FEYNMAN DIAGRAMS AND THE S-MATRIX, AND OUTER SPACE (SUMMER 2020)

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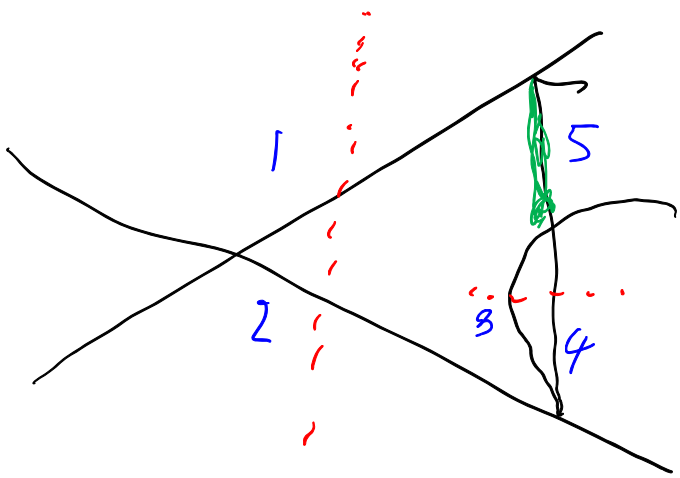
1. Cutkosky graphs

1.1. **The general formula for $H_C^{(0)}$.** Consider a Cutkosky graph $G \in H_C^{(0)}$ generated by a unique forest F and associated set of edges E_{on} with $e \in E_{on} \Leftrightarrow e \notin E_F$ so that $E_{on} \dot\cup E_F = E_G$.



$E_F = \{e_5\}$

$E_{on} = \{e_1, e_2, e_3, e_4\}$

$E_F \cup E_{on} = E_G$

How do evaluate ?

Then,

$$\Phi(G) = \int \prod_{j=1}^{|G|} d^D k_j \underbrace{\left(\frac{1}{\prod_{e\in E_F} Q_e}\right)}_{\cap_{f\in E_{on}}(Q_f=0)} .$$

$$\frac{1}{Q_S}\Bigg|_{\cap\{Q_{e_1}=0 \,\cap\, Q_{e_2}=0 \,\cap\, Q_{e_3}=0 \,\cap\, Q_{e_4}=0\}}$$

$$Q_{e_1} : k_1^2 - m_1^2 + i\}$$

$$k_{01}^2 = \vec{k}_1^2 + m_1^2 - i\}$$

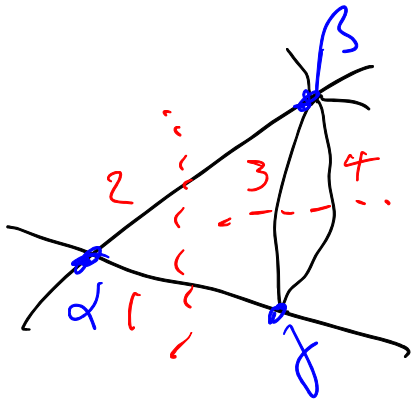
$$Q_{e_3} : k_2^2 - m_3^2 + i\} \quad \ldots$$

$$\left.\begin{array}{l} l_1 : l_1, l_2, l_3, l_4 \\ l_2 : l_3, l_4 \end{array}\right\} 2 \text{ loops}$$

It remains to describe the threshold divisor prescibed by $\cap_{f \in E_{on}}(Q_f = 0)$.

We first note that $|E_{on}| \gneq |G|$. Next, consider G/E_F. This graph is a Cutkosky graph which has all its edges cut.

$G/E_S \cong$



$\{\alpha\}, \{\beta\}, \{\gamma\}$

$Q_1 = 0 \Rightarrow X_{10}$ fixed

$Q_3 = 0 \Rightarrow X_{20}$ fixed

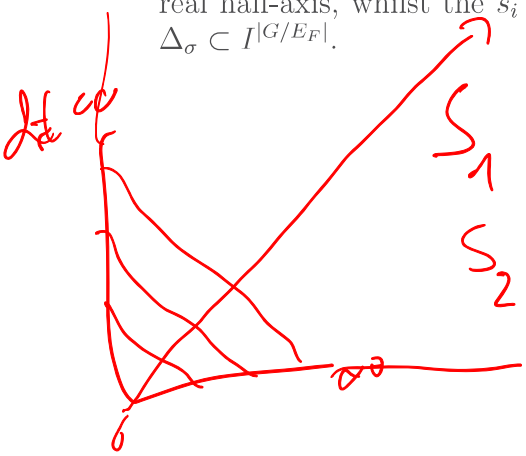
$Q_2 = 0 \Rightarrow S_1 \sim \vec{X}_1 \cdot \vec{X}_3$ fixed

