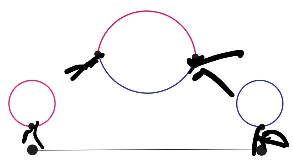
FEYNMAN DIAGRAMS AND THE S-MATRIX, AND OUTER SPACE (SUMMER 2020)

DIRK KREIMER (LECT. APRIL 22, 2020)

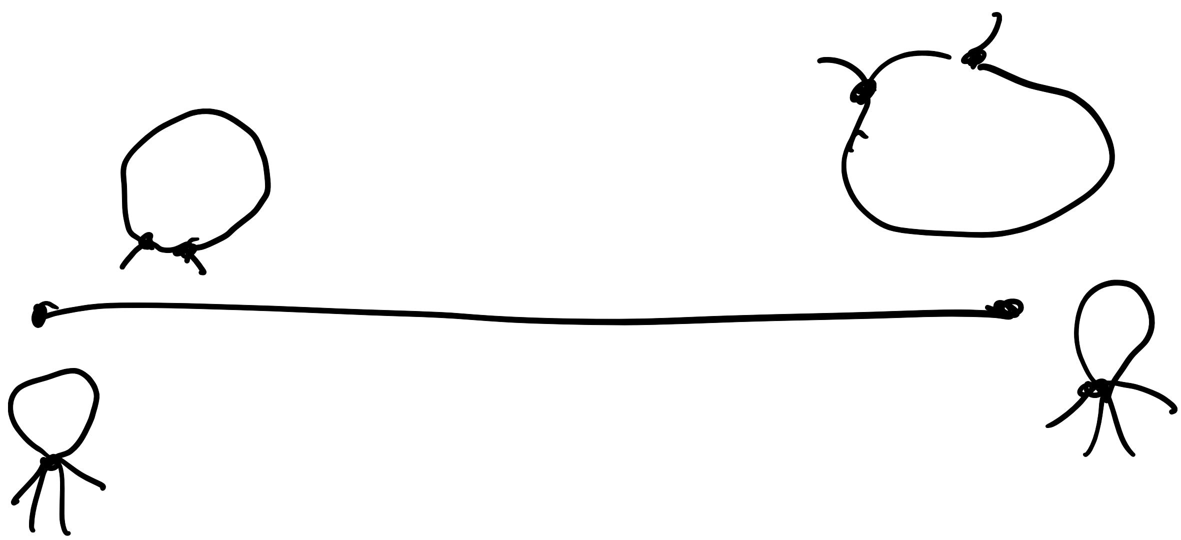
1. Motivation: Outer Space

We first discuss the elementary monodromy of the simplest one-loop graph. So we start with the 2-edge banana b_2, a bubble on two edges with two different internal masses m_b, m_r, indicated by two different colors:



The incoming external momenta at the two vertices of b_2 are $q, -q$.

We assign to b_2 a one-dimensional cell, an open line segment, and glue in its two boundary endpoints, to which the two tadpoles on the two different masses are assigned, obtained by either shrinking the blue or red edge. The vertex at each tadpole is then 4-valent, with no external momentum flow through the graph.



The fundamental group

$$\Pi_1(b_2) \sim \mathbb{Z}$$

of b_2 has a single generator. This matches with the monodromy of the function $\Phi_R(b_2)$ as we see in a moment.

Indeed, the Feynman integral we consider is coming from renormalized Feynman rules $\Phi_R(b_2)$, where we implement a kinematic renormalization scheme by subtraction at $\mu^2 < (m_b - m_r)^2$ (so that the subtracted terms does not have an imginary part, as μ^2 is even below the pseudo threshold):

$$\Phi_R(b_2) = \int d^4k \left(\underbrace{\frac{1}{k^2 - m_r^2}}_{Q_1} \underbrace{\frac{1}{(k+q)^2 - m_b^2}}_{Q_2} - \{q^2 \to \mu^2\} \right).$$

We set $s := q^2$ and demand $s > 0$, and also set $s_0 := \mu^2$.

We write $k = (k_0, \vec{k})^T$, $t := \vec{k} \cdot \vec{k}$. As the 4-vector q is assumed time-like (as $s > 0$) we can work in a coordinate system where $q = (q_0, 0, 0, 0)^T$ and get

$$\Phi_R(b_2) = 4\pi \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} \sqrt{t} dt \left(\frac{1}{k_0^2 - t - m_r^2} \frac{1}{(k_0 + q_0)^2 - t - m_b^2} - \{s \to s_0\} \right).$$

