FEYNMAN DIAGRAMS AND THE S-MATRIX, AND OUTER SPACE (SUMMER 2020)

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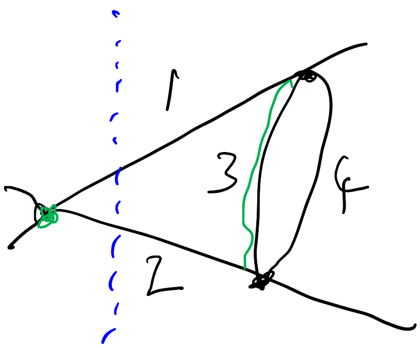
1. Sets or edges

Given a spanning k-forest F of a cut graph G, there are a number of different sets of edges which will be important. First the edges $e \in E_F$ of the forest themselves are important.

Second are the edges $e \in C_G$ of G which are not in F but join distinct components of F. If we view F as a spanning tree T with some edges removed then all the edges of $T - F$ are in this second class, as well, typically, as others.



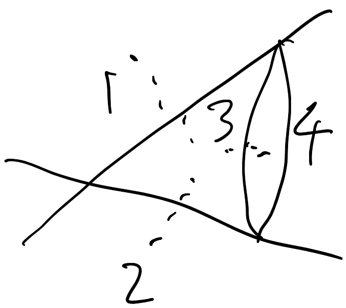
$E_F = \{3\}$

$\check{E}_F = \{1, 2\} = C_G$

$G = ((H_G, V_G, \mathcal{E}_G), (H_G, V_G, \mathcal{E}_{+1}))$

third set $\{4\}$

Third are the edges $e \in (E_G - E_T)$ of G which are not in F but have both ends in the same tree of F.



$C_G = \{1, 2, 3\}$

$\breve{E}_F \neq C_G \quad \forall F$

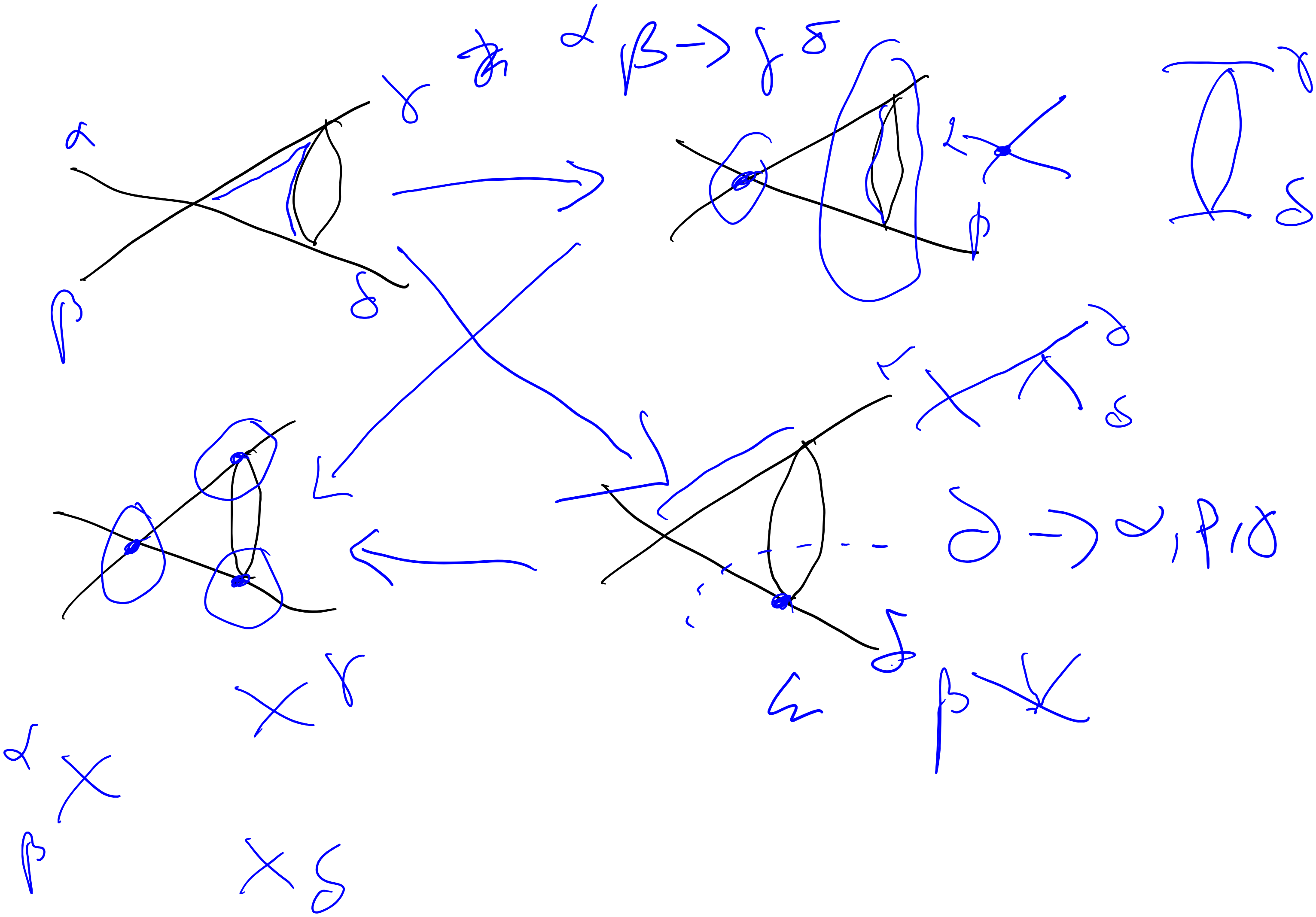
"Third set" is $\breve{E}_F = F_{on}$

The second and third sets of edges above are those which will ultimately be put on-shell and define the set E_{on}, while those in the first set remain off-shell whilst we will use the notation $\breve{E}_F$ for the second of the above sets of edges.