$Q_4 = 0 \Rightarrow S_2 \sim \vec{X}_2 \cdot \vec{X}_2$ fixed

independent domain $\mathcal{D}^{\vee}$

$X_{i,0}$, S_i non-compact

Any chosen refinement of L_G wth which F is compatible defines a refinement of V_{G/E_F} and therefore a set of variables $k_{i,0}$ and s_i which are determined by the set E_{on}. As $|E_{on}| \gneq ||G||$, all $k_{i,0}$ are fixed and so is t, where we set $s_i = t\tilde{s}_i$ for all i and integrate t over the positive real half-axis, whilst the $\tilde{s}_i$ are integrated in each ordering over the corresponding simplex $\Delta_\sigma \subset I^{|G/E_F|}$.

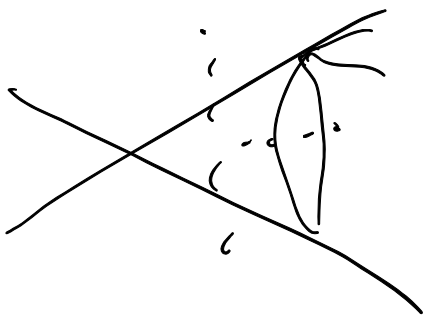


$$s_1 = t_1 \tilde{s}_1$$
$$s_2 = t_2 \tilde{s}_2 \rightarrow \int_0^\infty dt \int_0^1 dt$$

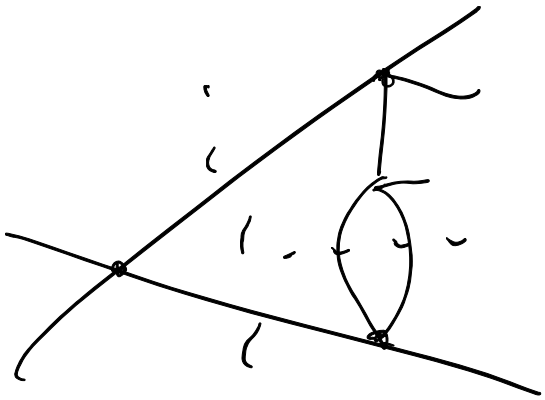
$$t_1 \quad u$$

$$\text{Pf.} \quad \sum_i (s_i) = t$$

As a result, the $|E_{on}|$ constraints make sure that the remaining integrals are over a compactum C_{G/E_f} and give the volume of C_{G/E_F}.



$$\frac{\vec{x}_1 \cdot \vec{x}_2}{|\vec{x}_1||\vec{x}_2|} = z$$

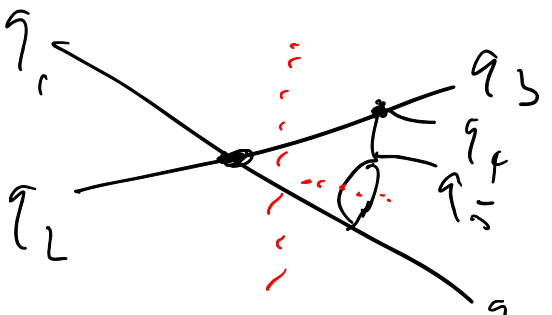


Now consider G itself. The side-constraints are unchanged. The compactum C_G fulfills

$$\dim(C_G) = \dim(C_{G/E_f}) - e_F.$$

Theorem 1.1. *For $G \in H_C^{(0)}$ with $h_0(F) \geq 2$, $\Phi(G)$ exists and determines a threshold $s_F(G)$ in the variable s_0 defined by the ~~2-refinement~~ of L_G.*

