1. Cutkosky graphs

1.1. The general formula for $H_C^{(0)}$. Consider a Cutkosky graph $G \in H_C^{(0)}$ generated by a unique forest $F$ and associated set of edges $E_{on}$ with $e \in E_{on} \iff e \notin E_F$ so that $E_{on} \cup E_F = E_G$.

\[ E_F = \{ e_5 \} \]

\[ E_{on} = \{ e_1, e_2, 1, 3, 4 \} \]

$E_F \cup E_{on} = E_G$

*How do evaluate?*
Then,

\[
\Phi(G) = \int \prod_{j=1}^{\left| \mathcal{G} \right|} d^D k_j \left( \frac{1}{\prod_{e \in E} Q_e} \right) \cap_{j \in \mathcal{R}^{\text{on}}(Q_j=0)} \]

\[
\frac{1}{Q_5} \left[ \begin{aligned}
Q_{e_1} &= 0 \cap Q_{e_2} = 0 \\
\cap Q_{e_3} &= 0 \cap Q_{e_4} = 0
\end{aligned} \right]
\]

\[
Q_{e_1} = x_1^2 - m_1^2 + i l_1 \tilde{Z}
\]

\[
Q_{e_2} = x_2^2 - m_2^2 + i l_2 \tilde{Z}
\]

\[
Q_{e_3} = x_3^2 - m_3^2 + i l_3 \tilde{Z}
\]

\[
Q_{e_4} = x_4^2 - m_4^2 + i l_4 \tilde{Z}
\]

\[
\left\{ \begin{array}{c}
l_1 = l_1, l_2, l_3, l_4 \end{array} \right\} \text{ 2 loops}
\]

\[
0_i = k_3, 0_4
\]
It remains to describe the threshold divisor prescribed by $\bigcap_{f \in E_{\text{on}}} (Q_f = 0)$.
We first note that $|E_{\text{on}}| \geq |G|$. Next, consider $G/E_F$. This graph is a Cutkosky graph which has all its edges cut.

\[
\begin{align*}
Q_1 &= 0 \implies \mathcal{X}_1 \text{ fixed} \\
Q_2 &= 0 \implies \mathcal{X}_2, \mathcal{X}_3 \text{ fixed} \\
Q_4 &= 0 \implies \mathcal{X}_4 \text{ fixed} \\
Q_6 &= 0 \implies \mathcal{X}_6 \text{ fixed} \\
Q_7 &= 0 \implies \mathcal{X}_7 \text{ fixed} \\
Q_8 &= 0 \implies \mathcal{X}_8 \text{ fixed}
\end{align*}
\]
Any chosen refinement of $L_G$ wth which $F$ is compatible defines a refinement of $V_{G/B_F}$ and therefore a set of variables $k_{i,0}$ and $s_i$ which are determined by the set $E_{en}$. As $|E_{en}| \geq ||G||$, all $k_{i,0}$ are fixed and so is $t$, where we set $s_i = t \tilde{s}_i$ for all $i$ and integrate $t$ over the positive real half-axis, whilst the $s_i$ are integrated in each ordering over the corresponding simplex $\Delta_\sigma \subset J^{G/B_F}$.

$$\begin{align*}
S_1 &= t \tilde{s}_1 \\
S_2 &= t \tilde{s}_2 \\
\sum_i |s_i| &= t
\end{align*}$$
As a result, the $|E_{on}|$ constraints make sure that the remaining integrals are over a compactum $C_{G/E'}$ and give the volume of $C_{G/E'}$. 

\[
\frac{x_1 - x_2}{\|x_1 - x_2\|} = z
\]
Now consider $G$ itself. The side-constraints are unchanged. The compactum $C_G$ fulfills
\[ \dim(C_G) = \dim(C_{G/E}) - e_F. \]

**Theorem 1.1.** For $G \in H_{\mathcal{G}}^{(0)}$ with $h_0(F) \geq 2$, $\Phi(G)$ exists and determines a threshold $s_F(G)$ in the variable $s_0$ defined by the 2-refinement of $L_G$. 

\[(\alpha, \beta, \gamma) \rightarrow (\alpha \beta, \beta \gamma) \rightarrow (\alpha \beta) \gamma\]
1.2. **Using the co-action.** Let $G$ be a Cutkosky graph with $h_0(G)$-partition $P$ of $LG$.

