1. Sets or edges

Given a spanning $k$-forest $F$ of a cut graph $G$, there are a number of different sets of edges which will be important. First the edges $e \in E_F$ of the forest themselves are important.

Second are the edges $e \in C_G$ of $G$ which are not in $F$ but join distinct components of $F$. If we view $F$ as a spanning tree $T$ with some edges removed then all the edges of $T - F$ are in this second class, as well, typically, as others.
Third are the edges $e \in (E_G - E_T)$ of $G$ which are not in $F$ but have both ends in the same tree of $F$.

$C_F = \{1, 2, 3, 3\}$

$E_F \cup C_F = F_{on}$
The second and third sets of edges above are those which will ultimately be put on-shell and define the set $E_{on}$, while those in the first set remain off-shell whilst we will use the notation $\mathcal{E}_F$ for the second of the above sets of edges.

**Definition 1.1.** A Cutkosky graph $G$ is a cut graph $G$ for which a compatible spanning forest $F$ such that $C_G = \mathcal{E}_F$ exists.

Compatibility ensures that the spanning forest is in accordance with the chosen refinements $\mathcal{E}_H$. 

Note $h_0(\tilde{G}) = h_0(F)$ for a compatible $F$ and note that an ordering of edges in a spanning tree of a Cutkosky graph $G$ induces a $(v_G - 1)$-refinement of $L_G$. 
We say that a spanning tree $T$ of $G$ with ordered edges is compatible with a given $(\nu_G - 1)$-refinement $R$ of $L_G$ if and only if the forests $T - \Pi_{i=1}^k e_i$ induce the $k + 1$-partition of $R$. 
We let $F^R_G$ be the set of ordered spanning trees of a graph $G$ compatible with a $(v_G - 1)$-refinement $R$ of $L_G$. Note that if $R = L_G$ is the trivial partition, then $F^R_G = T_G$, the set of spanning trees of $G$.

**Definition 1.2.** For a Cutkosky graph $G$ we let $G^{\text{red}} := \tilde{G}/E_{\tilde{G}}$ be the reduced graph. It is a collection of $h_0(\tilde{G})$ corollas and hence a Cutkosky graph $(\tilde{G}^{\text{red}}, G^{\text{red}})$ with $\tilde{G}^{\text{red}} = \tilde{G}/E_{\tilde{G}}$.

We call $\tilde{G}$ the co-reduced graph.
Definition 1.3. We define $H_C$ to be the $\mathbb{Q}$-vectorspace generated by Cutkosky graphs.

Definition 1.4. We define $H_{core}$ to be the $\mathbb{Q}$-vectorspace generated by graphs without cuts.
2. Hopf algebras

We have to define two Hopf algebras $H_{\text{core}}$ and $H_{\text{ren}}$. Both will co-act on $H_C$ defined above.

2.1. graph insertion and graph decomposition. Consider graphs $f = (H_f, V_f, E_f)$ and $g = (H_g, V_g, E_g)$ ($f$ can also be a Cutkosky graph, but $g$ is uncut).

We define the insertion of $g$ into $f$, first by specifying an insertion place and a suitable bijection.

Assume $l_g \geq 3$. Assume $v \in V_f$ such that $\text{val}(v) = l_g$. Choose a bijection $\sigma$ between $C_v$ and $L_g$.

Define

$$f \ast_{v, \sigma} g = (H_{f \ast_{v, \sigma} g}, V_{f \ast_{v, \sigma} g}, E_{f \ast_{v, \sigma} g}),$$

where

$$H_{f \ast_{v, \sigma} g} = (H_f - C_v) \dot{\cup} H_g,$$

$$V_{f \ast_{v, \sigma} g} = V_g \dot{\cup} (V_f - C_v),$$

$$E_{f \ast_{v, \sigma} g} = (E_f - \dot{\cup}_{e \in E_f \cap C_v} e) \dot{\cup} E_g \dot{\cup} \sigma_e,$$

where $\sigma_e$ is the set of edges induced by the bijection $\sigma$, each consisting of a half-edge in $C_v$ and a half-edge in $L_g$. 
Now assume \( l_g = 2 \). Choose \( e \in E_f \). Choose one of the two possible bijections between \( L_g \) and the edge \( e \) regarded as a set of two half-edges.

Define

\[
f_{e, \sigma} g = (H_{f_{e, \sigma} g}, V_{f_{e, \sigma} g}, E_{f_{e, \sigma} g}),
\]

where

\[
H_{f_{e, \sigma} g} = H_f \cup (H_g \setminus e),
V_{f_{e, \sigma} g} = V_g \cup V_f,
E_{f_{e, \sigma} g} = (E_f - e) \cup E_g \cup \sigma_e,
\]

where \( \sigma_e \) is the set of two edges induced by the bijection \( \sigma \), each consisting of a half-edge in \( e \) and a half-edge in \( L_g \).