1-partition



$q_3 + q_4 = r$

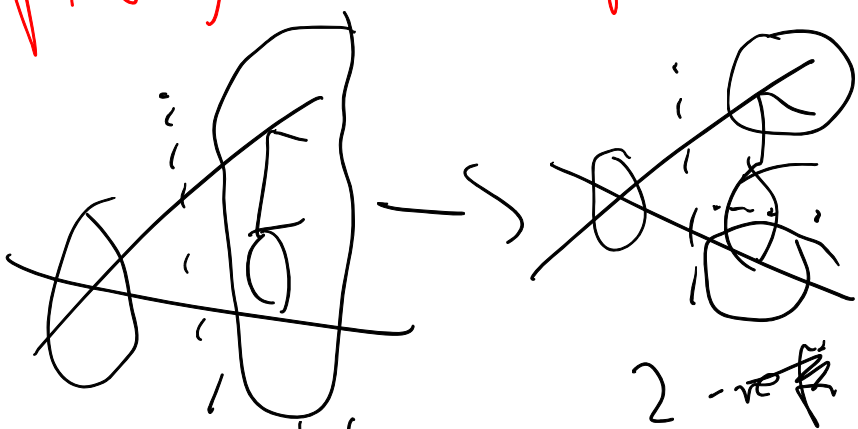
$q_1 + q_2 = q$

q_1, r, q_5, q_6

q_1^2, r^2, q_5^2, q_6^2

$q \cdot r \quad q_5 \cdot q_6$

$(\alpha, \beta, \gamma) \to (\alpha)(\beta, \gamma) \to (\alpha)(\beta)(\gamma)$



2-~~ref~~ partition

q^2

1-partition

$r^2 = (m_1 + m_2)^2$

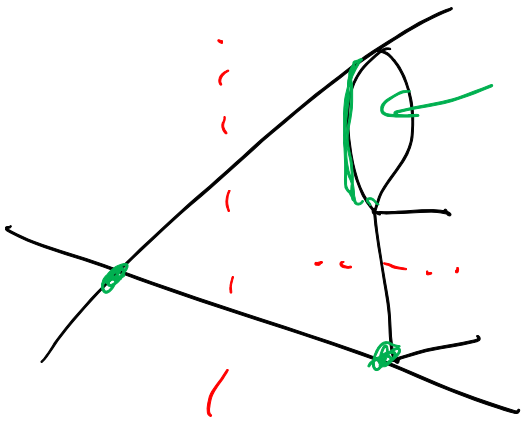
1.2. **Using the co-action.** Let G be a Cutkosky graph with $h_0(\tilde{G})$-partition P of L_G.

Consider a forest F compatible with that partition so that we get a pair of a forest F, $h_0(F) = h_0(\tilde{G})$, and a graph G. For any such pair there is an associated triple (G_0, g, F_0) where $g \in H_{core}$ and $G_0 \in H_C^{(0)}$ so that $||G_0|| = 0$, which determines F_0 uniquely. The set $\mathcal{F}_P$ of all compatible forests F can be described as

$$\mathcal{F}_P = F_0 \dot{\cup} \mathcal{T}(g). \tag{1.1}$$

The set $E_{on}^{G} = E_G - E_F$ so that $E_{on}^{G/g} = E_{G/g} - E_{F_0}$.

$H_C^{(0)} \vee$

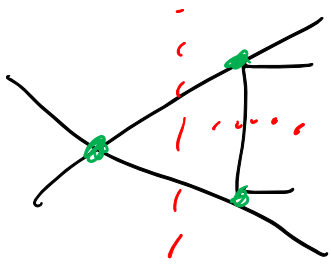


loop intact

$\in H_C^{(1)}$

$g \sim G \beta$

G/g



$\sim F_0$

$G\ell$

$\| $

G_0

$G/g \in H_C^{(0)}$

Then,

$$\Phi(G) = \sum_{F\in\mathcal{F}_P} \int \prod_{j=1}^{|G|} d^D k_j \left(\frac{1}{\prod^R_{e\in E_F} Q_e}\right)_{\cap_{f\in E^G_{on}}(Q_f=0)} .$$

sums of products

Note that this is a variant of Fubini's theorem by Eq.(1.1):

$$\Phi(G) = \int \prod_{j=1}^{|G/g|} d^D k_j \left(\frac{1}{\prod_{e\in E_{F_0}} Q_e}\right)_{\cap_{f\in E^{G/g}_{on}}(Q_f=0)} \underbrace{\sum_{t\in\mathcal{T}(g)} \int \prod_{j=1}^{|g|} d^D k_j \left(\frac{1}{\prod^R_{e\in E_t} Q_e}\right)_{\cap_{f\in(E_g-E_t)}(Q_f=0)}}_{\Phi_R(g)} .$$