We define the Kȧllen function

$$\lambda(a, b, c) := a^2 + b^2 + c^2 - 2(ab + bc + ca),$$

and find by explicit integration

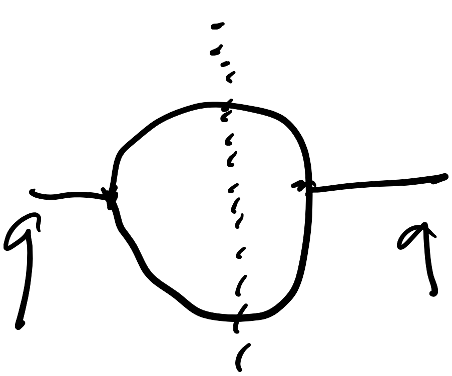
$$\Phi_R(b_2)(s, s_0; m_r^2, m_b^2) =$$

$$= \left(\underbrace{\frac{\sqrt{\lambda(s, m_r^2, m_b^2)}}{2s} \ln \frac{m_r^2 + m_b^2 - s - \sqrt{\lambda(s, m_r^2, m_b^2)}}{m_r^2 + m_b^2 - s + \sqrt{\lambda(s, m_r^2, m_b^2)}} - \frac{m_r^2 - m_b^2}{2s} \ln \frac{m_r^2}{m_b^2}}_{B_2(s)} \right.$$

$$\left. - \underbrace{\{s \to s_0\}}_{B_2(s_0)} \right).$$

The principal sheet of the above logarithm is real for $s \leq (m_r + m_b)^2$ and free of singularities at $s = 0$ and $s = (m_r - m_b)^2$. It has a branch cut for $s \geq (m_r + m_b)^2$.

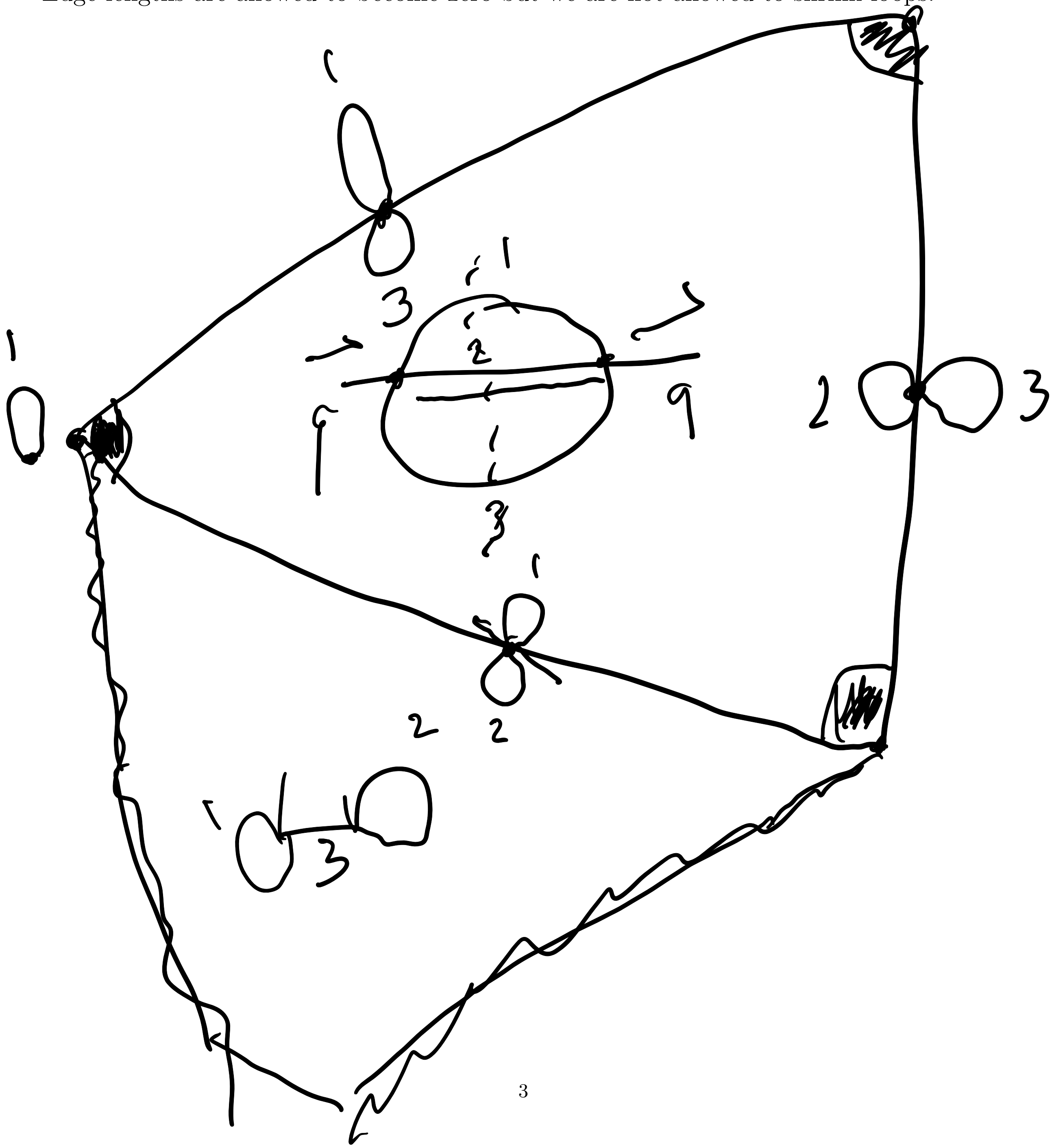
The threshold divisor defined by the intersection $Q_1 \cap Q_2$ where the zero locii of the quadrics meet is at $s = (m_b + m_r)^2$.