Summing over vertices (for \( l_g = 3 \)) or edges (for \( l_g = 2 \)) and over bijections defines a map \( f \star g \) which gives a pre-Lie product on graphs

\[
\sum_{G \in \mathcal{H}_{\text{core}}} n(g, f, G)G,
\]

where \( n(g, f, G) \) counts the number of appearances of \( G \) in those sums.
Ex:  Basically, this is already valid.

\[(6_1 \times 6_2) \times 6 - 6_1 \times (6_2 \times 6_3)\]

\[(6_1 \times 6_2) \times 6 - 6_1 \times (6_2 \times 6_3)\]

\[= (6_1 \times 6_3) \times 6_2 - 6_1 \times (6_3 \times 6_2)\]

\[\text{Product: } (6_1 \times 6_2) - (6_2 \times 6_1) = \{6_1, 6_2\}\]

\[\sum_{C_7} \left[\{6_1, 6_2\}, 6_3\right] = 0\]

\[\text{\textcopyright 2020}\]
2.2. The core Hopf algebra $H_{\text{core}}$. The core Hopf algebra $H_{\text{core}}$ is based on the $\mathbb{Q}$-vectorspace generated by connected bridgeless Feynman graphs.

We define a commutative product

$$m : H_{\text{core}} \otimes H_{\text{core}} \to H_{\text{core}}, \quad m(G_1, G_2) = G_1 \cup G_2,$$

by disjoint union. The unit $\mathbb{I}$ is provided by the empty set so that we get a free commutative $\mathbb{Q}$-algebra with bridgeless graphs as generators.

We define a co-product by

$$\Delta_{\text{core}}(G) = G \otimes \mathbb{I} + \mathbb{I} \otimes G + \sum_{g \subseteq G} g \otimes G/g,$$

where the sum is over all $g \in H_{\text{core}}$ such that $g \subsetneq G$. Hence there are bridgeless graphs $g_i$ such that $g = \dot{\cup}_i g_i$, and $G/g$ denotes the co-graph in which all internal edges of all $g_i$ shrink to zero length in $G$.

We have a co-unit $\hat{\mathbb{I}} : H_{\text{core}} \to \mathbb{Q}$ which annihilates any non-empty graph and $\hat{\mathbb{I}}(\mathbb{I}) = 1$ and we have the antipode $S : H_{\text{core}} \to H_{\text{core}}$, $S(\mathbb{I}) = \mathbb{I}$

$$S(G) = -G - \sum_{g \subseteq G} S(g)G/g.$$

Furthermore our Hopf algebras are graded,

$$H_{\text{core}} = \bigoplus_{j=0}^{\infty} H_{\text{core}}^{(j)}, \quad H_{\text{core}}^{(0)} \cong \mathbb{Q}\mathbb{I}, \quad \text{Aug}_{\text{core}} = \bigoplus_{j=1}^{\infty} H_{\text{core}}^{(j)},$$

and $h \in H_{\text{core}}^{(j)} \iff |h| = j$. 
We define structure coefficients \( n(g, G/g, G) \in \mathbb{N}_0 \) by setting
\[
\Delta_{\text{core}}(G) = \sum_g n(g, G/g, G') \left( g \otimes G/g \right).
\]
We can then define \( \ast : H_{\text{core}} \otimes H_{\text{core}} \to H_{\text{core}} \).

(2.1)
\[
G_1 \ast G_2 := \sum_{G \in H_{\text{core}}} \frac{n(G_2, G_1, G)}{|G|_v} G.
\]

Here,
\[
|G|_v := |\{ F \in H_{\text{core}} | \tilde{F} = \tilde{G} \}|
\]
is the number of graphs \( F \) which have the same amputated graph \( \tilde{G} = \tilde{F} \) as \( G \). That is \( |G|_v \) is the number of different ways of attaching an ordered set of external edges to the amputated graph of \( G \). Note that this number is finite as \( l_G \leq l_{G_1} + l_{G_2} \) and we assume \( l_{G_1}, l_{G_2} \leq \infty \).

We have \( H_G = H_{G} - L_{G} \), \( H_F = H_{F} - L_{F} \).
Lemma 2.1. The map $\ast$ is pre-Lie.

Proof. Known. $\square$

Proof in old style by $\mathfrak{l} \times \mathfrak{gl}$
This Hopf algebra has an extension operating on pairs \((G, F)\) of a graph \(G\) and a spanning forest \(F\).

Let \(\mathcal{F}_G\) be the set of all spanning forests of \(G\). The empty graph \(\mathbb{I}\) has an empty spanning forest also denoted by \(\mathbb{I}\).

We define a \(\mathbb{Q}\)-Hopf algebra \(H_{GF}\) for such pairs \((G, F), F \in \mathcal{F}_G\) by setting
\[
\Delta_{GF}(G, F) = (G, F) \otimes (\mathbb{I}, \mathbb{I}) + (\mathbb{I}, \mathbb{I}) \otimes (G, F) + \sum_{g \in G, F - (F \cap g) \in \mathcal{F}_{G/g}} (g, g \cap F) \otimes (G/g, F - (F \cap g)).
\]

Note that the condition \(F - (F \cap g) \in \mathcal{F}_{G/g}\) ensures that only terms contribute such that \(G/g\) has a valid spanning forest.