$$\prod^R_{e\in E_{..}} \to \prod^R_{e\in E_g} \frac{1}{Q_e}$$

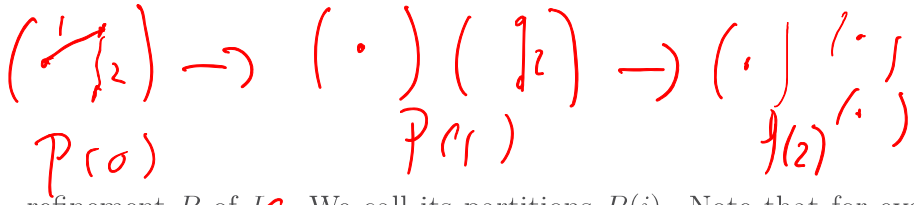
due to renormalization

use Hopf alg. of renormalization:

$$\frac{1}{Q_\alpha Q_\beta} = \frac{1}{Q_\alpha(l,k)\, Q_\beta(l)}$$

$$\frac{1}{Q_\alpha Q_\beta} \to \left(\frac{1}{Q_\alpha Q_\beta} - \frac{1}{\bar{Q}_\alpha \bar{Q}_\beta}\right)$$

$$\bar{Q}_\alpha = Q_\alpha(l, \bar{k})$$



G

Now consider a v_G-refinement R of $\cancel{L}G$. We call its partitions $P(i)$. Note that for every $T \in \mathcal{T}(G)$, such a redinement induces an ordering o_T of its edges.

Accomapying the partitions $P(i)$ are Cutkosky graphs $G(i)$, forests $F_0(i)$, core subgraphs $g(i)$, and sets $\mathcal{F}_{P(i)} = F_0(i) \dot\cup \mathcal{T}(g(i))$.

We get a sequence $\Phi(G(i))$ of evaluations of Cutkosky graphs. They define Hodge matrices for Feynman graphs to be discussed in June.

More in June.

1.3. **The pre-Lie product and the cubical chain complex.** So consider the pair (G, F) of a pre-Cutkosky graph with compatible forest F with ordered edges. Assume there are graphs G_1, G_2 and forests F_1, F_2 such that

$$(G, F) = (G_1, F_1) \star (G_2, F_2).$$

Here, $\star$ is the pre-Lie product which is induced by the co-product Δ_{GF} by the Milnor–Moore theorem.