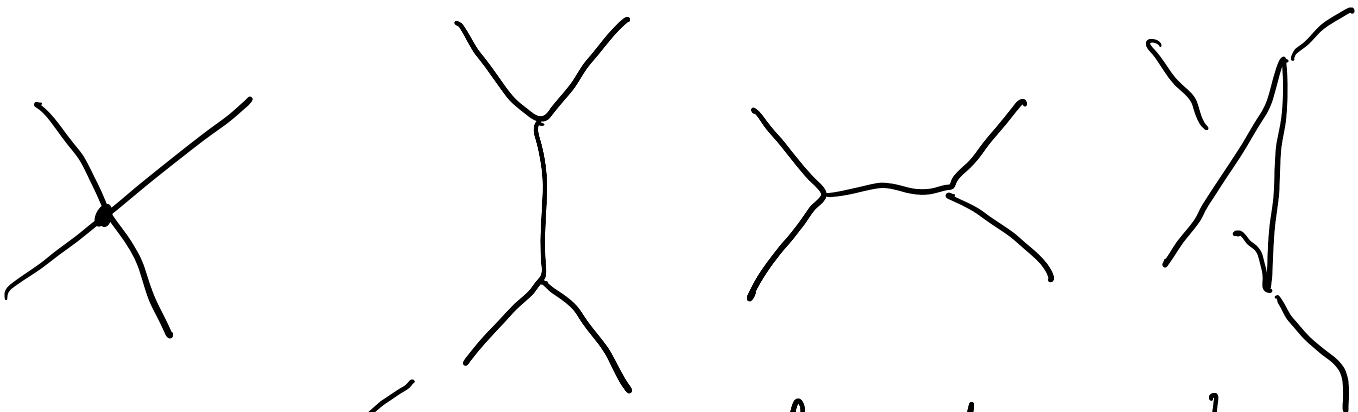
We now first describe Outer Space. What we use is actually a variant in which there are external edges at vertices, and internal edges are colored to allow for different types of internal propagators. Here, different colors indicate generic different internal masses, but could also be used as placeholders for different spin and more.

1.1. **The set-up of colored Outer Space.** Outer Space can be regarded as a collection of open simplices. For a graph with k edges, we assign an open simplex of dimension $k-1$. We can either demand that the sum of edge lengths (given by parametric variables A_e) adds to unity, or work in projective space $\mathbb{P}^{k-1}(\mathbb{R}_+)$ in such a cell. Each graph comes with a metric, and one moves around the cell by varying the edge lengths.

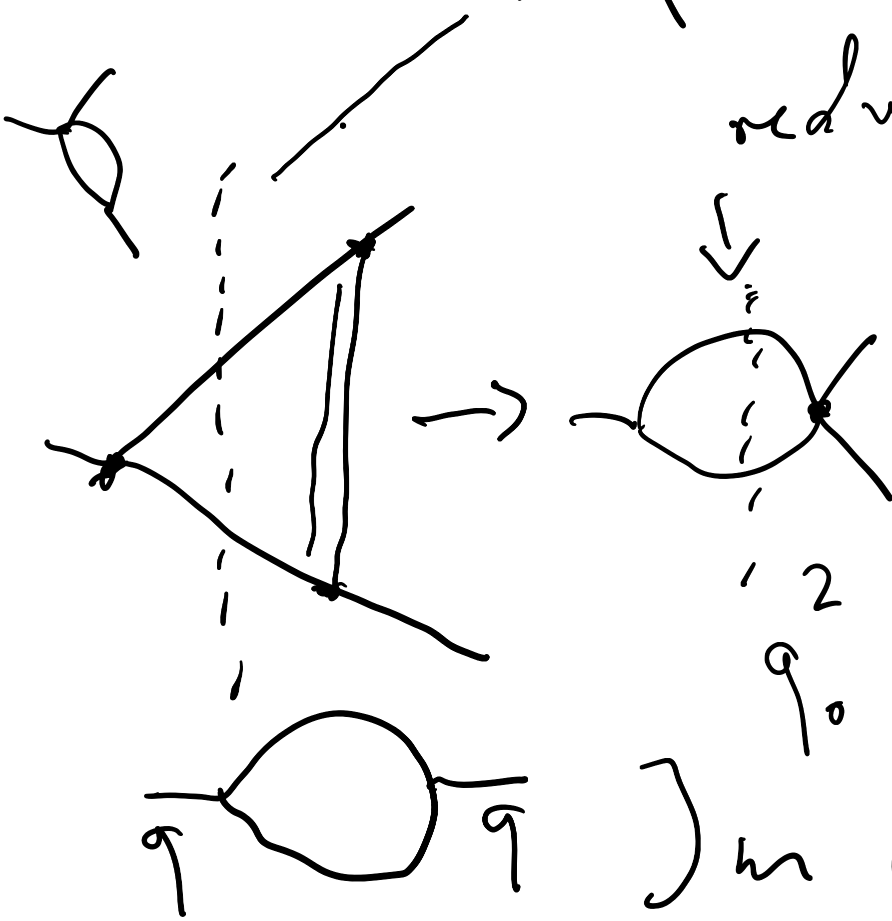
Edge lengths are allowed to become zero but we are not allowed to shrink loops.



