Critical Exponents and Invariant Charges

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Declaration

I hereby declare that this thesis is my own work and effort and that it has not been submitted anywhere for any award. Where other sources of information have been used, they have been acknowledged.

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CHAPTER 1

Introduction

One of the greatest mysteries to physicists is the behaviour of the smallest constituents of matter, the fundamental particles and their interactions. Over the better part of the 20th century many efforts were made to give a better understanding of fundamental particle dynamics, culminating in the standard model. The underlying framework, quantum field theory, is however not only used in particle physics, but can be employed for a variety of problems ranging from statistical mechanics and condensed matter physics to combinatorics, number theory and even biology (see for example [Con00; Con01; Del85; Kre06b; Wal74; Zin96]). Although the framework of quantum field theory is widely excepted and successful in its methods (with the standard model being the most rigorously tested model in physics), mathematically it is far from being well understood. A consistent and all encompassing groundwork for quantum field theory giving rise to descriptions of all desired physically observed phenomena is still elusive, despite many endeavours and hard work by physicists and mathematicians alike. In recent years new methodology was found by D. Kreimer and collaborators connecting combinatorial Hopf algebras and Dyson-Schwinger equations with particle scattering phenomena described by perturbative Feynman diagrams, giving precise mathematical meaning to the process of renormalisation and opening the gate for new research endeavours [Ber05; Bro01; Bro11; Kre06a; Kre06b; Kre03; Kre97; Kre09; Kre06c; Yea17] . From another branch in the quantum field theory community J. Gracey found formulation of universality classes, described in statistical physics by equal critical exponents, by tower theories. A *tower theory* is constructed from a base Lagrangian in a certain space-time dimension by a rather specific construction prescription, then by increasing the space-time dimension it connects different Lagrangians and thus (physical) theories lying in the same universality class [Gra; Gra17a; Gra17b]. The aim of this work is to give an explanation to why this peculiar approach indeed does define universality classes in the usual sense by constructing the renormalisation Hopf algebra of a massless scalar theory and its Dyson-Schwinger equations and analysing their structure. In chapter 2 the needed foundations are defined, followed by an introduction to tower theories in chapter 3. In chapter 4 the connection between tower theories and universality classes will be approached, while in chapter 5 results and consequences will be discussed. All graphs in this work have been compiled with the TikZ-Feynman package from J.P. Ellis [Ell17] for Latex.

CHAPTER 2

Prerequisits

In the following chapter the fundamentals of the (renormalisation) Hopf Algebra H, Rooted Trees, (combinatorial) Dyson-Schwinger equations (DSEs) and the correspondence between symmetries and Hopf-Ideals will be presented. The aim is to lay the groundwork for derivations in later chapters.

2.1 Hopf Algebra

A Hopf algebra is a graded bialgebra with an antipode, which is the inverse element with respect to character or convolution group (2.12). The exact definitions follow below. Let $\mathbb{F}$ be a field of characteristic zero (think of $\mathbb{Q}$ or $\mathbb{C}$) and the following tensor-products are taken over said field (which means $\otimes$ actually stands for $\otimes_{\mathbb{F}}$)

Algebras

Definition 2.1 ***Algebras*** *A (associative) $\mathbb{F}$-algebra is a collection (A,m,u) of a $\mathbb{F}$-vector space A together with a bilinear map $m : A \otimes A \to A$ the multiplication and another linear map $u : \mathbb{F} \to A$ the unit map, satisfying*

$$m \circ (id \otimes m) = m \circ (m \otimes id) \tag{2.1}$$

$$m \circ (u \otimes id) = m \circ (id \otimes u) \tag{2.2}$$

which is the same as providing that the following diagram commutes

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes id} & A \otimes A \\ \Big\downarrow{\scriptstyle id \otimes m} & & \Big\downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{\quad m \quad} & A \end{array}$$

The algebra is also commutative, if for the twist map $\tau : A_1 \otimes A_2 \to A_2 \otimes A_1$ of vector spaces A_1, A_2 with elements $a_1 \in A_1$ and $a_2 \in A_2$, such that $\tau(a_1 \otimes a_2) = a_2 \otimes a_1$ and m satisfies

$$m = m \circ \tau$$

as well. From here on commutativity is always assumed.

Although this is not the standard definition of an algebra, it serves our purpose well.

Coalgebras

By inverting the arrows in (2.1), we get a dual algebra or coalgebra holding maps to invert the operations of that algebra.

Definition 2.2 ***Coalgebras*** *A (coassociative) $\mathbb{F}$-coalgebra is a collection $(C,\varDelta,\varepsilon)$ of a $\mathbb{F}$-vector space C together with a linear map*

$$\varDelta : C \to C \otimes C$$

the coproduct and another linear map $\varepsilon : C \to \mathbb{F}$ the counit, satisfying

$$(id \otimes \varDelta) \circ \varDelta = (\varDelta \otimes id) \circ \varDelta \tag{2.3}$$

$$(id \otimes \varepsilon) \circ \varDelta = (\varepsilon \otimes id) \circ \varDelta \tag{2.4}$$

or in the form of commutative diagrams

$$\begin{tikzcd} C \otimes C \otimes C & C \otimes C \arrow[l, "\varDelta\otimes id"'] \\ C \otimes C \arrow[u, "id\otimes\varDelta"] & C \arrow[l, "\varDelta"] \arrow[u, "\varDelta"'] \end{tikzcd}$$

Cocommutativity is defined as

$$\tau \circ \varDelta = \varDelta$$

Bialgebras

Bialgebras are very closely related to Hopf algebras (actually every graded bialgebra is dual to a Hopf algebra ([Kas95]).

Definition 2.3 ***(Co-)Algebra Morphisms*** *A linear map $\phi : A_1 \to A_2$ between algebras (A_1,m_1,u_1) and (A_2,m_2,u_2) is a algebra morphism if it satisfies*

$$\phi \circ u_1 = u_2 \tag{2.5}$$

$$\phi \circ m_1 = m_2 \circ (\phi \otimes \phi)\,. \tag{2.6}$$

And analogously for coalgebras; A coalgebra morphism $\psi : C_1 \to C_2$ between coalgebras $(C_1,\varDelta_1,\varepsilon_1)$ and $(C_2,\varDelta_2,\varepsilon_2)$ such that

$$\varepsilon_2 \circ \psi = \varepsilon_1 \tag{2.7}$$

$$\varDelta_2 \circ \psi = (\psi \otimes \psi) \circ \varDelta_1 \tag{2.8}$$

Definition 2.4 ***Bialgebras*** *Given a unital algebra (B,u) with $\mathbb{F}$-vector space B and unit $u : \mathbb{F} \mapsto B$, such that $1_B := u(1_\mathbb{F})$. Then $(B,m,u,\varDelta,\varepsilon)$ is a bialgebra iff the following compatibility* *conditions are satisfied*

$$\Delta \circ m = (m \otimes m) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta) \tag{2.9}$$
$$u \otimes u = \Delta \circ u \tag{2.10}$$
$$\varepsilon \otimes \varepsilon = \varepsilon \circ m \tag{2.11}$$
$$\varepsilon \circ u = 1_F \tag{2.12}$$
$$u \circ \varepsilon = 1_B \tag{2.13}$$