ordered edges

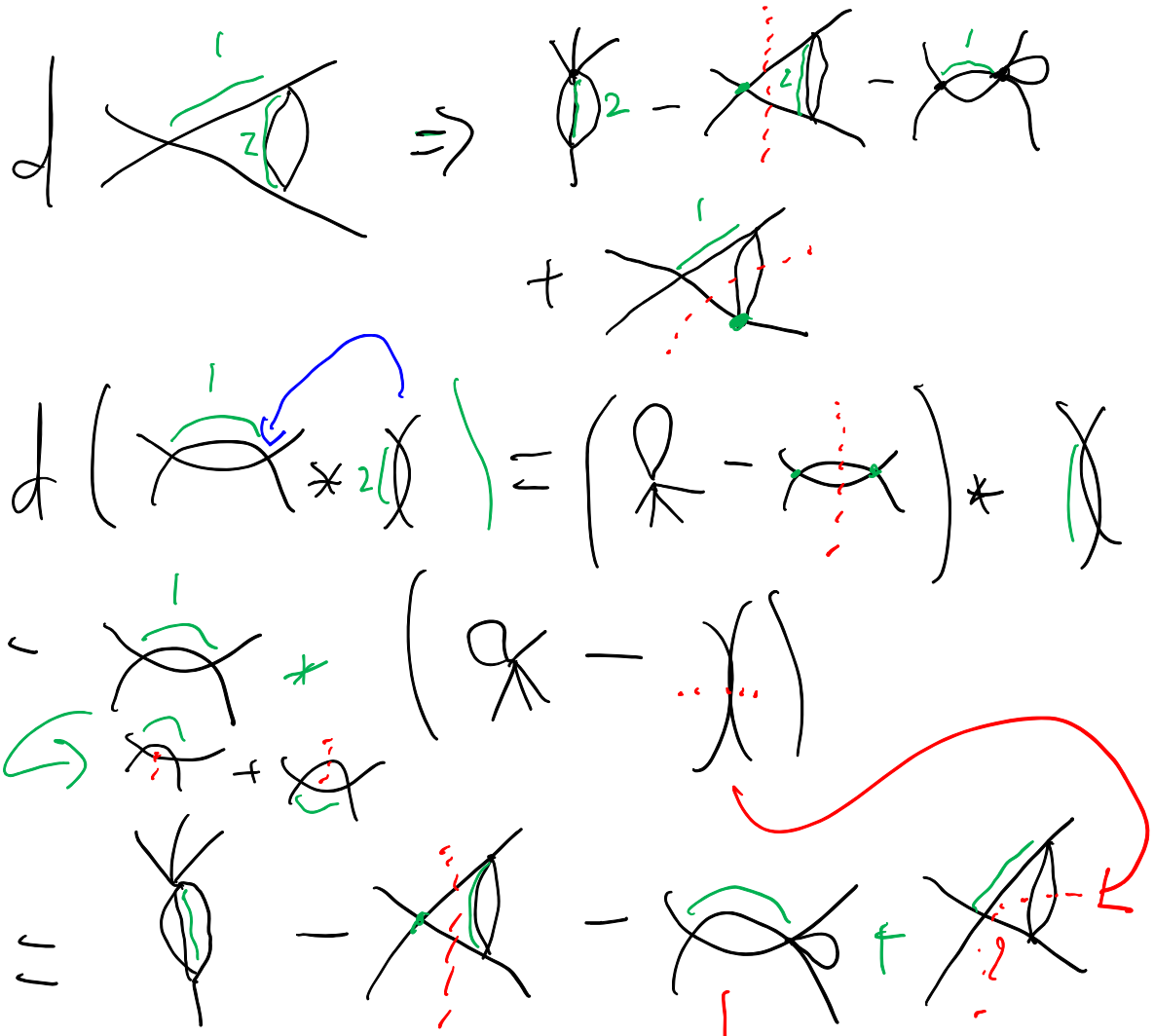
$$d(G, F) = \sum_{i=1}^{|E_F|} (-1)^i \left\{ (G/e_i, F/e_i) - (G, F - e_i) \right\}$$

$$d \circ d = 0$$

Theorem 1.2. [?] *We can reduce the computation of the homology of the cubical chain complex for large graphs to computations for smaller graphs by a Leibniz rule:*

$$d\left((G_1, F_1) \star (G_2, F_2)\right) = (d(G_1, F_1)) \star (G_2, F_2) + (-1)^{|E_{F_1}|}(G_1, F_1) \star (d(G_2, F_2)).$$

Here, $d = d_0 + d_1$ is the boundary operator which either shrinks edges or cuts a graph.



summing over all $\Gamma(G)$ makes this true,

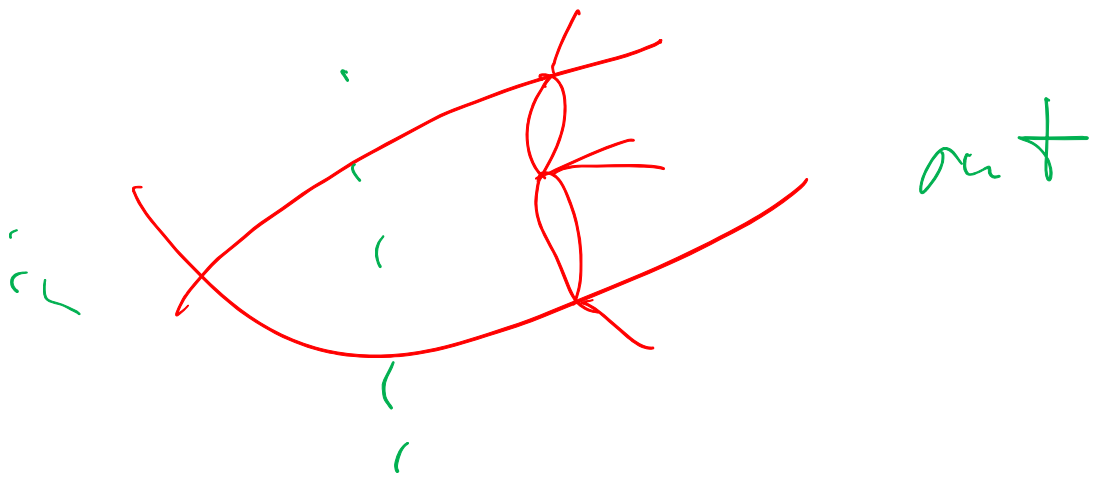
Example:

1.4. **Cutkosky graphs in general: iterated integral structure.** Consider a sequence of Cutkosky graphs $G(i)$ defined by a v_G-refinement R of the set L_G of external edges of a core graph $G = G(0)$.