Definition 1.1. A Cutkosky graph G is a cut graph G for which a compatible spanning forest F such that $C_G = \breve{E}_F$ exists.

Compatibility ensures that the spanning forest is in accordance with the chosen refinements $\mathcal{E}_H$.

Note $h_0(\tilde{G}) = h_0(F)$ for a compatible F and note that an ordering of edges in a spanning tree of a Cutkosky graph G induces a $(v_G - 1)$-refinement of L_G.

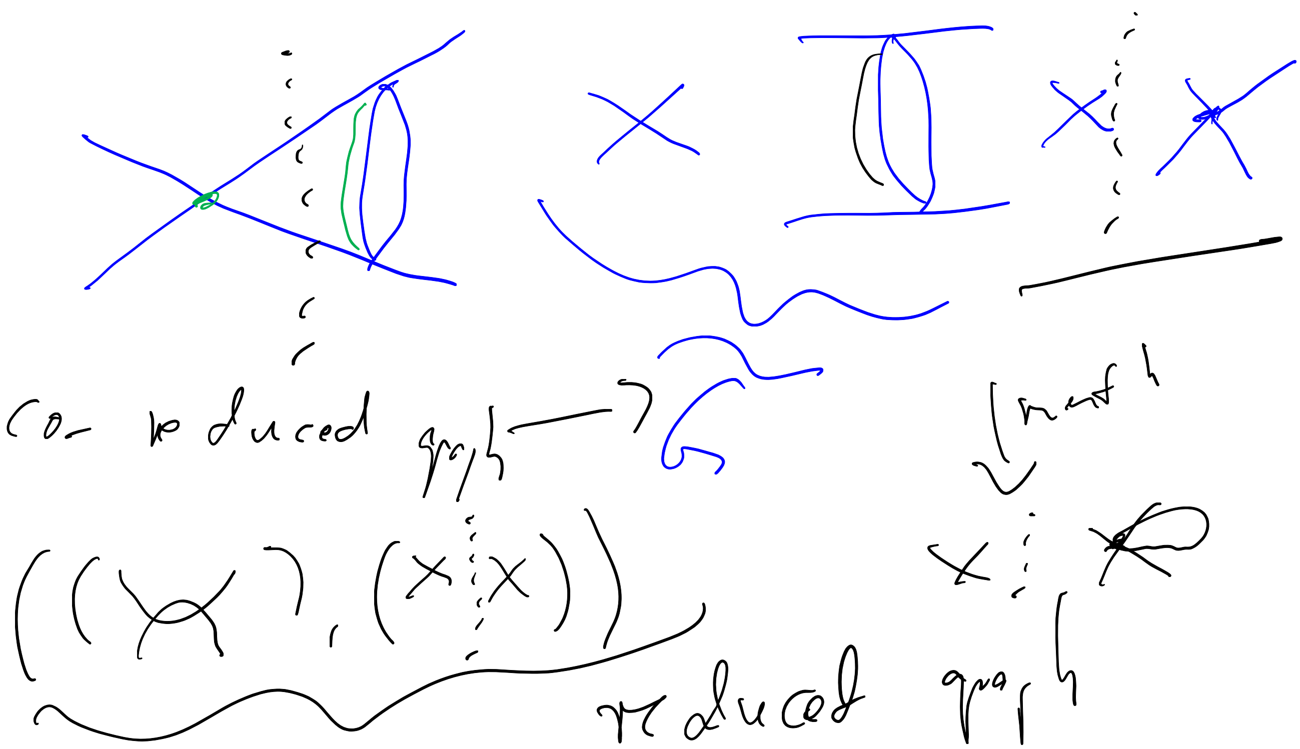


We say that a spanning tree T of G with ordered edges is compatible with a given $(v_G - 1)$-refinement R of L_G if and only if the forests $T - \amalg_{i=1}^{k} e_i$ induce the $k+1$-partition of R.

We let $\mathcal{F}_G^R$ be the set of ordered spanning trees of a graph G compatible with a $(v_G - 1)$-refinement R of L_G . Note that if $R = L_G$ is the trivial partition, then $\mathcal{F}_G^R = \mathcal{T}_G$, the set of spanning trees of G.