When an edge say between two three-valent vertices shrinks to zero length, there are several ways to resolve the resulting 4-valent vertex to obtain a new nearby graph: assume we have a 4-valent vertex in a graph G sitting in a $(k-1)$-dimensional cell. Then, this cell can be glued in as a common boundary of three other k-dimensional cells with corresponding graphs G_i, $i \in \{s,t,u\}$, which have an edge e connecting two 3-valent vertices, such that $G_i/e = G$, where G_i/e is the graph where edge e shrinks to zero length.



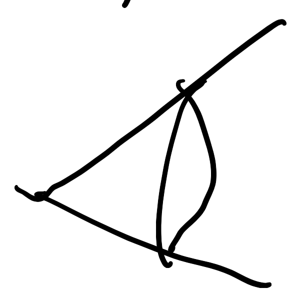
reduced graphs



$q = (q_0, \vec{0})$

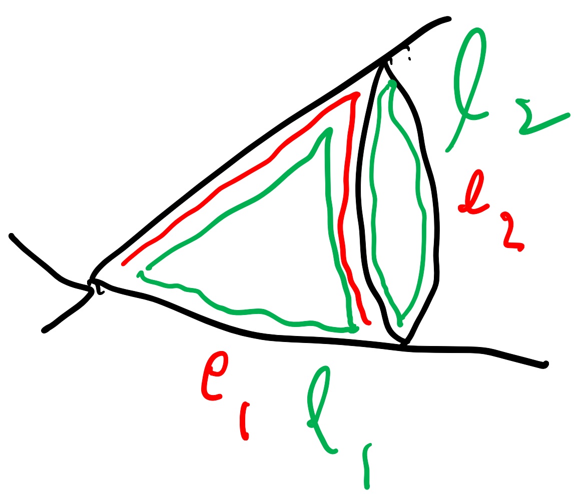
$q_0^2 < (m_1 + m_2)^2$

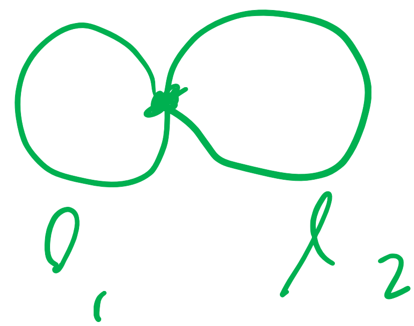
$\mathrm{Im}(G)$?