For the corresponding reduced co-product we have
\[
\tilde{\Delta}_{GF}(G, F) = + \sum_{g \in G, F - (F \cap g) \in \mathcal{F}_{G/g}} (g, g \cap F) \otimes (G/g, F - (F \cap g)),
\]

We define the commutative product to be
\[
m_{GF}((G_1, F_1), (G_2, F_2)) = (G_1 \cup G_2, F_1 \cup F_2),
\]
whilst \(\mathbb{I}_{GF} = (\mathbb{I}, \mathbb{I})\) serves as the obvious unit which induces a co-unit through \(\hat{\mathbb{I}}_{GF}(\mathbb{I}_{GF}) = 1\) and \(\hat{\mathbb{I}}_{GF}((G, F)) = 0\).

**Theorem 2.2.** This is a graded commutative bi-algebra graded by \(\vert G \vert\) and therefore a Hopf algebra \(H_{GF}(\mathbb{I}_{GF}, \hat{\mathbb{I}}_{GF}, m_{GF}, \Delta_{GF}, S_{GF})\).

We have \(H_{GF} = \oplus_{j=0}^{\infty} H^{(j)}_{GF}\) with \(H^{(0)}_{GF} \sim \mathbb{Q}\mathbb{I}_{GF}\) and \(\text{Aug}_{GF} = \oplus_{j=1}^{\infty} H^{(j)}_{GF}\). \((G, F) \in H^{(j)}_{GF} \iff \vert G \vert = j\).
2.3. The renormalization Hopf algebra $H_{\text{ren}}$. This is a quotient Hopf algebra of $H_{\text{core}}$.

\[ \Delta_{\text{ren}} g = \sum g \otimes g/\alpha \]

\[ \Delta_{\text{ren}} g = \sum g \otimes g/\alpha \]

$\omega g \geq 0$

$g = \frac{11}{g}$
2.4. The vector space $H_C$. Consider a Cutkosky graph $G$ with a corresponding $v_G$-refinement $P$ of its set of external edges $L_G$. It is a maximal refinement of $V_G$.

The core Hopf algebra co-acts on the vector-space of Cutkosky graphs $H_C$.

$$ \Delta_{\text{core}} : H_C \to H_{\text{core}} \otimes H_C. $$

We say $G \in H_C^{(n)} \iff |G| = n$ and define $\text{Aut}_C = \bigoplus_{i=1}^{\infty} H_C^{(i)}$. 

\[ \xymatrix{ H_C \ar[r] & H_{\text{core}} \otimes H_C } \]
Note that the sub-vectorspace $H_C^{(0)}$ is rather large: it contains all graphs $G = ((H_G, V_G, E_G), (H_H, V_H, E_H)) \in H_C$ such that $||G|| = 0$. These are the graphs where the cuts leave no loop intact.

For any $G \in H_C$ there exists a largest integer $\text{cor}_C(G) \geq 0$ such that

$$\bar{\Delta}_{\text{core}}^{\text{cor}_C(G)}(G) \neq 0, \bar{\Delta}_{\text{core}}^{\text{cor}_C(G)}(G) : H_C \to H_{\text{core}}^{\text{cor}_C(G)} \otimes H_C^{(0)},$$

whilst $\bar{\Delta}_{\text{core}}^{\text{cor}_C(G)+1}(G) = 0.$
Proposition 2.3.

\[ \text{cor}_C(G) = ||G||. \]

Proof. The primitives of \( H_{\text{core}} \) are one-loop graphs.

In particular there is a unique element \( g \otimes G / g \in H_{\text{core}} \otimes H_C^{(0)}: \)

\[ \Delta_{\text{core}}(G) \cap \left( H_{\text{core}} \otimes H_C^{(0)} \right) = g \otimes G / g, \]

with \( |g| = ||G||. \)

For any graph \( G \) we let \( G = \sum_{T \in \mathcal{T}_G} (G, T) \). Here \( \mathcal{T}_G \) is the set of all spanning trees of \( G \) and we set for \( G = \bigcup_i G_i, \mathcal{T}_G = \bigcup_i \mathcal{T}_{G_i}. \)

The maximal refinement \( P \) induces for each partition \( P(i), 0 \leq i \leq v_G \) a unique spanning forest \( f_i \) of \( G / g \). The set \( \mathcal{F}_{G, P(i)} \) of spanning forests of \( G \) compatible with \( P(i) \) is then determined by \( f_i \) and the spanning trees in \( \mathcal{T}_g. \)

Define \( G_i := \sum_{F \in \mathcal{F}_{G, P(i)}} (G, F). \)

\[ (2.5) \quad \tilde{\Delta}^{||G||} G_i = \sum_{i=1}^{||G||} G_i^{(1)} \otimes \cdots \otimes G_i^{(||G||+1)}. \]

Note that \( |G_i^{k}| = 1, \forall k \leq (||G|| + 1) \) and \( |G_i^{||G||+1}| = 0. \)