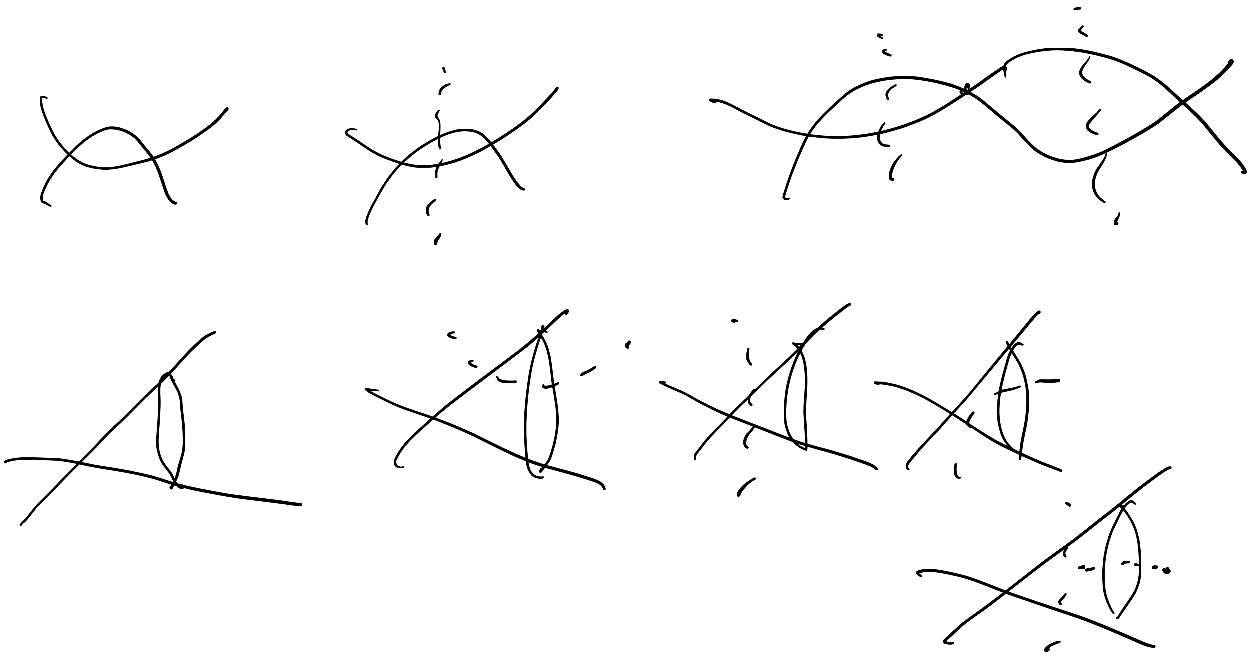
Definition 1.2. For a Cutkosky graph G we let $G^{red} := \tilde{G}/E_{\tilde{G}}$ be the reduced graph. It is a collection of $h_0(\tilde{G})$ corollas and hence a Cutkosky graph $(\hat{G}^{red}, G^{red})$ with $\hat{G}^{red} = \hat{G}/E_{\tilde{G}}$.

We call $\tilde{G}$ the co-reduced graph.



Definition 1.3. We define H_C to be the $\mathbb{Q}$-vectorspace generated by Cutkosky graphs.

Definition 1.4. We define H_{core} to be the $\mathbb{Q}$-vectorspace generated by graphs without cuts.



2. Hopf algebras

We have to define two Hopf algebras H_{core} and H_{ren}. Both will co-act on H_C defined above.

2.1. graph insertion and graph decomposition. Consider graphs $f = (H_f, \mathcal{V}_f, \mathcal{E}_f)$ and $g = (H_g, \mathcal{V}_g, \mathcal{E}_g)$ (f can also be a Cutkosky graph, but g is uncut).

We define the insertion of g into f, first by specifying an insertion place and a suitable bijection.

Assume $l_g \geq 3$. Assume $v \in V_f$ such that $\mathbf{val}(v) = l_g$. Choose a bijection σ between C_v and L_g.

Define

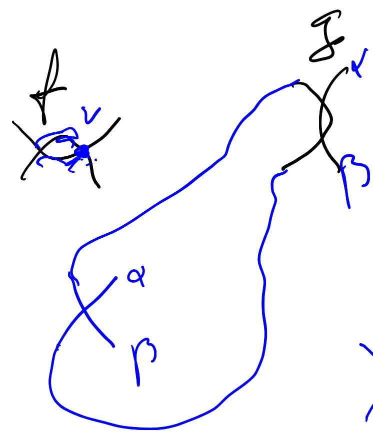
$$f *_{v,\sigma} g = (H_{f*_{v,\sigma}g}, \mathcal{V}_{f*_{v,\sigma}g}, \mathcal{E}_{f*_{v,\sigma}g}),$$

where

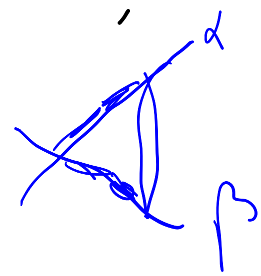
$$H_{f*_{v,\sigma}g} = (H_f - C_v)\dot{\cup} H_g,$$
$$\mathcal{V}_{f*_{v,\sigma}g} = \mathcal{V}_g \dot{\cup}(\mathcal{V}_f - C_v),$$
$$\mathcal{E}_{f*_{v,\sigma}g} = (\mathcal{E}_f - \dot{\cup}_{e\cap C_v \neq \emptyset} e)\dot{\cup}\mathcal{E}_g \dot{\cup}\sigma_e,$$

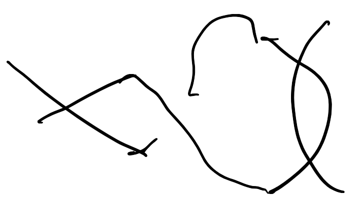
where σ_e is the set of edges induced by the bijection σ, each consisting of a half-edge in C_v $\cap \mathcal{E}_f$
and a half-edge in L_g.