Hopf Algebras

Definition 2.5 ***Hopf Algebras*** *Given a bialgebra* $(H,m,u,\Delta,\varepsilon)$*, a Hopf algebra* $(H,m,u,\Delta,\varepsilon,S)$ *is a bialgebra which has an antipode (inverse map with respect to the convolution group of 2.12)* $S : H \mapsto H$ *satisfying*

$$m \circ (S \otimes id) \circ \Delta = m \circ (id \otimes S) \circ \Delta = u \circ \varepsilon. \tag{2.14}$$

If an antipode exists it is unique (anticipating the notation of definition 2.10) and assuming antipodes S *and* S' *exist, then*

Proof 2.6

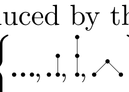
$$S = S \star e = S \star (id \star S') = (S \star id) \star S' = e \star S' = S'. \tag{2.15}$$

Therefore the antipode is unique $\square$

Connectedness bases on more abstract concepts like filtrations and gradings of Hopf algebras. It can be shown under fairly general circumstances that a coradical filtration exists which then permits for grading and connectedness. Absorbing all of this we can define a connected Hopf algebra by

Definition 2.7 ***graded and connected Hopf algebra*** *A Hopf algebra* $(H,m,u,\Delta,\varepsilon,S)$ *is graded and connected if it is endowed with the following structure, where* $j,k,l \in \mathbb{N}$

$$H = \bigoplus_{j=0}^{\infty} H_j \tag{2.16}$$
$$H_0 \simeq \mathbb{F} \tag{2.17}$$
$$m\,(H_k \otimes H_l) = H_{k+l} \tag{2.18}$$
$$\Delta\,(H_j) = \bigoplus_{k+l=j} H_k \otimes H_l. \tag{2.19}$$

As examples for gradings, the number of loops in case of the Hopf algebra of Feynman graphs or the number of nodes in case of the Hopf algebra of rooted trees will be referred to as the grade. To go into more detail by example of rooted trees: the grading (2.16) is induced by the number k of nodes of trees which are elements of a vector space H_k, e.g. $H_3 = \Big\{$  $\Big\}$. In this example H_0, which only consists of the empty tree $\mathbb{1}$, is isomorphic to the underlying field $\mathbb{F}$. (2.18) and (2.19) just tell us, that the multiplication and comultiplication respect the grading.

Definition 2.8 ***Hopf Ideals***

ideal *An algebra ideal I is (in this case) a proper subalgebra of an algebra (A,m,u), such that it lies in the kernel of an algebra morphism $\varphi : A \mapsto A$*

$$0 \in I \tag{2.20}$$
$$\varphi(I) = 0 \tag{2.21}$$
$$\varphi(ai) = \varphi(ia) = 0 \tag{2.22}$$

for any $a \in A$ and $i \in I$.

coideal *A coideal I is (in this case) a sub coalgebra of a coalgebra (C,Δ,ε), such that it respects the coproduct in the following sense*

$$\Delta(I) \subset H \otimes I + I \otimes H \tag{2.23}$$

and gives zero when inserted into the counit ε

$$\varepsilon(I) = 0. \tag{2.24}$$

Hopf ideal *An Ideal I of a Hopf algebra $(H,m,u,\Delta,\varepsilon,S)$ is a Hopf ideal if it additionally to (2.20) and (2.23) satisfies*

$$\varepsilon(I) = 0, \tag{2.25}$$
$$S(I) \subset I. \tag{2.26}$$

In this setting the quotient algebra $\mathcal{H} := H/I$ is a Hopf subalgebra inheriting its grading from H and is connected if H is connected. Note that $\mathbb{1} \not\subseteq I$ since $\varepsilon(I) = 0$ is required.

Augmentation ideal *In any bialgebra or Hopf algebra B the kernel of the co-unit ε gives an ideal, the augmentation ideal Aug.*

Definition 2.9 ***Convolution Product*** *For an algebra (A,m,u) and a coalgebra (C,Δ,ε) the operation*

$$f \star g \ = m \circ (f \otimes g) \circ \Delta \tag{2.27}$$

for algebra homomorphisms $f,g \in \mathrm{Hom}\,(C,A)$ is the convolution product.

Definition 2.10 ***Convolution Algebra*** $(\mathrm{Hom}\,(C,A)\,,\star,e)$ *is an associative, unital algebra, if for counit ε and unit u, give the neutral element $e := u \circ \varepsilon$.*

Proof 2.11 *For* f,g,h $\in$ Hom(C, A) *follows:*
Associativity

$$f \star (g \star h) = m \circ [f \otimes (m \circ (g \otimes h)\,\Delta)]\,\Delta \tag{2.28}$$
$$= m \circ (id \otimes m) \circ (f \otimes g \otimes h) \circ (id \otimes \Delta) \circ \Delta \tag{2.29}$$
$$= m \circ (m \otimes id) \circ (f \otimes g \otimes h) \circ (\Delta \otimes id) \circ \Delta \tag{2.30}$$
$$= m \circ [(m \circ f \otimes g \circ \Delta) \otimes h] \circ \Delta \tag{2.31}$$
$$= (f \star g) \star h \tag{2.32}$$

Unitality

$$e \star f = m \circ [(u \circ \varepsilon) \otimes f] \circ \Delta = m \circ (u \otimes id) \circ (id \otimes f) \circ (\varepsilon \otimes id) \circ \Delta \tag{2.33}$$
$$= m \circ (u \otimes id) \circ (id \otimes f) \circ (1 \otimes id) = m \circ (\mathbb{1} \otimes f) = f \tag{2.34}$$
$$= m \circ (f \otimes \mathbb{1}) = m \circ (id \otimes u) \circ (f \otimes id) \circ (id \otimes 1) \tag{2.35}$$
$$= m \circ (id \otimes u) \circ (f \otimes id) \circ (id \otimes \varepsilon) \circ \Delta = m \circ [f \otimes (u \circ \varepsilon)] \circ \Delta = f \star e \tag{2.36}$$
$$\square \tag{2.37}$$

Remark 2.12 ***Convolution Group*** *If the convolution algebra* $(\mathrm{Hom}\,(H,H)\,,\star, e)$ *gets augmented by the antipode* S *from definition 2.5, this gives a group structure with inverse elements* $f^{-1} = f \circ S,\ f \in \mathrm{Hom\,(H,H)}$*, called the convolution group* $G^H_{\mathbb{F}}$*.*

Proof 2.13 *Since in 2.11 the algebra structure is proven, only the inverse element has to be proven to exist, to permit for a group.*
Inverse Element

$$f \star f^{-1} = m \circ (f \otimes f \circ S) \circ \Delta = m \circ (f \otimes f) \circ (id \otimes S) \circ \Delta \tag{2.38}$$
$$= f \circ m \circ (id \otimes S)\,\Delta = f \circ u \circ \varepsilon = u \circ \varepsilon = e \tag{2.39}$$

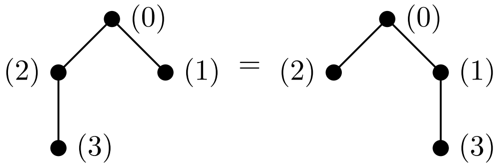
where properties (2.6) and (2.5) were used. $\square$

2.2 Rooted trees

Since the algebra structure of Feynman graphs maps onto the Hopf algebra of rooted trees [Foi10; Kre02], some definitions and remarks will follow. Rooted trees are certain graphs which are connected collections of nodes and edges and have a distinct node (the root), which will always be drawn at the top of a graph. The nodes which are connected to the root or any other node by edges are called children and the nodes, which do not have any children are called leafs. A collection of trees is called, naturally, a forest.