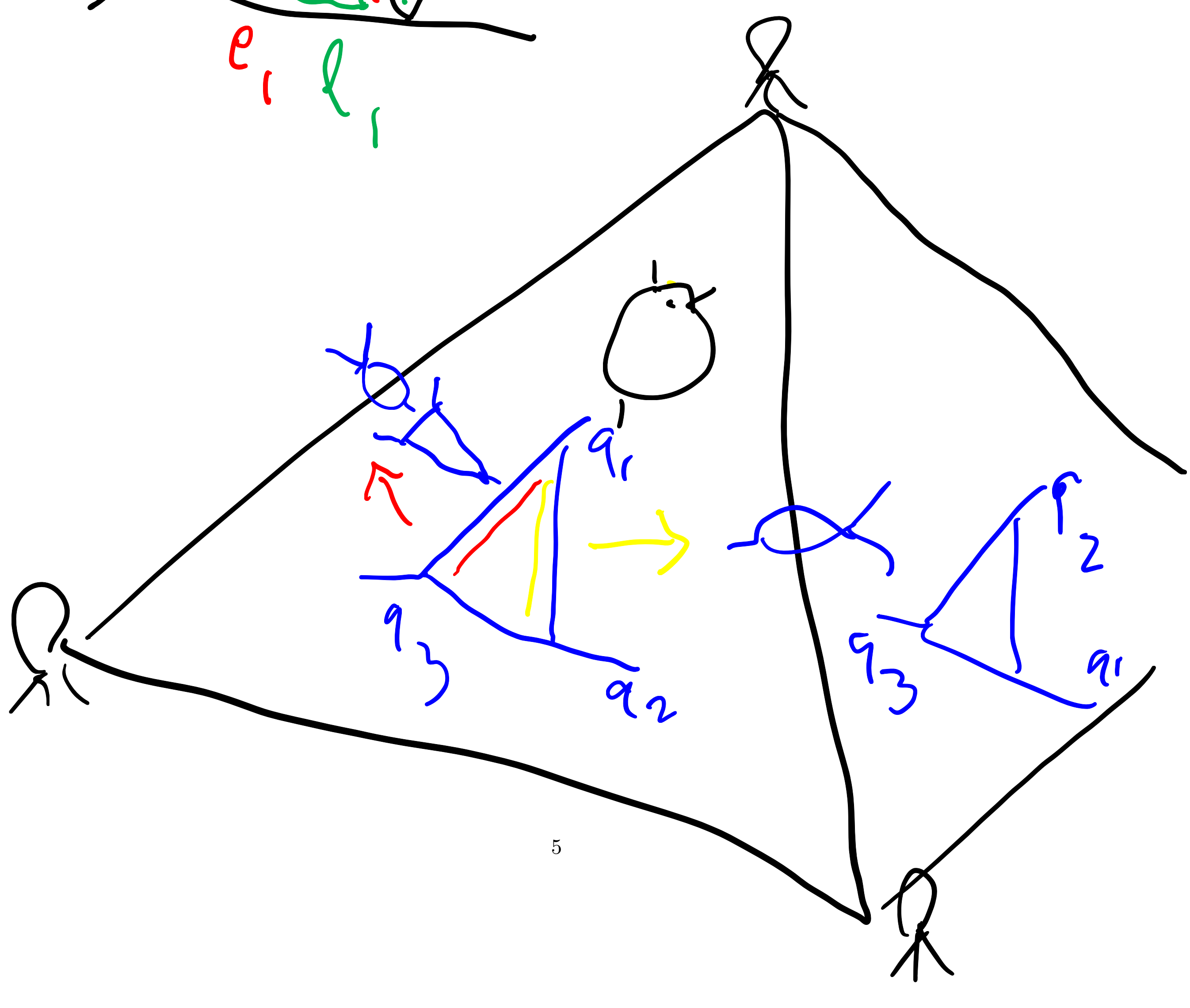


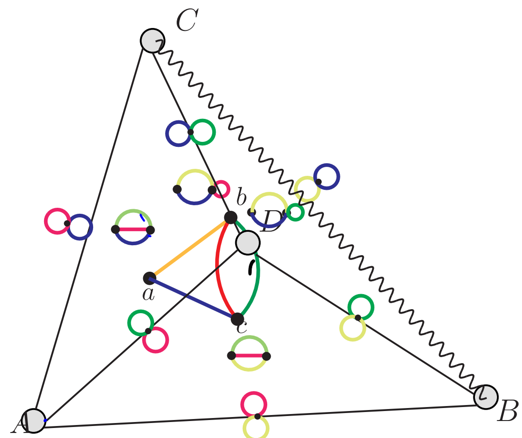
A choice of a spanning tree T of a graph with m independent loops l_i determines m edges e_i not in the spanning tree. The loops $l_i = l_i(e_i)$ are uniquely given by the edge e_i and the path in T connecting the two endponts of e_i. An orientation of e_i orients the loop, and shrinking all edges of T to zero length gives a rose R_m, a graph with one vertex and m oriented petals e_i.

In Outer Space graphs are metric graphs, where the metric comes from assigning an edge length to each edge, and using the parametric integrand for Feynman graph, the Feynman integral becomes an integral over the volume of the open simplex assigned to the graph, with a measure defined by the parametric representation. All vertices we assume to be of valence three or higher.








C
b
D
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c
A
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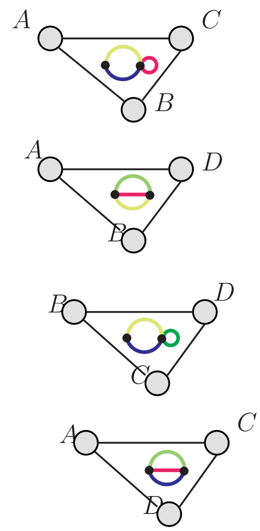
A
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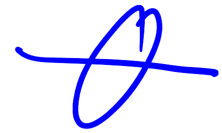