$f *_{v,\sigma} g$



$l_g = 4$





Now assume $l_g = 2$. Choose $e \in E_f$. Choose one of the two possible bijections between L_g and the edge e regarded as a set of two half-edges.

Define

$$f *_{e,\sigma} g = (H_{f*_{e,\sigma}g}, \mathcal{V}_{f*_{e,\sigma}g}, \mathcal{E}_{f*_{e,\sigma}g}),$$

where

$$H_{f*_{e,\sigma}g} = H_f \dot\cup (H_g \cancel{- L_g}),$$
$$\mathcal{V}_{f*_{e,\sigma}g} = \mathcal{V}_g \dot\cup \mathcal{V}_f,$$
$$\mathcal{E}_{f*_{e,\sigma}g} = (\mathcal{E}_f - e) \dot\cup \mathcal{E}_g \dot\cup \sigma_e,$$

where σ_e is the set of two edges induced by the bijection σ, each consisting of a half-edge in e and a half-edge in L_g.

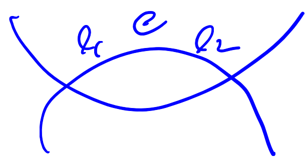
Summing over vertices (for $l_g = 3$) or edges (for $l_g = 2$) and over bijections defines a map $f \star g$ which gives a pre-Lie product on graphs

$$f * g = \sum_{G \in H_{core}} \underline{n(g, f, G)} G,$$

where $n(g, f, G)$ counts the number of appearances of G in those sums.

$L_g = l_1, l_2$

$e = \{e_1, e_2\}$



$*_{e,\sigma}$

$\sigma_1 \; (l_1, e_1)(l_2, e_2)$

$\sigma_2 \; (l_1, e_2)(l_2, e_1)$

(e_1, l_1)

$\sum_{v,\sigma} *_{v,\sigma} \qquad \sum_{e,\sigma} *_{e,\sigma}$

Ex: Basically, this is already pre-Lie.

$$(G_1 * G_2) * G_3 - G_1 * (G_2 * G_3)$$

$$= (G_1 * G_3) * G_2 - G_1 * (G_3 * G_2)$$

"pre-Lie" product

$$(G_1 * G_2) - (G_2 * G_1) = [G_1, G_2]$$

$$\sum_{\text{cyl}} [[G_1, G_2], G_3] \equiv 0$$

Jacobi

2.2. **The core Hopf algebra** H_{core}**.** The core Hopf algebra H_{core} is based on the $\mathbb{Q}$-vectorspace generated by connected bridgeless Feynman graphs.

We define a commutative product

$$m : H_{core} \otimes H_{core} \to H_{core},\ m(G_1, G_2) = G_1 \dot{\cup} G_2,$$

by disjoint union. The unit $\mathbb{I}$ is provided by the empty set so that we get a free commutative $\mathbb{Q}$-algebra with bridgeless graphs as generators.

We define a co-product by

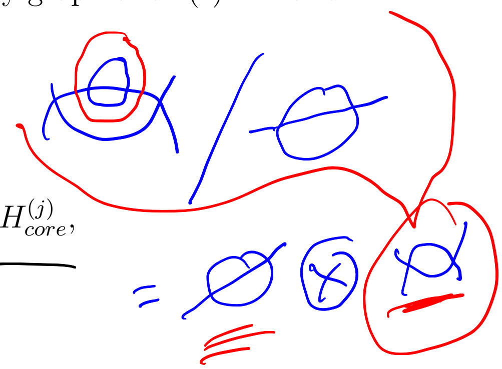
Δ_{core} is co-assoc.

$$\Delta_{core}(G) = G \otimes \mathbb{I} + \mathbb{I} \otimes G + \sum_{g \subsetneq G} g \otimes G/g,$$

where the sum is over all $g \in H_{core}$ such that $g \subsetneq G$. ~~Hence there~~ are bridgeless graphs g_i such that $g = \dot{\cup}_i g_i$, and G/g denotes the co-graph in which all internal edges of all g_i shrink to zero length in G.

We have a co-unit $\hat{\mathbb{I}} : H_{core} \to \mathbb{Q}$ which annihilates any non-empty graph and $\hat{\mathbb{I}}(\mathbb{I}) = 1$ and we have the antipode $S : H_{core} \to H_{core}$, $S(\mathbb{I}) = \mathbb{I}$

