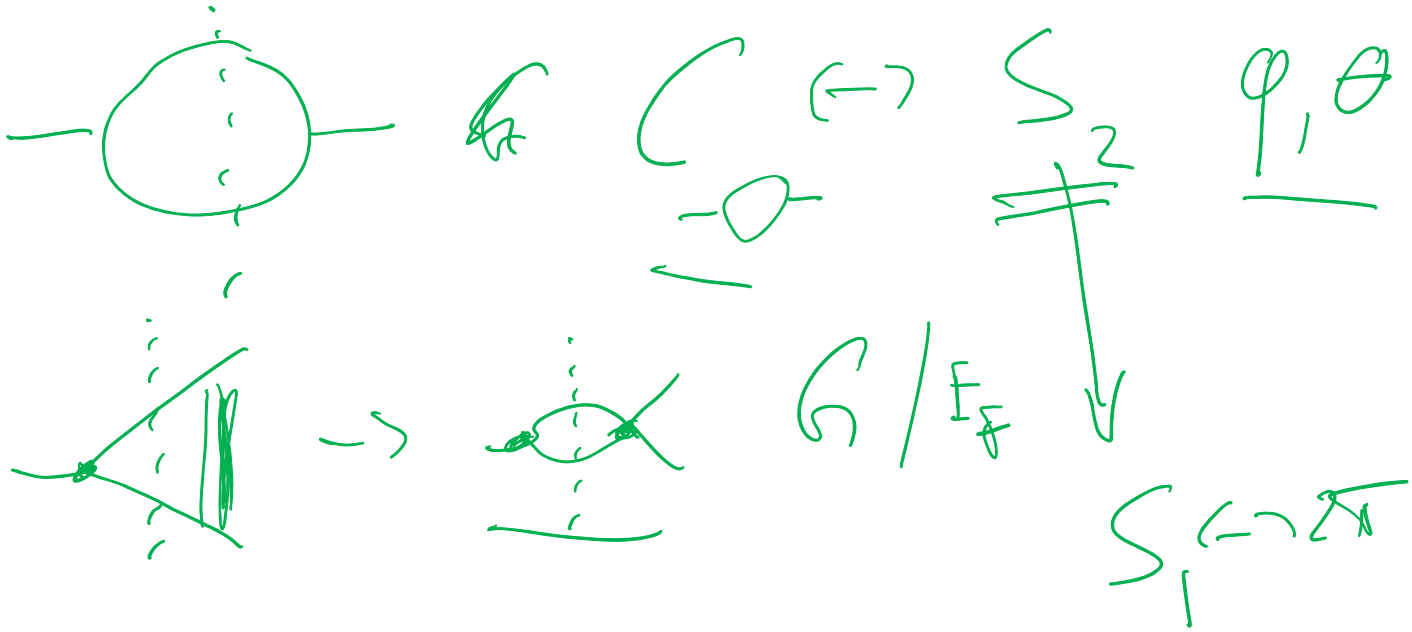
fixed

finite one compact manifolds but complicated

Now consider G itself. The side-constraints are unchanged. The compactum C_G fulfills

$$\dim(C_G) = \dim(C_{G/E_f}) - e_F.$$

Theorem 1.1. *For $G \in H_C{}^{(0)}$ with $h_0(F) \geq 2$, $\Phi(G)$ exists and determines a threshold $s_F(G)$ in the variable s_0 defined by the 2-refinement of L_G.*



2. THE TRIANGLE

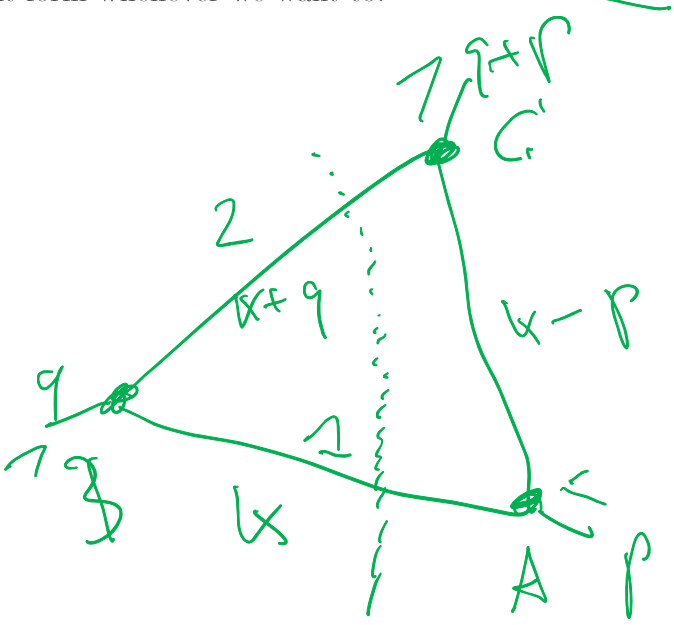
Consider the one-loop triangle with vertices $\{A, B, C\}$ and edges $\{(A, B), (B, C), (C, A)\}$, and quadrics:

$$P_{AB} = k_0^2 - k_1^2 - k_2^2 - k_3^2 - M_1,$$
$$P_{BC} = (k_0 + q_0)^2 - k_1^2 - k_2^2 - k_3^2 - M_2,$$
$$P_{CA} = (k_0 - p_0)^2 - (k_1)^2 - (k_2)^2 - (k_3 - p_3)^2 - M_3.$$

Here, we Lorentz transformed into the rest frame of the external Lorentz 4-vector $q = (q_0, 0, 0, 0)^T$, and oriented the space like part of $p = (p_0, \vec{p})^T$ in the 3-direction: $\vec{p} = (0, 0, p_3)^T$.