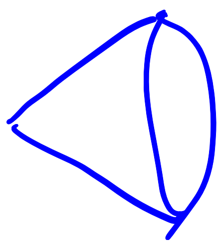

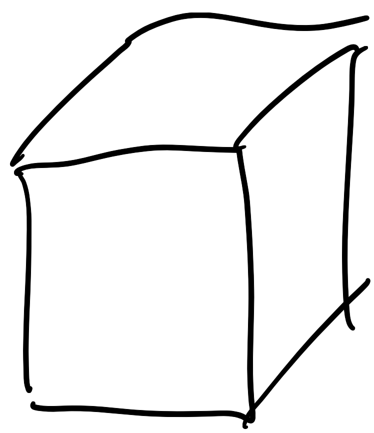

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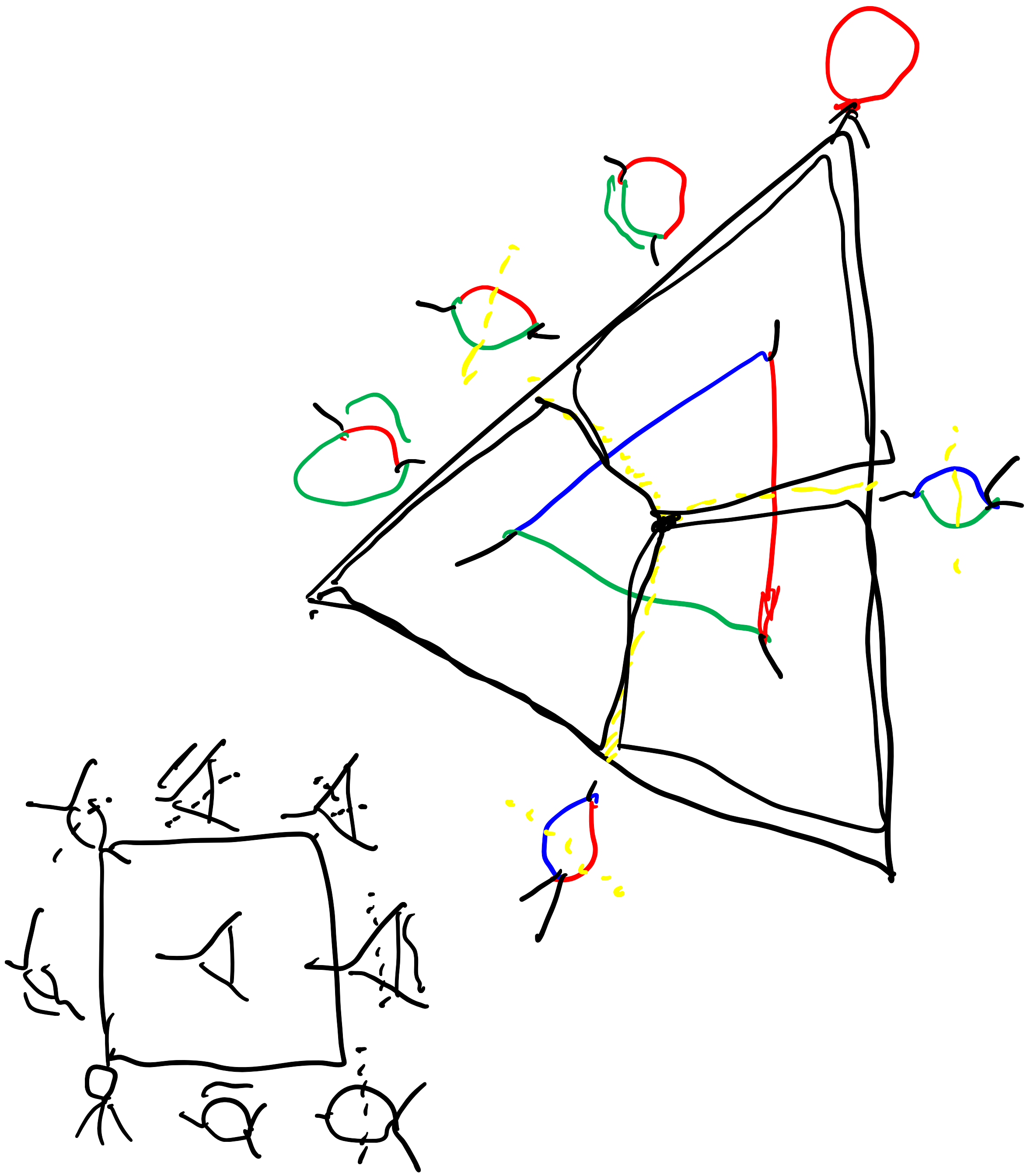
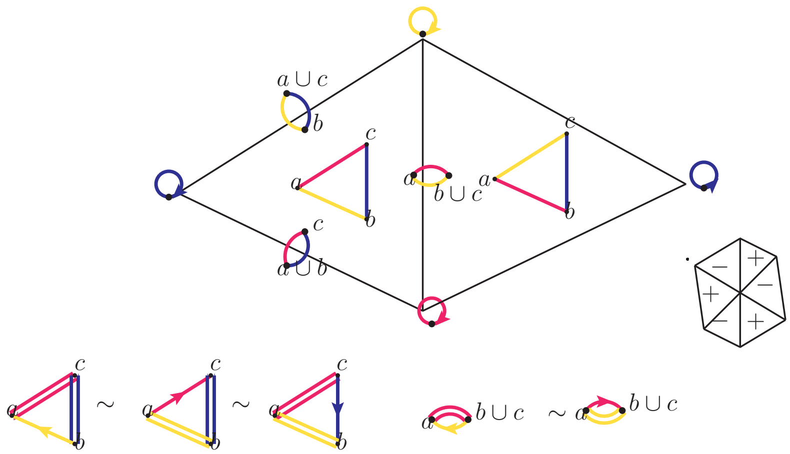
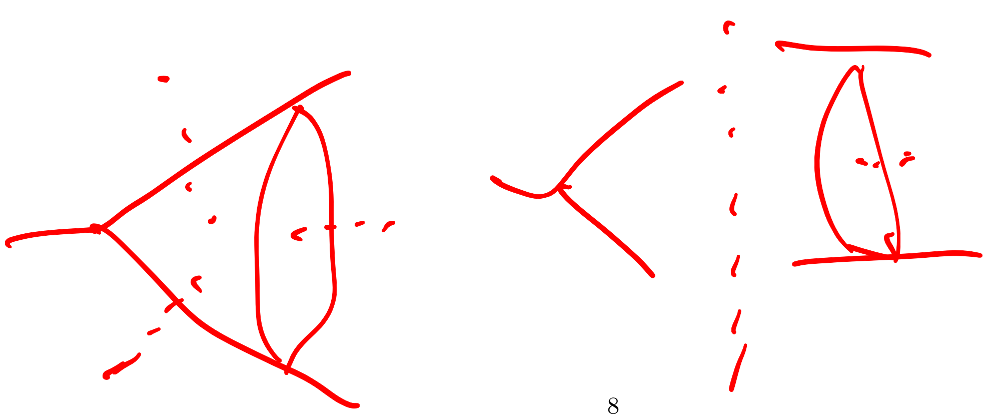