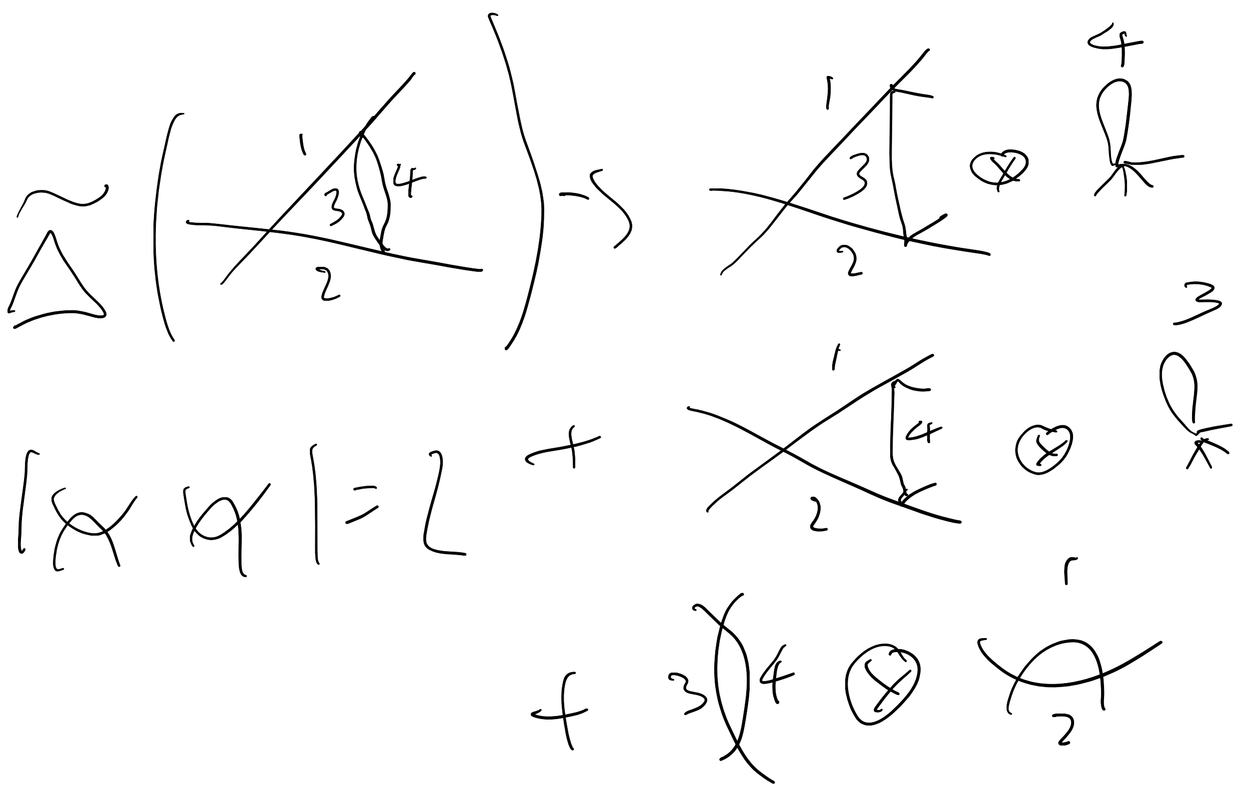
$$S(G) = -G - \sum_{g \subsetneq G} S(g) G/g.$$



Furthermore our Hopf algebras are graded,

$$H_{core} = \oplus_{j=0}^{\infty} H_{core}^{(j)},\ H_{core}^{(0)} \cong \mathbb{Q}\mathbb{I},\ \mathrm{Aug}_{core} = \oplus_{j=1}^{\infty} H_{core}^{(j)},$$

and $h \in H_{core}^{(j)} \Leftrightarrow |h| = j$.



We define structure coefficients $n(g, G/g, G) \in \mathbb{N}_0$ by setting

$$\Delta_{core}(G) = \sum_g n(g, G/g, G)\left(g \otimes G/g.\right)$$

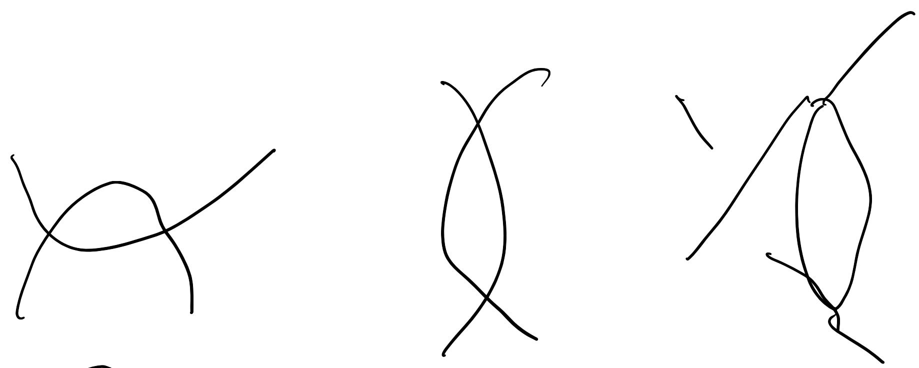
We can then define $\star : H_{core} \otimes H_{core} \to H_{core}$.

$$G_1 \star G_2 := \sum_{G \in H_{core}} \frac{n(G_2, G_1, G)}{|G|_v} G. \tag{2.1}$$

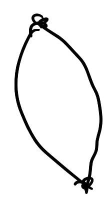
Here,

$$|G|_v := |\{F \in H_{core} | \bar{F} = \bar{G}\}|,$$

is the number of graphs F which have the same amputated graph $\bar{G} = \bar{F}$ as G. That is $|G|_v$ is the number of different ways of attaching an ordered set of exernal edges to the amputated graph of G. Note that this number is finite as $l_G \lneq l_{G_1} + l_{G_2}$ and we assume $l_{G_1}, l_{G_2} \lneq \infty$. We have $H_{\bar{G}} = H_G - L_G$, $H_{\bar{F}} = H_F - L_F$



G_1 G_2 G_3

$\bar{G}_1 = \bar{G}_2 = \bar{G}_3 =$ 

Lemma 2.1. *The map $\star$ is pre-Lie.*

Proof. Known. $\square$

Proof in old paper by

Kr Connes

This Hopf algebra has an extension operating on pairs (G, F) of a graph G and a spanning forest F.