The ~~2~~ 1-partition $V_{G(1)} = V_{in} \dot\cup V_{out}$ defines

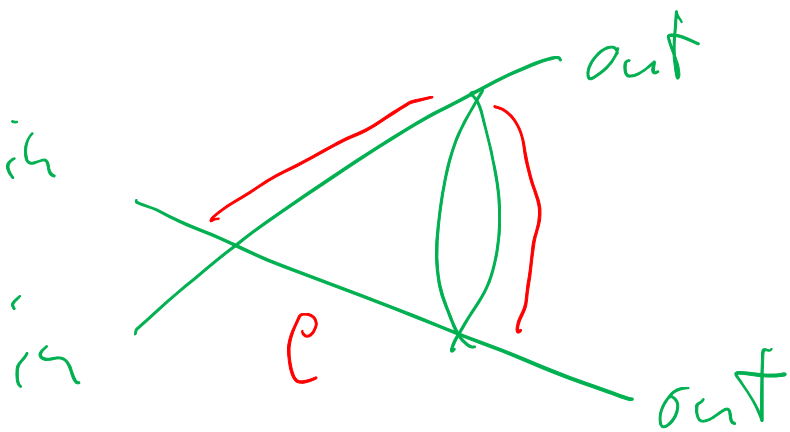
$$s = \left(\underbrace{\sum_{v \in V_{in}} q_v}_{q_{in}} \right)^2 = \left(\underbrace{\sum_{v \in V_{out}} q_v}_{q_{out}} \right)^2 ,$$

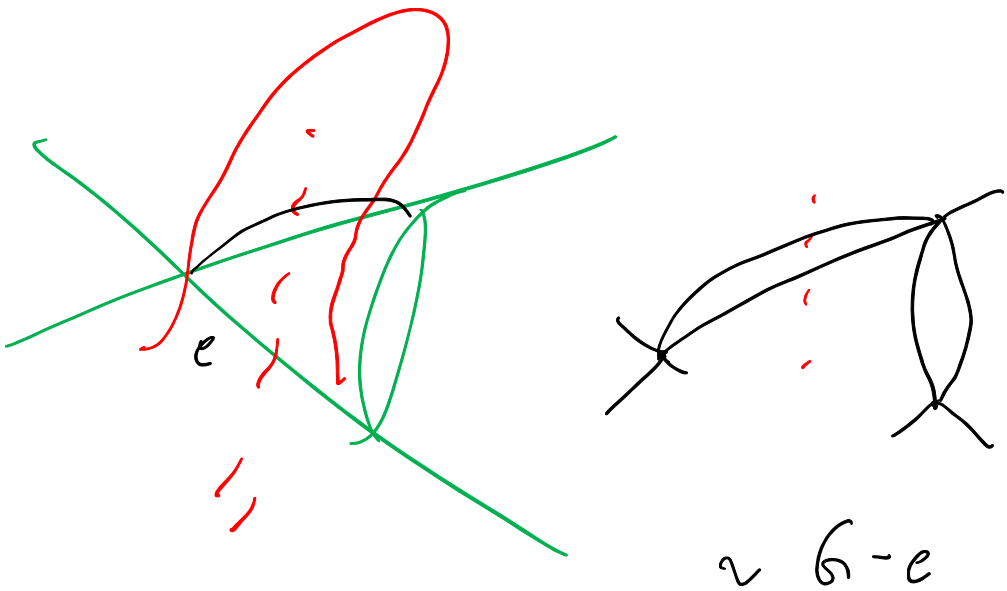
where q_v is the external four-momentum at v.



For $G(i)$, $i \geq 1$, consider a compatible forest $F(i)$ with corresponding spanning tree $T = F(0)$.