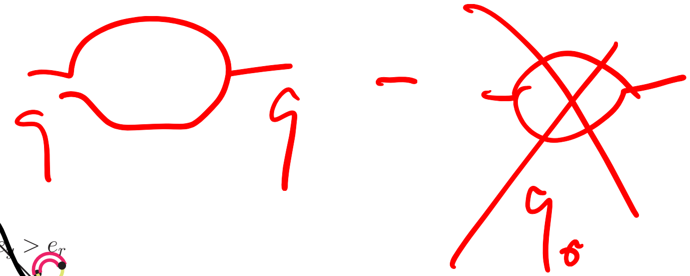
1.2. **Example: the triangle graph.** The above example discusses the structure of one cell together with its boundary components. We now look at the example of a triangle graph, and discuss its appearance in different cells.

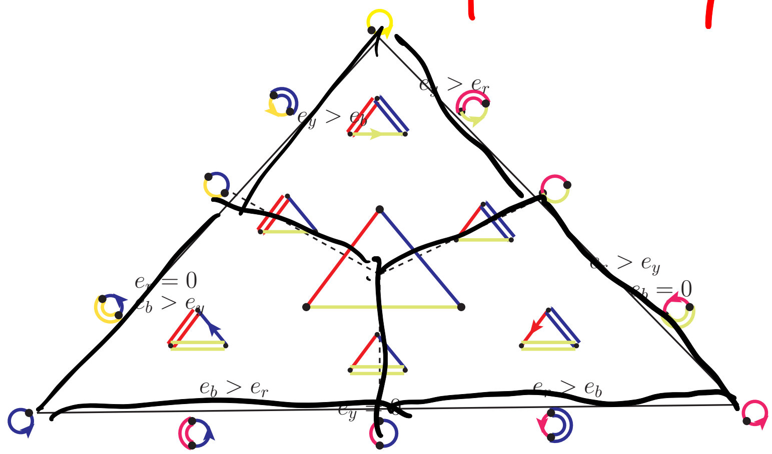


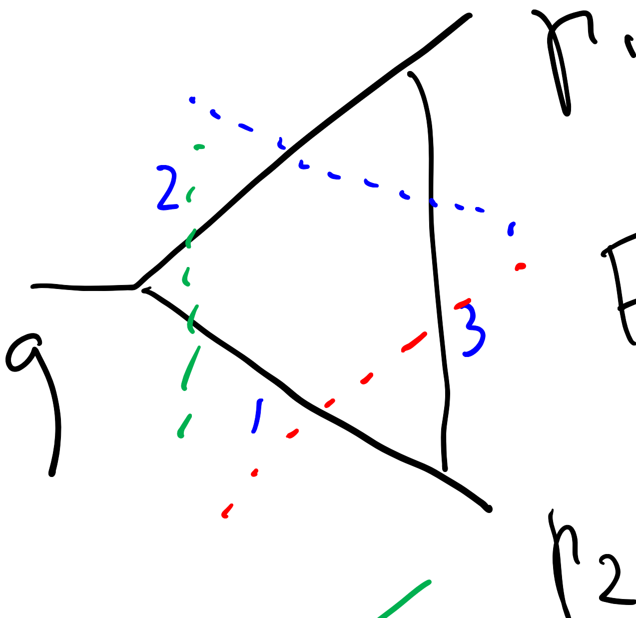
b

Here, the boundaries of the triangular cell belong themselves to OS: the three edges of the triangular cell are a cell for the indicated 1-loop graphs on two graph-edges, the vertices correspond to colored 1-petal roses.