Let $\mathcal{F}_G$ be the set of all spanning forests of G. The empty graph $\mathbb{I}$ has an empty spanning forest also denoted by $\mathbb{I}$.

We define a $\mathbb{Q}$-Hopf algebra H_{GF} for such pairs (G, F), $F \in \mathcal{F}_G$ by setting

$$\begin{aligned} \Delta_{GF}(G,F) &= (G,F)\otimes(\mathbb{I},\mathbb{I}) + (\mathbb{I},\mathbb{I})\otimes(G,F) + \\ &+ \sum_{\substack{g \subsetneq G \\ F-(F\cap g)\in\mathcal{F}_{G/g}}} (g, g\cap F)\otimes(G/g, F-(F\cap g)). \end{aligned} \tag{2.2}$$

Note that the condition $F - (F \cap g) \in \mathcal{F}_{G/g}$ ensures that only terms contribute such that G/g has a valid spanning forest.

For the corresponding reduced co-pruduct we have

$$\tilde{\Delta}_{GF}(G,F) = + \sum_{\substack{g \subsetneq G \\ F-(F\cap g)\in\mathcal{F}_{G/g}}} (g, g\cap F)\otimes(G/g, F-(F\cap g)), \tag{2.3}$$

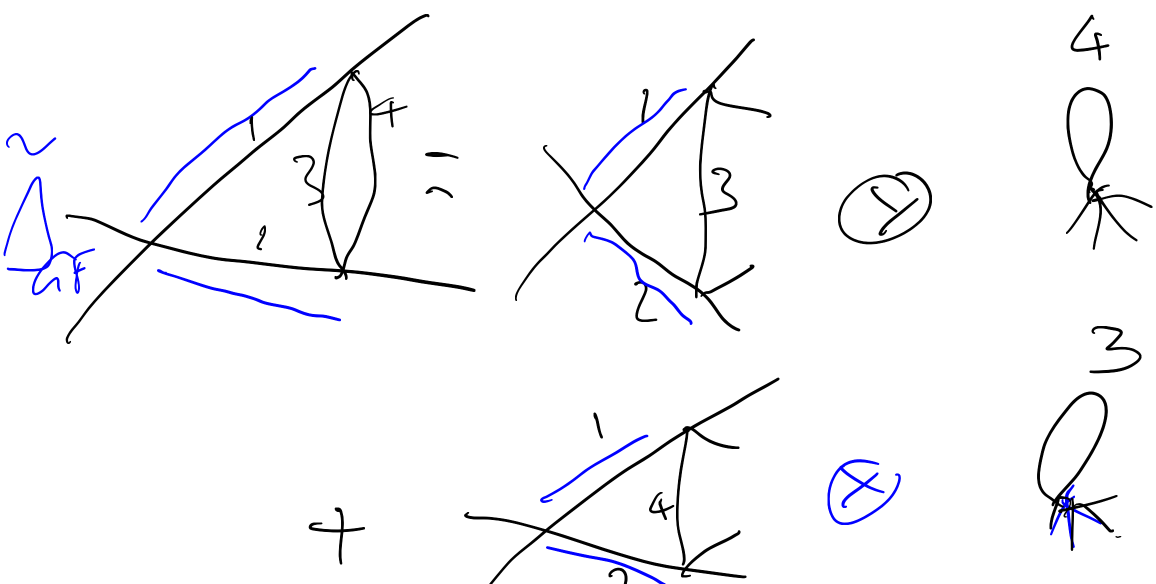
We define the commutative product to be

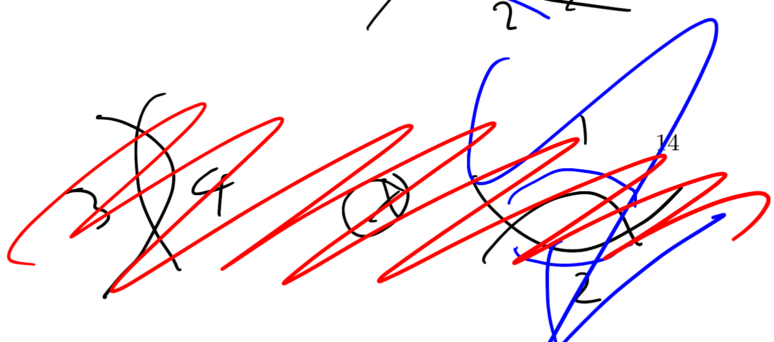
$$m_{GF}((G_1,F_1),(G_2,F_2)) = (G_1 \dot\cup G_2, F_1 \dot\cup F_2),$$

whilst $\mathbb{I}_{GF} = (\mathbb{I},\mathbb{I})$ serves as the obvious unit which induces a co-unit through $\hat{\mathbb{I}}_{GF}(\mathbb{I}_{GF}) = 1$ and $\hat{\mathbb{I}}_{GF}((G,F)) = 0$.