Definition 2.14 ***Rooted Trees*** *A tree* T *consists of a set of nodes (or in regard to QFT, vertices)* $V(T))$ *and a set of edges* $E(T)$ *which connects two nodes* $v, w\ \varepsilon\ V(T)$*, such that* T *is connected (all nodes are connected via subsequent edges) and simply connected (does not have any loops). A distinguished node, the root* r*, will be placed atop of the graph by convention. A pair* (T, r) *is called labelled rooted tree.*

An isomorphism $\phi : (T,r) \to (T',r')$ of labelled rooted trees fixes the root $\phi(r) = r'$. All trees belonging to the same isomorphism class are considered equal. For example, the trees

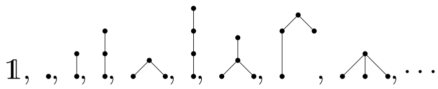


(2.40)

are equal. The isomorphism $\phi(0,1,2,3) = (0,2,1,3)$ maps the tree on the left hand side to the tree on the right hand side. Thus both trees belong to the same isomorphism class and are considered equal.

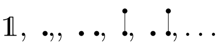
Definition 2.15 ***Rooted Forests*** *Let $\mathcal{T}_n$ be the set of rooted trees with n vertices and let*

$$\mathcal{T} = \bigcup_{n=0}^{\infty} \mathcal{T}_n = \left\{ \mathbb{1}, \ldots \right\} \tag{2.41}$$

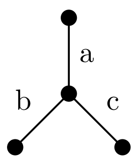


with $\mathbb{1} := \varnothing$ the empty tree, be the set of all rooted trees. Then the disjoint union of rooted trees graded by their vertices gives a rooted forest

$$\mathcal{F} = \left\{ \mathbb{1}, \ldots \right\} \tag{2.42}$$



Before introducing the Hopf algebra of rooted trees H_R, we will shortly explain the notion of admissible cuts.
On a tree T cuts are admissible, if it is only cut once along a branch. A cut tree then disconnects into a tree which contains the root $R^{\mathcal{C}}$ and a polynomial of trees $P^{\mathcal{C}}$. Let us take a look at an example:
The tree (edges are labelled, to make clear which edge gets cut)

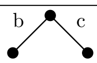
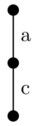
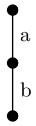
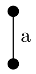


can be cut in six different ways, of which four are admissible, since they cut every branch only once. Results from cutting are collected in table 2.1.

Definition 2.16 ***Hopf Algebra of Rooted Trees*** *The Hopf algebra of rooted trees H_R is the algebra generated by rooted trees, the grading is induced by the node number. The multiplication of trees $T_1, T_2 \in \mathcal{T}$ is defined by juxtaposition*

$$m(T_1 \otimes T_2) = T_1 T_2. \tag{2.43}$$

Table 2.1: Example for admissible cuts

cut	$P^{\mathfrak{C}}$	$R^{\mathfrak{C}}$	admissible
a		$\bullet$	✓
b	$\bullet$		✓
c	$\bullet$		✓
a,b	$\bullet\ \bullet$	$\bullet$	✗
a,c	$\bullet\ \bullet$	$\bullet$	✗
b,c	$\bullet\ \bullet$		✓

The coproduct of a tree $T \in \mathcal{T}$ is defined by

$$\Delta(T) = \sum_{\mathfrak{C}} P^{\mathfrak{C}}(T) \otimes R^{\mathfrak{C}}(T) \tag{2.44}$$

where $\sum_{\mathfrak{C}}$ sums over admissible cuts $\mathfrak{C}$.
The antipode of a tree $T \in \mathcal{T}$ is recursively defined by

$$S(T) = -T - \sum_{\mathfrak{C}} S(P^{\mathfrak{C}})R^{\mathfrak{C}}. \tag{2.45}$$

Definition 2.17 ***Insertion Operator*** *The Hopf algebra morphism B_+ that connects a new root to a tree $\mathcal{T}_n \in \mathcal{T}$ or forest up to m nodes $\mathcal{F}_m \subset \mathcal{F}$ is called the insertion operator.*

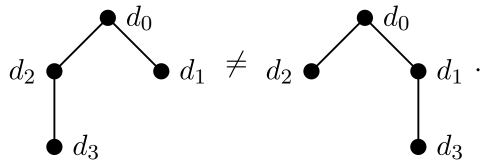
For example

$$B_+(a\ \mathbb{1} + b\ \overset{\bullet}{\underset{\bullet}{|}} + c\ \bullet\bullet\bullet) = a\ \bullet + b\ \overset{\bullet}{\underset{\bullet}{|}}\!\!\underset{\bullet}{|} + c\ \bullet\!\diagup\!\!\overset{\bullet}{|}\!\!\diagdown\!\bullet. \tag{2.46}$$

B_+ is also a Hochschild one-cocycle of the Hopf algebra, which means it fulfils the commutation relation

$$\Delta \circ B_+ = B_+ \otimes \mathbb{1} + (id \otimes B_+)\Delta. \tag{2.47}$$

Remark 2.18 *Trees can be endowed with decorations $d_i \in \mathcal{D}$, $i \in \mathbb{N}$ belonging to a set of decorations $\mathcal{D}$, similar to the labels of (2.40), but in general loosing the equality in (2.40) for* *decorations* $d_0, \ldots, d_3$



(2.48)

The insertion operator (2.47) respects decorations and gets assigned a decoration as a superscript. In above tree the insertion operator would be $B_+^{d_0}$.

2.3 Algebra Morphisms and Integrals

To be able to express Feynman graphs via the Hopf algebra of rooted trees it is necessary to establish a connection between the two. The connection arises from the universal property of connected commutative Hopf algebras in a natural way and will be explained below.

Theorem 2.19 ***Universal Property*** *The pair* (H, B_+) *of the connected commutative Hopf algebra* H *and Hochschild one-cocycle* B_+ *defined by (2.47), is unique up to algebra isomorphisms among all such pairs* $\left(\tilde{H}, \tilde{B}_+\right)$. *This means that for any Hopf algebra* $\tilde{H}$ *and one-cocyle* $\tilde{B}_+$ *there exists an unique isomorphism* $\rho : H \mapsto \tilde{H}$ *such that*

$$\rho \circ B_+ = \tilde{B}_+ \circ \rho \tag{2.49}$$

which also means that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\rho} & \tilde{H} \\ \downarrow{\scriptstyle B_+} & & \downarrow{\scriptstyle \tilde{B}_+} \\ H & \xrightarrow{\rho} & \tilde{H} \end{array}$$

commutes [Con98; Kre].

Proof 2.20 *See [Con98].*

With the universal property (2.49) we can establish a connection between the Hopf algebra of rooted trees H_R and the Hopf algebra of Feynman graphs H_{FG} discussed below.