Consider a forest $F$ compatible with that partition so that we get a pair of a forest $F$, $h_0(F) = h_0(G)$, and a graph $G$. For any such pair there is an associated triple $(G_0, g, F_0)$ where $g \in H_{core}$ and $G_0 \in H^{(0)}_C$ so that $\|G_0\| = 0$, which determines $F_0$ uniquely. The set $\mathcal{F}_P$ of all compatible forests $F$ can be described as

\[(1.1) \quad \mathcal{F}_P = F_0 \cup \mathcal{T}(g).\]

The set $E_{on}^G = E_G - E_F$ so that $E_{on}^{G|g} = E_{G|g} - E_{F_0}$.
Then,

$$\Phi(G) = \sum_{F \in \mathcal{F}_P} \int \prod_{j=1}^{\lvert G \rvert} d^Dk_j \left( \prod_{e \in E_P} \frac{1}{Q_e} \right) \prod_{f \in E_n} (Q_f = 0)$$

Note that this is a variant of Fubini’s theorem by Eq. (1.1):

$$\Phi(G) = \prod_{j=1}^{\lvert G/G \rvert} d^Dk_j \left( \prod_{e \in E_{P_0}} \frac{1}{Q_e} \right) \prod_{f \in E_{n_0}} (Q_f = 0) \int \sum_{T \in T(G)} \prod_{j=1}^{\lvert T \rvert} d^Dk_j \left( \prod_{e \in E_{T_1}} \frac{1}{Q_e} \right) \prod_{f \in E_{T_2}} (Q_f = 0)$$

Use Hopf of a renormalization:

$$\frac{1}{Q \cdot Q} = Q \cdot (h \cdot k) \cdot Q \cdot (l)$$

$$\frac{1}{Q \cdot Q} - \frac{1}{Q \cdot Q} = \left( \frac{1}{Q \cdot Q} - \frac{1}{Q \cdot Q} \right)$$

$$Q = Q \cdot (l, k)$$
Now consider a $v_T$-refinement $R$ of $G$. We call its partitions $P(i)$. Note that for every $T \in T(G)$, such a refinement induces an ordering $o_T$ of its edges.

Accompanying the partitions $P(i)$ are Cutkosky graphs $G(i)$, forests $F_0(i)$, core subgraphs $g(i)$, and sets $F_{P(i)} = F_0(i) \cup T(g(i))$.

We get a sequence $F(G(i))$ of evaluations of Cutkosky graphs. They define Hodge matrices for Feynman graphs to be discussed in June.
1.3. **The pre-Lie product and the cubical chain complex.** So consider the pair \((G, F)\) of a pre-Cutkosky graph with compatible forest \(F\) with ordered edges. Assume there are graphs \(G_1, G_2\) and forests \(F_1, F_2\) such that

\[
(G, F) = (G_1, F_1) \ast (G_2, F_2).
\]

Here, \(\ast\) is the pre-Lie product which is induced by the co-product \(\Delta_{GF}\) by the Milnor–Moore theorem.

\[
d \left( (G, F) \right) = \sum_{i=1}^{\text{on } \text{edges} \not\in F} (-1)^{|e_i|} \left( (G, F \cup e_i), F \cup e_i \right) - (G, F - e_i)
\]

\[
d \circ d = 0
\]
**Theorem 1.2.** We can reduce the computation of the homology of the cubical chain complex for large graphs to computations for smaller graphs by a Leibniz rule:

\[
d((G_1, F_1) \ast (G_2, F_2)) = (d(G_1, F_1)) \ast (G_2, F_2) + (-1)^{|E_{F_1}|} (G_1, F_1) \ast (d(G_2, F_2)).
\]

Here, \(d = d_0 + d_1\) is the boundary operator which either shrinks edges or cuts a graph.
Example:
14. **Cutkosky graphs in general: iterated integral structure.** Consider a sequence of Cutkosky graphs $G(t)$ defined by a $v_G$-refinement $R$ of the set $L_G$ of external edges of a core graph $G = G(0)$.