Using $q_0 = \sqrt{q^2}$, $q_0 p_0 = q_\mu p^\mu \equiv q.p$, $\vec{p} \cdot \vec{p} = \frac{q.p^2 - p.p q.q}{q^2}$, we can express everything in covariant form whenever we want to.

$$\int_{-\infty}^{\infty} dk_0 \int_0^{\infty} ds \int_{-1}^{1} dz$$



We consider first the two quadrics P_{AB}, P_{BC} which intersect in $\mathbb{C}^4$.

The real locus we want to integrate is $\mathbb{R}^4$, and we split this as $\mathbb{R} \times \mathbb{R}^3$, and the latter three dimensional real space we consider in spherical variables as $\mathbb{R}_+ \times S^1 \times [-1, 1]$, by going to coordinates $k_1 = \sqrt{s} \sin\phi \sin\theta$,$k_2 = \sqrt{s} \cos\phi \sin\theta$, $k_3 = \sqrt{s} \cos\theta$, $s = k_1^2 + k_2^2 + k_3^2$, $z = \cos\theta$.

$$\delta_+ \left(k_1^2 - s - m_1^2 + i\eta \right)$$

$$\delta_+ \left((k_1 + q_0)^2 - s - m_2^2 + i\eta \right)$$

We have

$$P_{AB} = k_0^2 - s - M_1,$$
$$P_{BC} = (k_0 + q_0)^2 - s - M_2.$$

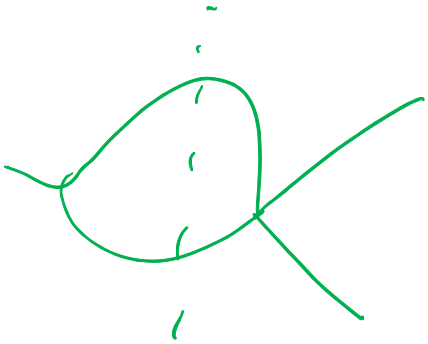
So we learn say $s = k_0^2 - M_1$ from the first and

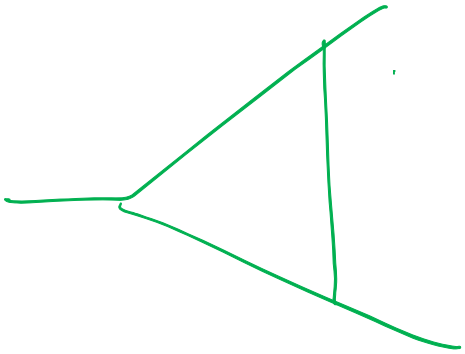
$$k_0 = k_r := \frac{M_2 - M_1 - q_0^2}{2q_0} = \frac{\lambda(M_1^2, M_2^2, q_0^2)}{4 s_r^2}$$

from the second, so we set

$$s_r := \frac{M_2^2 + M_1^2 + (q_0^2)^2 - 2(M_1 M_2 + q_0^2 M_1 + q_0^2 M_2)}{4q_0^2}.$$

like the bubble



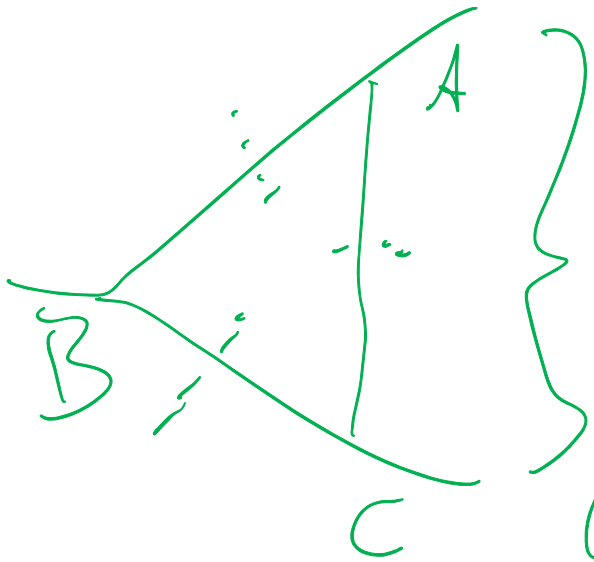


2π

The integral over the real locus transforms to

$$\int_{\mathbb{R}^4} d^4k \to \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}_+} \sqrt{s}\delta_+(P_{AB})\delta_+(P_{BC})dk_0 ds \times \int_0^{2\pi}\int_{-1}^{1} d\phi\delta_+(P_{CA})dz.$$

We consider k_0, s to be base space coordinates, while P_{CA} also depends on the fibre coordinate $z = \cos\theta$. Nothing depends on ϕ (for the one-loop box it would).