2.3.1 Algebra of Formal Integrals and Renormalisation

The algebra of formal integrals permits for a connection between Feynman integrals and the Hopf algebra of rooted trees, which in turn lets us switch between those two representations in a well defined manner and work in the representation that is more convenient to work in. Let H

be the connected, commutative Hopf algebra of rooted trees, which by 2.19 has the universal property and define integrals over differential forms ω

$$\int_a^\infty \omega,\ a \in \mathbb{R},\ a > 0 \tag{2.50}$$

to be formal pairs

$$\left(\int_X, \omega\right), \tag{2.51}$$

with interval $X \subset \mathbb{R}$ and $\inf(X) = a > 0$. The algebra structure is then given through regular addition between integrals and multiplication as

$$\left(\int_X, \omega_1(x)\right)\left(\int_Y, \omega_2(y)\right) = \left(\int_X \int_Y, \omega_1(x)\omega_2(y)\right), \tag{2.52}$$

with differential forms ω_1, ω_2 and neutral element $(\emptyset, 1)$. $\left(\int_X \int_Y, \omega(x)\omega(y)\right)$ evaluates to independent integrals if they are well-defined (which they are not necessarily for the time being). Now consider a commutative target algebra $\tilde{H}$ and linear operator $\tilde{B}_+ : \tilde{H} \to \tilde{H}$. Then following 2.19, there exists a unique algebra morphism $\Phi_a : H \to \tilde{H}$, which after 2.3 is an Hopf algebra homomorphisms (also called Feynman rules in the form of a (Hopf) character), such that

$$\Phi_a \circ B_+ = \tilde{B}_+ \circ \Phi_a. \tag{2.53}$$

Let the "integral" insertion operator $\tilde{B}_+$ be defined by its action on the neutral element of formal pairs

$$\tilde{B}_+(\emptyset, \mathbb{1})(a) = \left(\int_a^\infty, \omega\right) \tag{2.54}$$

and on formal pairs in general

$$\tilde{B}_+\left(\int_x^\infty, \omega_T\right)(a) = \left(\int_a^\infty \int_x^\infty, \omega(x)\omega_T(x)\right), \tag{2.55}$$

where the differential form ω_T is dependent on the variables of the decorations of nodes $V(T)$ of the tree T. Then Φ_a is defined by

$$\Phi_a(\mathbb{1}) = (\emptyset, 1) \tag{2.56}$$

and

$$\Phi_a \circ B_+(T) = \left(\int_a^\infty, \omega(x)\Phi_x(T)\right) \tag{2.57}$$

with $\Phi_x(T) = \left(\int_x^\infty, \omega_T\right)$, where ω_T is the associated differential form to tree T.

2.3.2 Renormalisation

The formal integrals defined in (2.51) can be ill-defined for differential forms such as $\omega(x) = \frac{1}{x+a}$, which yields a logarithm when integrated. It is however possible to assign such integrals finite values through renormalisation. For $\frac{1}{x+a}$ it is sufficient to subtract at a renormalisation point μ (such a renormalisation scheme is called kinematic renormalisation) i.e.

$$\int_0^\infty \left\{ \frac{1}{x+a} - \frac{1}{x+\mu} \right\} dx = \int_a^\infty \frac{1}{x} dx - \int_\mu^\infty \frac{1}{x} dx = \int_a^\mu \frac{1}{x} dx = \ln(\frac{\mu}{a}). \tag{2.58}$$

The infinity at the upper integral boundary should be understood as a limiting process $\lim\limits_{\Lambda\to\infty}$ for a regulator Λ.
In general to find well-defined, finite expressions for formal integrals (2.51), it might be necessary to submit them to a renormalisation operator S_R^Φ, depending on a renormalisation scheme R. The following description considers kinematic renormalisation schemes, in which the integrand is due to a subtraction at the singular point (renormalisation point μ). It is possible to define R differently to accommodate for different renormalisation schemes, as long as they obey the Rota-Baxter property [Con99]

$$R[ab] + R[a]R[b] = R[R[a]b] + aR[b], \tag{2.59}$$

but is not of interest in this work. To yield a finite integral, the singular point from the ill-defined integral associated to Φ_a gets subtracted by the antipode S, following a renormalisation scheme R. Which means evaluating a singular character Φ_a at renormalisation point μ by $R\Phi_a = \Phi_\mu \Rightarrow R\Phi_a(S(B_+(1))) = -\left(\int_\mu^\infty, \omega\right)$. The universal property then gives the connection between the algebraic language, a simple integral without subdivergences:

$$\Phi_a(B_+(\mathbb{1})) + R\Phi_a(S(B_+(\mathbb{1}))) = \left(\int_a^\infty, \omega\right) - \left(\int_\mu^\infty, \omega\right) = \left(\int_a^\infty - \int_\mu^\infty, \omega\right) = \left(\int_a^\mu, \omega\right). \tag{2.60}$$

A Hopf algebra morphism, the renormalisation operator, $S_R^\Phi : H \to \tilde{H}$ can then be defined recursively by

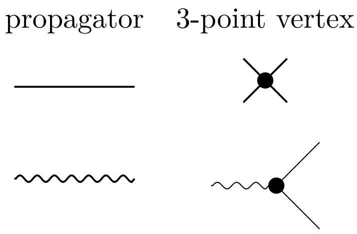
$$S_R^\Phi(h) = -R[S_R^\Phi \star (\Phi_a \circ P)](h),\ h \in Aug, \tag{2.61}$$

with projection P into the augmentation ideal and the convolution product $\star$ (2.27) in this case for characters from $Hom(H,\tilde{H})$. The well-defined integrals are then yielded by the renormalised Feynman rules

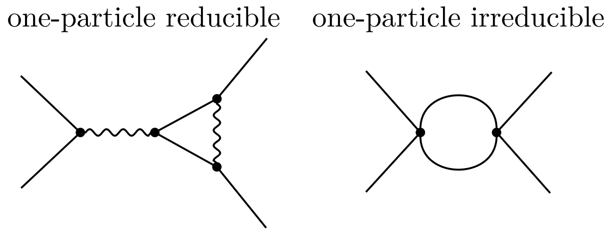
$$\Phi_{a,R} := S_R^\Phi \star \Phi_a. \tag{2.62}$$

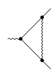
2.4 Hopf Algebra of Feynman Graphs

Feynman graphs are a special type of graphs, which have their roots in physics. They are comprised of edges, called propagators (in the context of this work, all propagators will be undirected) and nodes called vertices, for example



Physics define different propagators, vertices and so called Feynman rules, which define what mathematical expression the propagators and vertices are decorated with (example in chapter 3). Propagators and vertices then stand for building blocks from which Feynman graphs can be build, describing scattering processes of particles for example. When constructing Feynman graphs by connecting vertices by appropriate propagators (i.e. propagators, which are incident to the vertices in question), two vertices can either be connected by one edge, which would deem them disconnected upon removal of said edge, or by more than one edge. When a graph consists only of vertices connected by more than one edge it will be called 1PI (one-particle irreducible). An example should make that clearer



The left hand graph can be disconnected into  and  upon removal of the left wavy edge, while the right hand graph can not be disconnected by removing a single edge.
We can then define the Hopf algebra of Feynman graphs, which has a one-to-one relation to the Hopf algebra of decorated rooted trees. Only divergent 1PI graphs will be regarded, since all non-1PI graphs can be constructed from 1PI graphs.