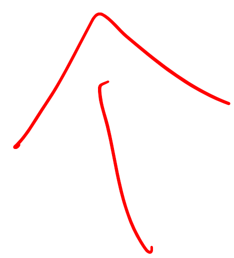
Theorem 2.2. *This is a graded commutative bi-algebra graded by $|G|$ and therefore a Hopf algebra $H_{GF}(\mathbb{I}_{GF}, \hat{\mathbb{I}}_{GF}, m_{GF}, \Delta_{GF}, S_{GF})$.*

We have $H_{GF} = \oplus_{j=0}^{\infty} H_{GF}^{(j)}$ with $H_{GF}^{(0)} \sim \mathbb{Q}\mathbb{I}_{GF}$ and $\mathrm{Aug}_{GF} = \oplus_{j=1}^{\infty} H_{GF}^{(j)}$. $(G,F) \in H_{GF}^{(j)} \Leftrightarrow |G| = j$.





Ex:



2.3. **The renormalization Hopf algebra H_{ren}.** This is a quotient Hopf algebra of H_{core}.

$$\tilde{\Delta}_{core} G = \sum_{g \subsetneq G} g \otimes G/g$$

$$\tilde{\Delta}_{ren} G = \sum_{g \subsetneq G} g \otimes G/\gamma$$

$$\omega_{g_i} \geq 0$$

$$* \quad g = \coprod g_i$$

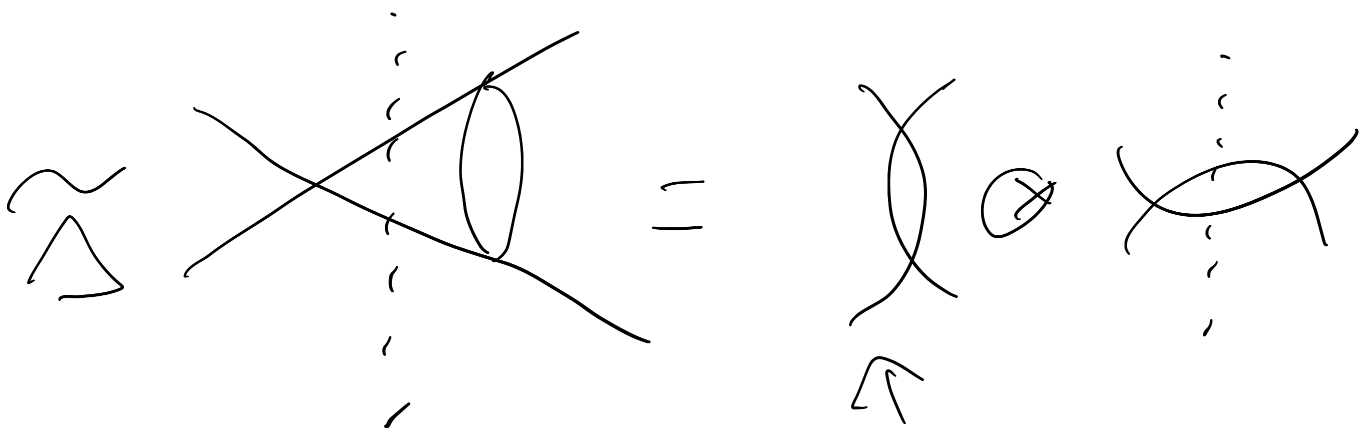
2.4. **The vectorspace H_C.** Consider a Cutkosky graph G with a corresponding v_G-refinement P of its set of external edges L_G. It is a maximal refinement of V_G.

The core Hopf algebra co-acts on the vector-space of Cutkosky graphs H_C.

$$\Delta_{core} : H_C \to H_{core} \otimes H_C. \tag{2.4}$$

We say $G \in H_C^{(n)} \Leftrightarrow |G| = n$ and define $\mathrm{Aut}_C = \oplus_{i=1}^{\infty} H_C^{(i)}$.

$H_C \to H_{core} \otimes H_C$ left co-action



might need ren.