The partition $V_{G(1)} = V_{in} \cup V_{out}$ defines

$$s = \left( \sum_{v \in V_{in}} q_v \right)^2 = \left( \sum_{v \in V_{out}} q_v \right)^2,$$

where $q_v$ is the external four-momentum at $v$. 
For $G(i)$, $i \geq 1$, consider a compatible forest $F(i)$ with corresponding spanning tree $T = F(0)$.

Let $e \in E_T$ be an edge which connects $V_{\text{in}}, V_{\text{out}}$. For all $G(i)$ we get $k_{e,0} = s_e + m^2_e - i\eta$ and $e \not\in F(i)$. 
The graph $G_e := G(i) - e$ can be written as

$$
\int_0^\infty d\Omega_e \sqrt{s_e} \Theta((\sqrt{s_e + m_e^2} - q_{in,0})^2 - 2k_e \cdot q_{in} - s_e - q_{in}^2 - t_{G_e}) F_{G_e} s_e
$$

We partially integrate with respect to $s_e$. There are no boundary contributions by power-counting. The derivative of the $\Theta$-function gives a Dirac distribution $\delta$ which fixes $s_e$.

$k_e$ is a function of $x$ as the long moment is defined with the long and $lq_e$.

$k_e - k_e - k_e = s_e$
As an example the 3-edge banana graph $b_3$ with spanning tree $e_2$ has quadrics

$Q(e_1) = k_0^2 - s - m_1^2 + i\eta$, $Q(e_2) = (k_0 - l_0 + q_0)^2 - (\bar{l} - \bar{k})^2 - m_2^2 + i\eta$, $Q(e_3) = l_0^2 - l - m_3^2 + i\eta$.

The $t$-integral with cut edges $e_2, e_3$ defines $G_{e_1} = b_3 - e_1$ and delivers

$L(k_0,s) := \Theta((k_0 + q_0)^2 - s - (m_2 + m_3)^2) \frac{\sqrt{\lambda((k_0 + q_0)^2 - s, m_2^2, m_3^2)}}{2((k_0 + q_0)^2 - s)}$.

Cutting $e_1$ delivers $k_0 = \sqrt{s + m_1^2}$. 
It remains the integral

$$
\Phi(b_3^C) = \int_0^\infty \sqrt{s} \Theta((\sqrt{s + m_1^2 + q_0})^2 - s - (m_2 + m_3)^2) \frac{\sqrt{\lambda((\sqrt{s + m_1^2 + q_0})^2 - s, m_2^2, m_3^2)}}{2((\sqrt{s + m_1^2 + q_0})^2 - s)} ds.
$$

A partial integration delivers

$$
\Phi(b_3^C) = \int_0^\infty \frac{1}{\sqrt{s + m_1^2}} \delta((\sqrt{s + m_1^2 + q_0})^2 - s - (m_2 + m_3)^2) \left( \int_0^s \sqrt{u} \frac{\sqrt{\lambda((\sqrt{u + m_1^2 + q_0})^2 - u, m_2^2, m_3^2)}}{2((\sqrt{u + m_1^2 + q_0})^2 - u)} du \right) \, dc_{c_1}(s).
$$

This fixes

$$
s = s_r := \frac{\lambda(q_0^2, m_1^2, (m_2 + m_3)^2)}{4q_0^2},
$$

where we need \( q_0^2 \geq (m_1 + m_2 + m_3)^2 \). The integral then delivers

$$
\Phi(b_3^C)(q_0^2) = G_{c_1}(s_r) \Theta(q_0^2 - (m_1 + m_2 + m_3)^2).
$$

Note that \( G_{c_1}(0) = 0 \) and hence \( \phi(b_3^C)((m_1 + m_2 + m_3)^2) = 0 \), using \( \lambda((\sqrt{b + c})^2, b, c) = 0 \).