P_{CA} involving z

$$(k_0 - p_0)^2 - \underbrace{\vec{k}\cdot\vec{k}}_{s} - \vec{p}\cdot\vec{p} - 2|\vec{p}|\sqrt{s}\,z - m_3^2$$

determined by $\delta_+(P_{BC})$ $\delta_+(P_{BA})$

Integrating in the base and integrating also ϕ trivially in the fibre gives

$$\frac{1}{2}\frac{\sqrt{s_r}}{2q_0}2\pi\int_{-1}^{1}\delta_+(P_{CA}(s=s_r,k_0=k_r))dz.$$

For P_{CA} we have

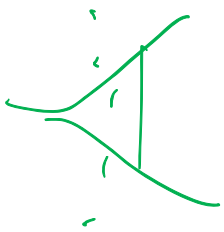
$$P_{CA}=(k_r-p_0)^2-s_r-\vec{p}\cdot\vec{p}-2|\vec{p}|\sqrt{s_r}z-M_3.$$

$\alpha + \beta z$

Integrating the fibre gives a very simple expression (the Jacobian of the δ-function is $1/(2\sqrt{s_r}|\vec{p}|)$, and we are left with

$$\frac{\pi}{4|\vec{p}|q_0}.$$

This is the famous Omnès factor.



$$\sim \frac{1}{\sqrt{\lambda(q^2, p^2, (p+q)^2)}}$$

$$\int_{-1}^{1} dz \, \frac{1}{\underbrace{P_{cA}}_{\alpha+\beta z}}$$

This contributes as long as the fibre variable

$$z = \frac{(k_r - p_0)^2 - s_r - \vec{p}\cdot\vec{p} - M_3}{2|\vec{p}|\sqrt{s_r}}$$

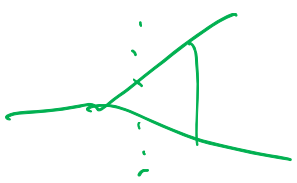
lies in the range $(-1, 1)$. This is just the condition that the three quadrics intersect.

An anomalous threshold below the normal theshold appears when $(m_1 - m_2)^2 < q^2 < (m_1 + m_2)^2$. In that range, s_r is negative, hence its square root imaginary. It follows that z can be real only for $z = 0$, and this delivers

$$s_r = (k_r - p_0)^2 - \vec{p}\cdot\vec{p} - M_3,$$

which is negative for sufficiently large M_3, as expected.

$q^2 \geq (m_1 + m_2)^2$ normal threshold



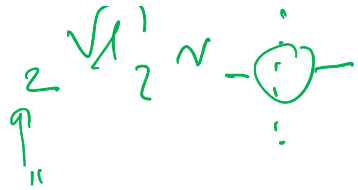
$$\int_{-1} \frac{1}{\alpha + \beta z} dz$$

As the integrand does not depend on ϕ, this gives a result of the form

$$2\pi C \underbrace{\int_{-1}^{1} \frac{1}{\alpha + \beta z} dz}_{:=J(z)} = 2\pi C \frac{\ln \frac{\alpha+\beta}{\alpha-\beta}}{\beta},$$

where C is intimitaly related to $\mathrm{Var}(\Phi_R(b_2)) = 2C$, and the factor 2 here is $\mathrm{Vol}(S^2)/\mathrm{Vol}(S^1)$.

$$C \sim \sqrt{\lambda(s, m_1^2, m_2^2)}$$



Here, α and β are given through $l_1 := \lambda(s, p_b^2, p_c^2)$ and $l_2 := \lambda(s, m_y^2, m_r^2)$ as

$$\alpha := (m_y^2 - m_r^2 - s - p_a.p_c)^2 - l_1 - l_2,\ \beta := 2\sqrt{l_1 l_2}.$$

Accordingly, we find a single generator for the monodromy in the complement of the threshold divisors: either for the normal threshold at $s_0 = (m_r + m_y)^2$ or for the anomalous threshold at s_1, with $l_r = p^2 - m_r^2 - m_b^2, l_y = (p+q)^2 - m_y^2 - m_b^2,\ \lambda_1 = \lambda(p^2, m_r^2, m_b v^2),\ \lambda_1 = \lambda((p+q)^2, m_y^2, m_b v^2)$ it is [?] given as,

$$s_1 = (m_r + m_y)^2 + \frac{4m_b^2(\sqrt{\lambda_2} m_r - \sqrt{\lambda_1} m_y)^2 - (\sqrt{\lambda_1} l_y + \sqrt{\lambda_2} l_r)^2}{4m_b^2 \sqrt{\lambda_1}\sqrt{\lambda_2}}. \tag{2.1}$$

We will derive it later using the parametric representation.

Today: $H_c^{(0)}$

$H_c^{(j)}$ parametric reps.

Quadric reps (Feynman rules)

good for $H_c^{(0)}$

H_{cov} $\longleftrightarrow$ parametrically

$H_c^{(j)}$ hybrid method

have interpretations in Orb space

3. Generic graphs: Using the co-action

Humboldt U. Berlin