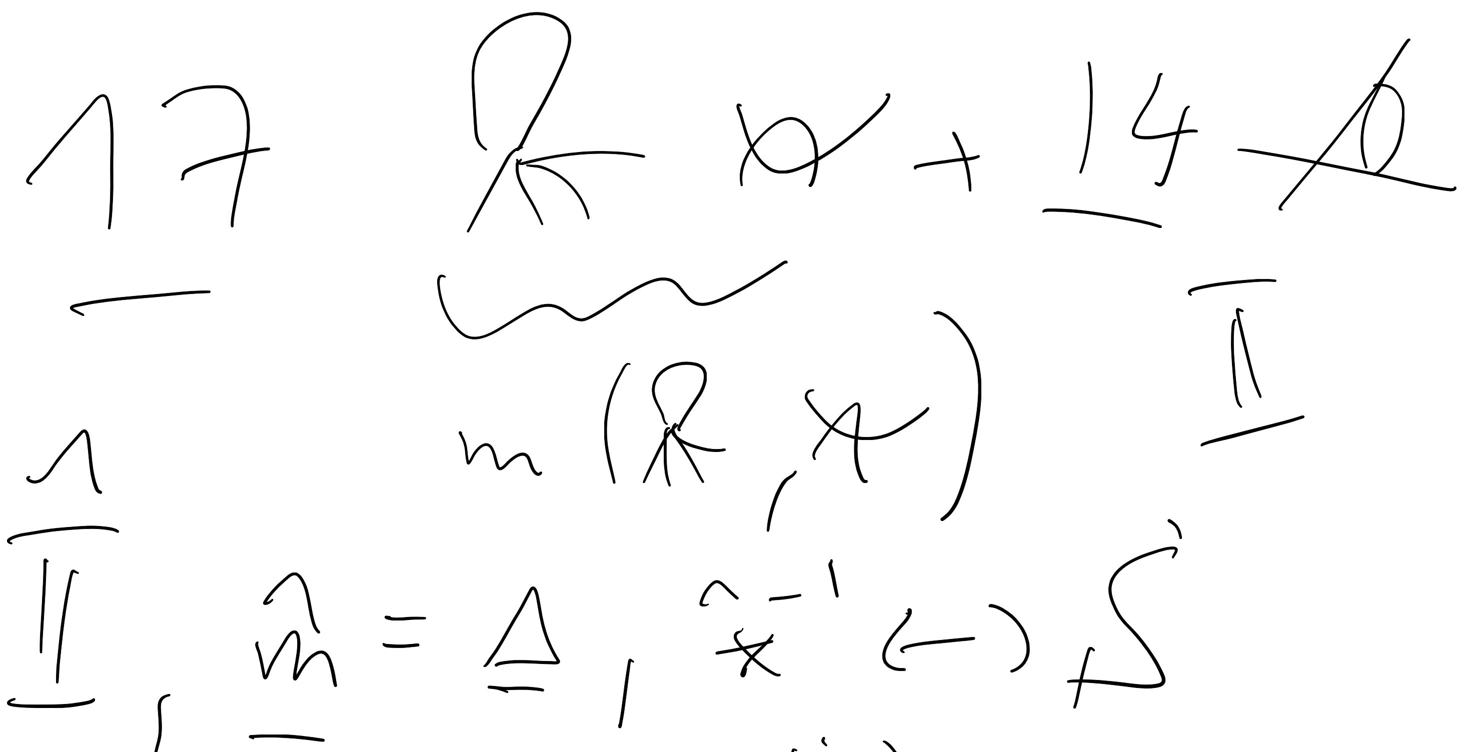
$\Delta_{ren} : H_C \to H_{core} \otimes H_C$

Note that the sub-vectorspace $H_C^{(0)}$ is rather large: it contains all graphs $G = ((H_G, \mathcal{V}_G, \mathcal{E}_G), (H_G, \mathcal{V}_G, \mathcal{E}_H)) \in H_C$ such that $||G|| = 0$. These are the graphs where the cuts leave no loop intact.

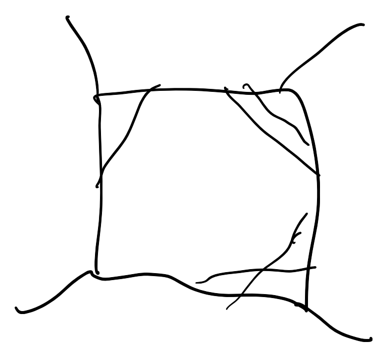
For any $G \in H_C$ there exists a largest integer $\mathrm{cor}_C(G) \geq 0$ such that

$$\tilde{\Delta}_{core}^{\mathrm{cor}_C(G)}(G) \neq 0,\ \tilde{\Delta}_{core}^{\mathrm{cor}_C(G)}(G) : H_C \to H_{core}^{\otimes cor_C(G)} \otimes H_C^{(0)},$$

whilst $\tilde{\Delta}_{core}^{\mathrm{cor}_C(G)+1}(G) = 0$.



$$H = \bigoplus_{j=0}^{\infty} H^{(j)} = \sum_{v,\rho}$$



Proposition 2.3.

$$\mathrm{cor}_C(G) = ||G||.$$

Proof. The primitives of H_{core} are one-loop graphs. $\square$

In particular there is a unique element $g \otimes G/g \in H_{core} \otimes H_C^{(0)}$:

$$\Delta_{core}(G) \cap \left(H_{core} \otimes H_C^{(0)}\right) = g \otimes G/g,$$

with $|g| = ||G||$.

For any graph G we let $\mathbf{G} = \sum_{T \in \mathcal{T}_G}(G,T)$. Here $\mathcal{T}_G$ is the set of all spanning trees of G and we set for $G = \dot{\cup}_i G_i$, $\mathcal{T}_G = \dot{\cup}_i \mathcal{T}_{G_i}$.

The maximal refinement P induces for each partition $P(i), 0 \leq i \leq v_G$ a unique spanning forest f_i of G/g. The set $\mathcal{F}_{G,P(i)}$ of spanning forests of G compatible with $P(i)$ is then determined by f_i and the spanning trees in $\mathcal{T}_g$.

Define $\mathbf{G}_i := \sum_{F \in \mathcal{F}_{G,P(i)}}(G,F)$.

$$\tilde{\Delta}^{||G||}_{G,F}\mathbf{G}_i = \sum_{i=1} \mathbf{G}_{\mathbf{i}}^{(\mathbf{1})} \otimes \cdots \otimes \mathbf{G}_{\mathbf{i}}^{(||G||+1)}. \tag{2.5}$$

Note that $|\mathbf{G}_i^k| = 1$, $\forall k \lneq (||G||+1)$ and $|\mathbf{G}_i^{||G||+1}| = 0$.

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