Definition 2.21 *The Hopf algebra of Feynman graphs* $H_{FG} = (H, m, \Delta, \mathbb{1}, \varepsilon, S)$ *consists of a vector spaces* H_n *and Feynman graphs* Γ_n*, where* n *denotes the number of loops of the graph. The algebra is graded by the number of loops*

$$H = \bigoplus_{n=0}^{\infty} H_n. \tag{2.63}$$

The multiplication m *is given by disjoint union of graphs*

$$m(\Gamma \otimes \gamma) := \Gamma \sqcup \gamma = \Gamma\gamma, \tag{2.64}$$

usually written by juxtaposition.
The comultiplication Δ *is defined, over proper subgraphs* $P(\Gamma) := \{\gamma \subsetneq \Gamma\}$*, by*

$$\Delta(\Gamma) = \Gamma \otimes \mathbb{1} + \mathbb{1} \otimes \Gamma + \sum_{\gamma \in P(\Gamma)} \gamma \otimes \Gamma/\gamma. \tag{2.65}$$

The unit is the empty graph

$$\mathbb{1} = \emptyset. \tag{2.66}$$

The counit is defined by

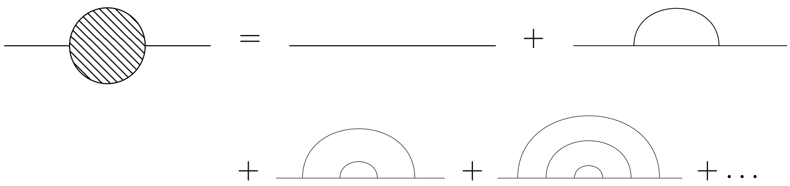
$$\varepsilon(\Gamma) = \begin{cases} \mathbb{1} & ,\Gamma = \emptyset \\ 0 & ,\Gamma \neq \emptyset \end{cases}. \tag{2.67}$$

Finally the antipode is defined by

$$S(\Gamma) = \begin{cases} \mathbb{1} & ,\Gamma = \mathbb{1} \\ -\Gamma - \sum_{\gamma \in P(\Gamma)} S(\gamma)\Gamma/\gamma & ,else \end{cases}. \tag{2.68}$$

2.5 Dyson-Schwinger Equations

It is possible to formulate iterated integrals, as fixed point equations called Dyson-Schwinger equations (analytic DSEs) of QFT, that often arise from perturbative expansions of scattering amplitudes in Feynman integrals. A simple example is the self energy of a scalar particle in ϕ^3 theory which can be expanded by propagator corrections, neglecting vertex corrections

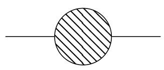
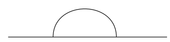


(2.69)

where the self similarity implies

(2.70)

If we then define

$G^{\text{---}} = \Phi\Big($  $\Big)$, $\quad 1 = \Phi\big($ $\big)$, $\quad \int \omega = \Phi\Big($  $\Big)$ (2.71)

it is possible to reformulate expansion (2.69) as a fixed point equation

$$\begin{aligned} G^{\text{---}} &= 1 + \int \omega + \int \omega \int \omega + \int \omega \int \omega \int \omega + \dots \\ &= 1 + \int \omega \left(1 + \int \omega + \int \omega \int \omega + \dots\right) \\ &= 1 + \int \omega G^{\text{---}}, \end{aligned} \tag{2.72}$$

the afore mentioned Dyson-Schwinger equation.

2.5.1 Combinatorial Dyson-Schwinger Equations

If we recall the definitions of 2.3 we see a natural way to reformulate the analytic DSEs as equations in a connected commutative Hopf algebra of Feynman graphs H_{FG} (combinatorial DSEs, or just DSEs). The (renormalised) 2-point Green's function $G^{\text{---}}$ is an evaluation of series of polynomials $X^{\text{---}}$ by (renormalised) Feynman rules Φ_R from (2.62), the self similarity of (2.69) is then implemented by the insertion operator (2.47)

$$G^{\text{---}}(\alpha) = \Phi_R(X^{\text{---}})(\alpha) = \Phi_R\left(\mathbb{1} + \alpha B_+(X^{\text{---}}(\alpha))\right), \tag{2.73}$$

with a parameter α referring to the grading of H. We may then work with the equation

$$X^{-}(\alpha) = \mathbb{1} + \alpha B_{+}\left(X^{-}(\alpha)\right) \tag{2.74}$$

without having to consider the specifics of the analytic representation obtained by evaluation with Φ_R. It is possible to generalize this approach to more than the single propagator residue above.

If a theory is defined by a Lagrangian, then every monomial in that Lagrangian corresponds to a residue graph, which is either a propagator type corresponding to a two-point Green's function (like —) or a n-point vertex (interaction) type corresponding to a n-point Green's function (like ✕ for the 4-point Green's function of ϕ^4). These residues construct a residue set $\mathcal{R} = \left\{\mathcal{R}^{[0]}, \mathcal{R}^{[1]}\right\}$ of vertex residues $\mathcal{R}^{[0]} = \{v_i\}$, $i \in \mathbb{N}$ and propagator residues $\mathcal{R}^{[1]} = \{p_j\}$, $j \in \mathbb{N}$. To properly correct the residue $r \in \mathcal{R}$ for all possible cases it is however necessary to introduce an invariant charge

$$Q^v(\alpha) = \frac{X^v}{\prod_{p \text{ incident to } r} \sqrt{X^p}} \tag{2.75}$$

for every vertex $v \in \mathcal{R}^{[0]}$ and where $\frac{1}{X^p}$ is shorthand notation for a formal geometric series $\frac{1}{X^p} = \sum_{n=0}^{\infty} (\mathbb{1} - X^p)^n$. The invariant charge is an equivalent of the charge renormalisation in the standard formulation of QFT.

A general DSE for a residue $r \in \mathcal{R}$ is thus

$$X^r = \mathbb{1} \pm \sum_k \alpha^k B_+^{k,r} \left(X^r \prod_{n_i, \in \mathcal{R}^{[0]}} (Q^v)^{n_i} \right) \tag{2.76}$$

with tupel $n_i = \{n_{v_i}(\gamma)\}$ $i \in \mathcal{R}^{[0]}$ where each $n_{v_i}(\gamma) = \#v_i - res_{v_i}(\gamma)$ counts the number $\#v_i$ of vertices v_i minus

$$res_{v_i}(\gamma) = \begin{cases} 0 & , res(\gamma) \neq v_i \\ 1 & , res(\gamma) = v_i \end{cases} \tag{2.77}$$

e.g. $n(\;) = 0 - = -1$, while $n(\;) = 3 - 0 = 3$. The elements to each order in α then generate a Hopf algebra (more on that in chapter 4). Therefore one could also just define a theory via its residues, without consulting the Lagrangian or even just by its representation in DSEs [**kreimer_dyson-schwinger_equations**]. A DSE of type (2.76) can be solved by the Ansatz

$$X^r = \mathbb{1} \pm \sum_{j \geq 1} \alpha^j c_j^r \tag{2.78}$$

where + again refers to vertex DSEs while − refers to propagator DSEs and the reduced Green's functions $c_j^r \in H$ are polynomials, which have a closed form under the coproduct

$$\Delta(c_n) = \sum_{k=0}^{n} P_k^n \otimes c_k \tag{2.79}$$

with homogeneous polynomials of deree $n-k$ in c_l, $l \leq n$ defined by

$$P_k^n = \sum_{l_1+\ldots+l_k+1=n-k} c_{l_1} \ldots c_{l_k+1}. \tag{2.80}$$

These c_k give rise to a sub-Hopf algebra [Ber05].
For completeness it shall be said, that a DSE can be non-self-similar, which means it has no residue term and therefore (2.76) lacks the constant term:

$$X^e = \sum_k \alpha^k B_+^{k,e}\left(X^r \prod_{n_i,\in\mathcal{R}^{[0]}} (Q^v)^{n_i}\right). \tag{2.81}$$