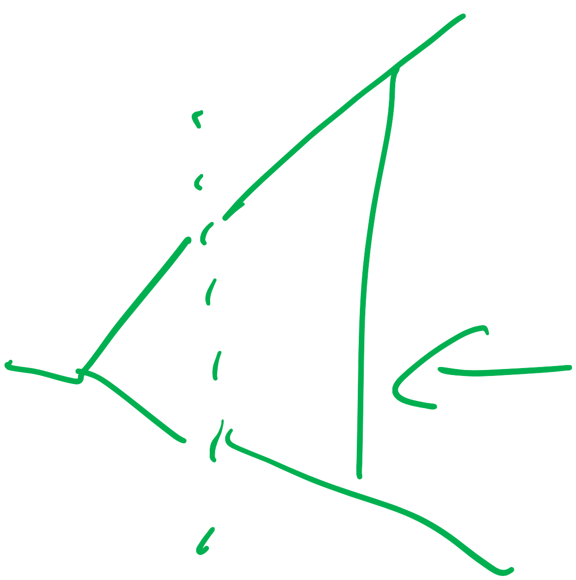


$$F_\Delta(p_1^2, p_2^2, q^2, m_1^2, m_2^2, m_3^2)$$

$$p_1^2 > (m_2+m_3)^2$$

$$p_2^2 > (m_1+m_3)^2$$

$$q^2 > (m_1+m_2)^2$$



normal thresholds

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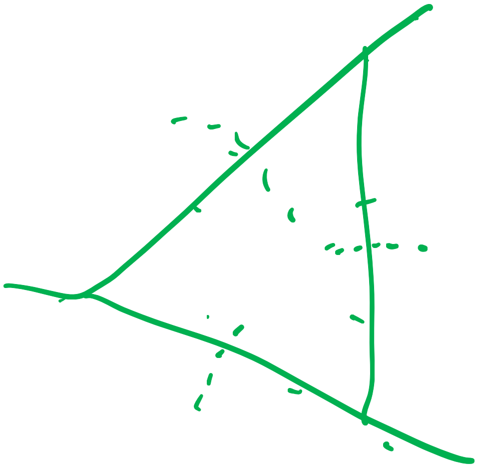
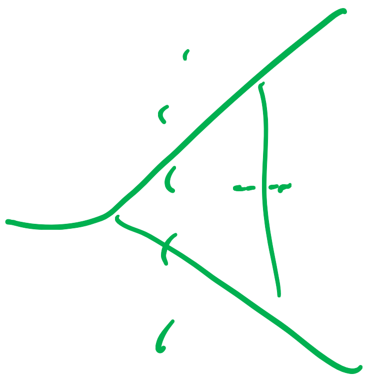
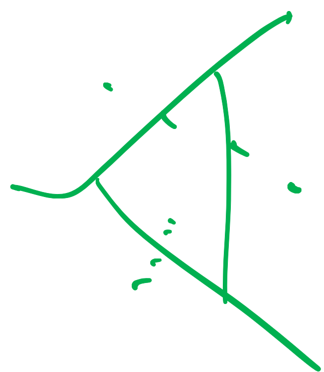


gives physical monodromy!

u

anomalous threshold

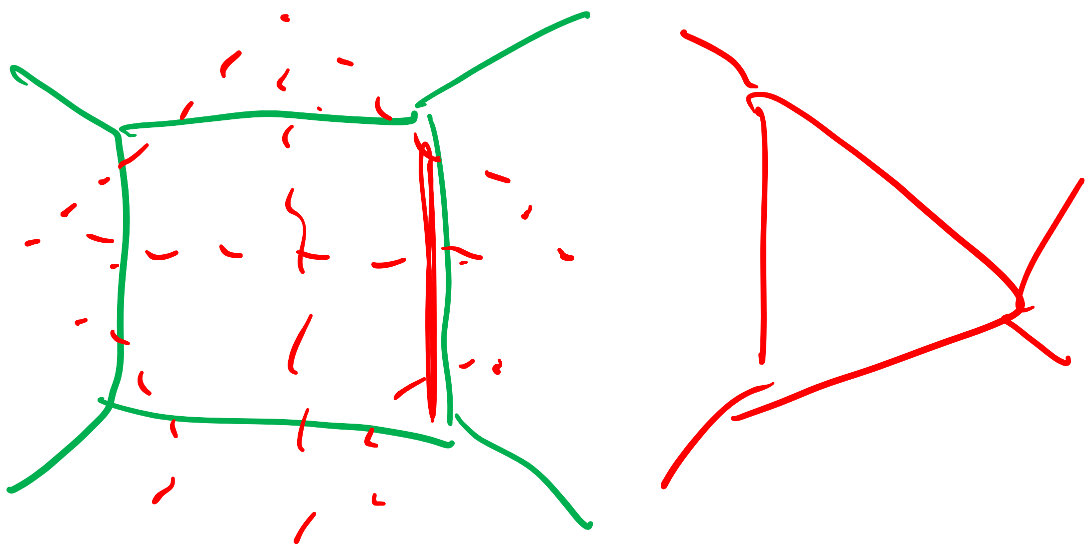




The triangle graph has a single loop and its fundamental group a single generator. Accordingly, we find a single generator for the monodromy in the complement of the threshold divisors: either for the normal threshold at $s_0 = (m_r + m_y)^2$ or for the anomalous threshold at s_1, with $l_r = p^2 - m_r^2 - m_b^2, l_y = (p+q)^2 - m_y^2 - m_b^2$, $\lambda_1 = \lambda(p^2, m_r^2, m_b v^2)$, $\lambda_1 = \lambda((p+q)^2, m_y^2, m_b v^2)$ it is given as,

$$s_1 = (m_r + m_y)^2 + \frac{4m_b^2(\sqrt{\lambda_2}m_r - \sqrt{\lambda_1}m_y)^2 - (\sqrt{\lambda_1}l_y + \sqrt{\lambda_2}l_r)^2}{4m_b^2\sqrt{\lambda_1}\sqrt{\lambda_2}}.$$

The function $J(z)$ has no pinch singularity and does not generate a new vanishing cycle. In general, a one-loop graph generates one pinch singularity through its normal threshold given by a reduced graph b_2, and as many anomalous thresholds as there are further edges in the graph.



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