Let $e \notin E_T$ be an edge which connects V_{in}, V_{out}. For all $G(i)$ we get $k_{e,0} = s_e + m_e^2 - i\eta$ and $e \notin F(i)$.





$$k_e^2 - 2k_e \cdot q + q^2$$

The graph $G_e := G(i) - e$ can be written as

$$\int_0^\infty \int d\Omega_e \sqrt{s_e}\Theta((\sqrt{s_e + m_e^2} - q_{in,0})^2 - 2\vec{k}_e \cdot \vec{q}_{in} - s_e - \vec{q}_{in}^2 - t_{G_e})F_{G_e} ds_e$$

We partially integrate with respect to s_e. There are no boundary contributions by power-counting. The derivative of the Θ-function gives a Dirac distribution δ which fixes s_e.

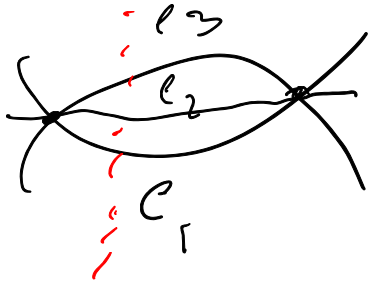
G_e is a function of k the loop momentum defined by e, q_{in} and $\{q_e\}$

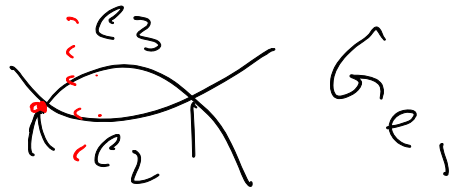
$$k_e \quad k_e \cdot k_e \equiv s_e$$

As an example the 3-edge banana graph b_3 with spanning tree e_2 has quadrics $Q(e_1) = k_0^2 - s - m_1^2 + i\eta, Q(e_2) = (k_0 - l_0 + q_0)^2 - (\vec{l} - \vec{k})^2 - m_2^2 + i\eta, Q(e_3) = l_0^2 - t - m_3^2 + i\eta$. The l-integral with cut edges e_2, e_3 defines $G_{e_1} = b_3 - e_1$ and delivers

$$L(k_0, s) := \Theta((k_0 + q_0)^2 - s - (m_2 + m_3)^2) \frac{\sqrt{\lambda((k_0 + q_0)^2 - s, m_2^2, m_3^2)}}{2((k_0 + q_0)^2 - s)}.$$

Cutting e_1 delivers $k_0 = \sqrt{s + m_1^2}$.





It remains the integral

$$\Phi(b_3^C) = \int_0^\infty \sqrt{s}\Theta((\sqrt{s+m_1^2}+q_0)^2 - s - (m_2+m_3)^2)\frac{\sqrt{\lambda((\sqrt{s+m_1^2}+q_0)^2 - s, m_2^2, m_3^2)}}{2((\sqrt{s+m_1^2}+q_0)^2 - s)}ds.$$

A partial integration delivers

$$\Phi(b_3^C) = \int_0^\infty \frac{1}{\sqrt{s+m_1^2}}\delta((\sqrt{s+m_1^2}+q_0)^2 - s - (m_2+m_3)^2)\underbrace{\left(\int_0^s \sqrt{u}\frac{\sqrt{\lambda((\sqrt{u+m_1^2}+q_0)^2 - u, m_2^2, m_3^2)}}{2((\sqrt{u+m_1^2}+q_0)^2 - u)}du\right)}_{G_{e_1}(s)}$$

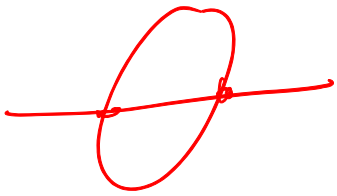
This fixes

$$s = s_r := \frac{\lambda(q_0^2, m_1^2, (m_2+m_3)^2)}{4q_0^2},$$

where we need $q_0^2 \geq (m_1+m_2+m_3)^2$. The integral then delivers

$$\Phi(b_3^C)(q_0^2) = G_{e_1}(s_r)\Theta(q_0^2 - (m_1+m_2+m_3)^2).$$

Note that $G_{e_1}(0) = 0$ and hence $\phi(b_3^C)((m_1+m_2+m_3)^2) = 0$, using $\lambda((\sqrt{b}+\sqrt{c})^2, b, c) = 0$.



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