Here $B_+^{k,e} = \sum_i B_+^{k,e_i}$ is a sum over all contributing insertion operators B^{k,e_i} with the same external leg structure e and k loops but with different internal loop structure.

2.6 Renormalisation Group

The renormalisation group helps analysing macroscopic behaviour emerging from the microscopic configuration and dynamics of a theory. In Wilson's approach the field dependence of correlation Green's functions gets split into small scale (often called ultra violet or UV-regime) and large scale (often called infrared or IR-regime) momentum shells, which consecutively get integrated out. After many iterations only large scale degrees of freedom remain. This approach goes by the name renormalisation group [Wil74]. However, the iterations of consecutive integrations can be viewed as repeated rescalings of the fields and parameters of the theory leading to the renormalisation group equation (RGE) (2.90) or Callan-Symanzik equation [Cal70; Sym70]. Callan and Symanziks approach will be reviewed below, essentially following a paper by Steven Weinberg [Wei73].

Every renormalised N-point Green's function of a massless scalar theory $G^{r_N}(\{g_i\}, \mu, d)$ of a graph r is dependent on the same parameters, the couplings g_i, the renormalisation scale μ (which is directly connected to the energy scale at which experiments are conducted) and possibly a regulator like a cutoff Λ or in our case the dimension $d = D - 2\varepsilon$. It is related to the bare Green's function $G_0^{r_N}(\{g_i\}, d)$ by $G^{r_N}(\{g_i\}, \mu, d) = (Z_{r_N})^{\frac{N}{2}} G_0^{r_N}(\{g_i\}, d)$ with the field renormalisation coefficient Z_{r_N} which implicitly depends on the renormalisation scale μ, while $G_0^{r_N}$ is independent of the renormalisation scale. The renormalisation group equation describes a change of these parameters and the Green's function under rescaling (e.g. infinitesimal change in parameter μ or change in the renormalisation prescription) i.e. simultaneous changes

$$\phi(x) \rightarrow \phi'(x') \tag{2.82}$$
$$g_i(\mu) \rightarrow g_i'(\mu') \tag{2.83}$$
$$\mu \rightarrow \mu'. \tag{2.84}$$

This combination of rescalings keeps the Lagrangian invariant and therefore does not alter the essence of the theory. The Green's function then inhibits a rescaling itself from the rescaling of the field ϕ, couplings g_i and renormalisation scale μ

$$G^{r_N} \rightarrow G'^{r_N}. \tag{2.85}$$

The change in G^{r_N} with respect to μ can thus be described by the variation of G^{r_N} with respect to μ

$$\mu \frac{d}{d\mu} G_N^r = \left[\sum_i \mu \frac{d}{d\mu} g_i(\mu) \frac{\partial}{\partial g_i} + \mu \frac{\partial}{\partial \mu}\right] G^{r_N} = 0, \tag{2.86}$$

changing to logarithmic derivatives in $L = \ln(\mu)$ and introducing the coefficients renormalisation function β and anomalous dimension γ

$$\beta(g, L) = -\frac{d}{dL} g(\mu) \tag{2.87}$$

$$\gamma(g, L) = \frac{d}{dL} \ln(Z_r) \tag{2.88}$$

$$\tag{2.89}$$

immediately leads to the Callan-Symanzik or renormalisation group equation (RGE)

$$\left[\frac{N}{2}\gamma(g, L) - \sum_i \beta_i(g, L)\frac{\partial}{\partial g_i} - \frac{\partial}{\partial L}\right] G^{r_N} = 0. \tag{2.90}$$

While the β function (there is one for each coupling, depending on all couplings) describes a change in the coupling (in this context running coupling, since it depends on the renormalisation scale) g_i, due to change of renormalisation scale μ, the anomalous dimension γ gives information on the accompanied change of the Green's function G^{r_N}. (2.90) tells us, that the Green's function is invariant under simultaneous rescalings of fields, couplings and scale.

2.7 Critical Exponents and Universality Classes

Critical behaviour and the associated critical exponents are concepts from statistical physics, which arose mostly by analysis of Ising and Ising like models and were expanded on by renormalisation group techniques from quantum field theory (QFT). This section will follow mostly [Fis74; Pes16; Sta99; Ton], if not mentioned otherwise. First we will give a short introduction with the statistical physics approach using the Ising model as an example, which is defined by a hamiltonian of the form

$$H_{ising} = -\frac{1}{2}\sum_{i,j} J_{ij} s_i . s_j - B . \sum_{i=1}^{N} s_i, \tag{2.91}$$

where s_i and B are D dimensional vectors for spin and external magnetic field respectively, J_{ij} is the coupling strength between spins s_i, s_j. If $J_{ij} > 0$ when spins are aligned the system described by (2.91) is called ferromagnetic. When $J_{ij} < 0$ in an antisymmetric spin configuration, the system is called anti-ferromagnetic.

Let us take a look at the ferromagnetic case $J_{ij} > 0$. The spins s_k are located at discrete lattice positions x^k, $k \in N$ on a lattice of length aN with a lattice constant a. Each spin can have the value $|s_k| = \pm 1$ and we do not yet make any assumptions on the correlation length between spin interactions J_{ij}. The partition function of the canonical ensemble then reads

$$Z(T,J,B) = \sum_{s_i} e^{-\beta H_{ising}(s_i)}, \tag{2.92}$$

where $\beta = \frac{1}{k_B T}$, with Boltzmann constant k_B and temperature T. From the partition function we can get the thermodynamic free energy

$$F(T,B) = \langle H \rangle - TS = -T \log(Z). \tag{2.93}$$

What we would really like to know, is the magnetisation of the system under influence of an external magnetic field B. To get information on the magnetisation m we change variables to

$$m = \frac{1}{N}\left\langle \sum_i s_i \right\rangle = \frac{1}{N\beta}\frac{\partial \log(Z)}{\partial B}, \tag{2.94}$$

here $m \in [-1, +1]$. Intuitively it is expected that under the influence of an external field $B > 0$ and temperatures T below the critical temperature T_c the magnetisation $m \to +1$, while above T_c the magnetisation would stay at $m = 0$.

If we sum over all spin configurations $\frac{1}{N}\sum_i s_i = m$ and then over all magnetisations m the partition function (2.92) becomes

$$Z = \sum_m e^{-\beta F(m)}. \tag{2.95}$$

In the large N limit the sum turns into an integral over a measure dm such that

$$Z = \int dm e^{-\beta F(m)}. \tag{2.96}$$

In the vicinity of the critical point where $T \approx T_c$ and $m \approx 0$ it is possible to give an ansatz for the free energy when respecting a number of physical constraints like locality, respected rotational and translational invariance as well as mirroring symmetry $m \to -m$ of the magnetisation when B is absent, we also take analyticity in m as given (which is at least not to far fetched for physical systems). Thus at the critical point the free energy (or Landau-Ginzburg free energy) can be written as the functional

$$F[m(x)] = \int d^D x \left[\frac{1}{2}(\nabla m(x))^2 + a_2(T)m^2(x) + a_4(T)m^4(x) - Bm(x) + \dots\right]. \tag{2.97}$$

Non trivial solutions exist only for $a_4 = b > 0$ and $a_2 < 0$ for $T < T_c$, therefore at $T \approx T_c$ we can approximate $a_2 = a(T - T_c)$. We now get access to parameters which can be experimentally measured, the (ground state $B = 0$) magnetisation m_0, magnetic susceptibility χ and specific heat c.
The ground state magnetisation can be gained by setting $B = 0$ and varying (2.97) and assuming $\nabla m(x) = const$ since at the critical point the system is homogeneously ordered, which yields extrema

$$m = \pm m_0 = \pm\sqrt{\frac{a(T - T_c)}{b}}. \tag{2.98}$$

When the external magnetic field $B \neq 0$ we need to incorporate the mediation of that field through clusters of magnetisations by allowing $\nabla m(x)$ to vary. Then using the functional derivative on (2.97) up to the quadratic term in m, and solving the arising equation yields approximate solutions for magnetisation $m(B,T)$

$$m \approx \begin{cases} \frac{B}{(T-T_c)} & , T \gtrsim T_c \\ m_0 + \frac{B}{(T_c-T)} & , T < T_c. \end{cases} \tag{2.99}$$

Which allows us to derive the magnetic susceptibility χ from (2.99) by differentiation

$$\chi = \frac{\partial m}{\partial B}|_T \sim |T - T_c|^{-1}. \tag{2.100}$$

The specific heat can be derived from the partition function at a the critical point

$$c = \frac{\beta^2}{V}\frac{\partial^2}{\partial \beta^2}\log(Z) \sim |T - T_c|^0, \tag{2.101}$$

with the volume of the system V
In general the critical exponents of m_0, χ and c near the critical point of a phase transition are different for different systems, however there are systems which may have very different

microscopic structures, but the same macroscopic critical exponents when scaled properly. These are then considered lying in the same universality class. In general the critical exponents for m, χ and c the reduced temperature $t = \frac{|T-T_c|}{T_c}$ are

$$m_0^2 \sim t^{2\beta} \tag{2.102}$$
$$c \sim t^{-\alpha} \tag{2.103}$$
$$\chi \sim t^{-\gamma} \tag{2.104}$$

where above $\beta = \frac{1}{2}$, $\alpha = 0$ and $\gamma = 1$. Moreover, after statistical physics lore, they are related by

$$\alpha + 2\beta + \gamma = 2. \tag{2.105}$$

To stay within the context of ferromagnets, we can describe the magnetisation as a function of external magnetic field H and reduced temperature t; $m = m(B,t)$. If we plot the scaled magnetisation $\tilde{m} = \frac{m}{B^{\delta_1}}$ vs scaled reduced temperature $\tilde{t} = \frac{t}{B^{\delta_2}}$, with $\delta_1, \delta_2 \in \mathbb{R}$, of different magnetic materials, it can be observed that for certain δ_1, δ_2 all plots "collapse" onto one line (see Fig. 2.1), this is the so called scaling hypothesis.
From the scaling hypothesis stems the idea to combine different materials which collapse onto the same scaling function with the same critical exponents near their critical point, into universality classes, akin to the periodic table of elements (see again Fig. 2.1) [Sta99].

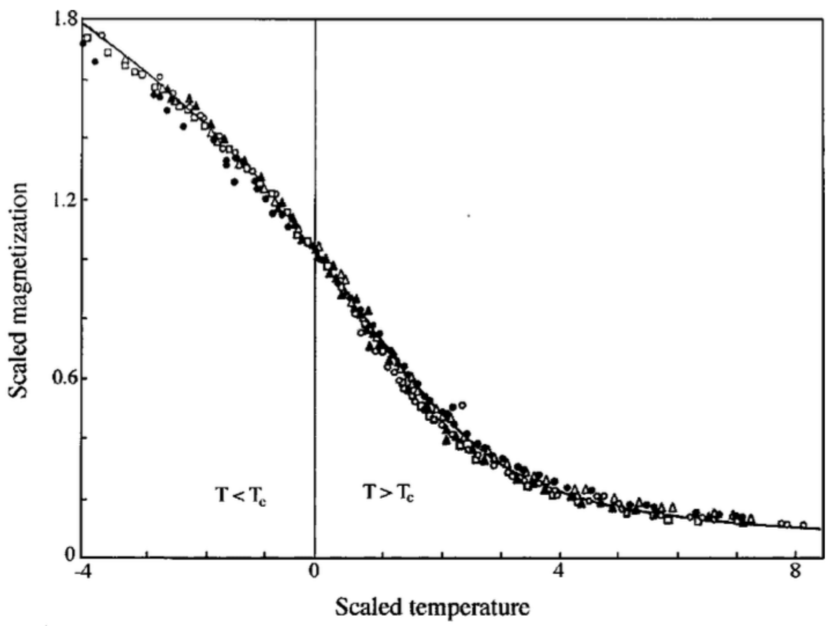


Figure 2.1: "collapsed" scaled magnetisation $\tilde{M}$ vs scaled reduced temperature $\tilde{t}$ plot of different magnetic materials data ($CrBr_3$, EuO, Ni, YIG, Pd_3Fe) to substantiate scaling hypothesis and universality classes hypothesis [Sta99]

2.7.1 Correlation Functions

If we see the partition function (2.96) as a functional of $B(x)$ rather than of $m(x)$ and then take the functional derivative of $-\log Z$ we get the expectation value for m with regard to $B(x)$

$$-\frac{1}{\beta}\frac{\delta \log Z}{\delta B(x)} = \langle m(x)\rangle_B \,. \tag{2.106}$$

If we take the second functional derivative and set $B = 0$ we get a correlation function for spatially separated magnetisations $m(x)$, $m(y)$

$$\frac{1}{\beta^2}\frac{\delta^2 \log Z}{\delta B(x)\delta B(y)}|_{B=0} = \langle m(x)m(y)\rangle \,. \tag{2.107}$$

These correlation functions can then be computed by solving the partition functions with Green's functions akin to the propagators in QFT by

$$\langle m(x)m(y)\rangle = \frac{1}{\beta}G(x-y), \tag{2.108}$$

where the Green's function only depends on the absolute value of vector x, due to rotational symmetry, and can be written as a Fourier transform

$$G(|x|) = \int \frac{d^Dk}{(2\pi)^D}\frac{e^{-ik.x}}{k^2+\frac{1}{\xi^2}} \tag{2.109}$$

with

$$\xi = \frac{1}{\sqrt{a\,(T-T_c)}} \tag{2.110}$$

the correlation length of magnetisation. When solving the Green's function the regimes $|x| << \xi$ and $|x| >> \xi$ have to be separated which in the vicinity of $T \sim T_c$ leads to the correlation function

$$\langle m(x)m(y)\rangle \sim \begin{cases} \frac{1}{|x|^{D-2}}, |x| << \xi \\ \frac{e^{-\frac{|x|}{\xi}}}{|x|^{\frac{D-1}{2}}}, |x| >> \xi. \end{cases} \tag{2.111}$$

2.7.2 Renormalisation Group Approach

When using the renormalisation group approach of chapter 2.6, the magnetisation $m(x)$ will be considered as a field $\phi(x)$, hence giving the free energy

$$F[\phi(x)] = \int d^Dx \left[\frac{1}{2}\left(\nabla\phi(x)\right)^2 + a_2(T)\phi^2(x) + a_4(T)\phi^4(x) - B\phi(x) + \dots\right]. \tag{2.112}$$

By setting the external magnetic field $B = 0$, we get an expression where the integrand is suspiciously similar to the ϕ^4 Lagrangian with "mass" $\mu^2 = a(T-T_c)$ and "coupling constant"

$\frac{g}{4!} = b$. The similarity goes beyond that, the partition function of statistical physics (2.96) and the path integral of quantum field theory

$$Z = \int \mathcal{D}\phi e^{\frac{i}{\hbar}\int d^D x \mathcal{L}} \underset{\text{Wick rotate, } it \to \tau}{=} \int \mathcal{D}\phi e^{-\frac{1}{\hbar}\int d^D_E x \mathcal{L}} \tag{2.113}$$

share a similar structure and can be used similarly. This useful connection permits to use QFT tools on statistical physics problems. This means we can analyse the free energy $F[\phi(x)]$ at the critical point $T \approx T_c$ with no external magnetic field B as an action over massless ϕ^4 Lagrangian

$$\mathcal{L}(\partial\phi(x), \phi(x)) = \frac{1}{2}(\partial\phi)^2 + \frac{g}{4!}\phi^4. \tag{2.114}$$

The correlation functions (2.111) can then be represented by the propagator of the ϕ field.
In chapter 2.6 rescaling behaviour of the theory was mentioned. If we rescale space, vectors will scale like $x \to x' = \frac{x}{\zeta}$, which introduces a new exponent η connected to the field scaling dimension $\triangle_\phi$. This in turn means, that the field rescaling (2.82) takes the form

$$\phi(x) \;\to\; \phi'(x') = \zeta^{\triangle_\phi}\phi(x) \tag{2.115}$$

where

$$\triangle_{\phi'} = \frac{D-2+\eta}{2}. \tag{2.116}$$

It is this expression were the quantum nature of field ϕ comes into play. While classically $\triangle_\phi = \frac{D-2}{2}$, in QFT the renormalised field $\phi = Z^{\frac{1}{2}}\phi_0$ depends on the bare (or unrenormalised) field ϕ_0, which is independent of the renormalisation scale μ and the renormalisation coefficient Z, which depends on the renormalisation scale μ. This means Z gives a contribution $\frac{\eta}{2}$ to $\triangle_\phi$ after rescaling changing it to (2.116). Thus $\frac{\eta}{2}$ exactly describes the deviation from the classical behaviour and has an implicit and complicated dependence on the fix point coupling g^* and therefore the space-time dimension D.

In (2.110) the correlation length has an inverse dependence on the temperature t

$$\xi \sim t^{-\nu} \tag{2.117}$$

which can be represented by critical exponent ν with $\nu = \frac{1}{2}$ in (2.110). If we invert this relation we get a scaling dimension $\triangle_t$ for t

$$t \to t^{\triangle_t},\ \triangle_t = \frac{1}{\nu} \tag{2.118}$$

from which it is possible to derive the critical exponents (2.102) - (2.104).
In (2.112) we have replaced the magnetisation m by the quantised field ϕ. So the relation for the ground state magnetisation (2.102) becomes

$$\phi_0^2 \sim t^{2\beta}. \tag{2.119}$$

If we now use relation (2.118) we get the critical exponent β for the ground state magnetisation ϕ_0

$$\triangle_\phi = \beta \triangle_t \Rightarrow \beta = \frac{(D-2+\eta)\,\nu}{2}. \tag{2.120}$$

The second critical exponent α of the specific heat comes from a second derivative of the Lagrangian with respect to the temperature. So we need to think of the scaling behaviour of the Lagrangian first.
Since the action $S = \int d^D x \mathcal{L}$ has to be scale invariant, the Lagrangian $\mathcal{L}$ needs to cancel any scales brought into the game by the integration measure. We now take the Lagrangian as temperature dependent $\mathcal{L} = \mathcal{L}(t)$. If we suppose to be near the critical point the correlation length is the only length that matters and therfore the Lagrangian scales like

$$\mathcal{L}(t) \sim t^{D\nu}. \tag{2.121}$$

From this expression we can derive the critical exponent of the specific heat

$$c \sim \frac{\partial^2 \mathcal{L}(t)}{\partial t^2} \sim t^{D\nu - 2} \Rightarrow \alpha = 2 - D\nu. \tag{2.122}$$

The final critical exponent γ is obtained from (2.105) $\alpha + 2\beta + \gamma = 2$

$$\gamma = 2 - \alpha - 2\beta = \nu\,(2-\eta)\,. \tag{2.123}$$

The parameter η is then equal to the renormalisation group anomalous dimension γ—(g^*) of the ϕ field propagator of the corresponding tower theory at the critical point g^* [Gra; Gra17b], while $\nu = \frac{1}{2}$. Since in this way the critical exponents are all dependent on the RGE anomalous dimension γ—(g^*), equal γ—(g^*) define universality classes.

CHAPTER 3

Tower Theories and Universality Classes

Universality classes (of QFTs) are characterized by the same critical exponents at a non-trivial fixed point of the renormalization group β-function, such as the Wilson-Fisher fixed point (WF-FP). It can be possible to formulate such theories in a way, that they inherit a common interaction term in the defining Lagrangian $\mathcal{L}$. The procedure is as follows: Start with a seed Lagrangian, replace two fields of every interaction term by an new auxiliary field and add a propagator expression for every auxiliary field. The coupling constants of the gained interaction terms, called core interactions, may then be rescaled into the propagator expressions. Those core interactions then persist through any dimensionality, while other terms, only depending on the auxiliary fields with dimensionless couplings, have to be added to ensure renormalisability [Gra; Gra18]. In the example below the universality class including massless scalar ϕ^4 theory, in $D = 4$ dimensions, and the non linear σ model, in $D = 2$ dimensions, will be considered, since ϕ^4 is a very familiar model.

3.1 Tower theory of ϕ^4 as an example for massless scalar theories

The starting point of the derivation can either be the non linear sigma model or ϕ^4. In its common form, on a manifold M with metric g_{ab}, the Lagrangian of the non-linear sigma model is given by

$$\mathcal{L}^2 = \frac{1}{2} g_{ab}(\phi)\, \partial_\mu \phi^a \partial^\mu \phi^b \tag{3.1}$$

where the length of ϕ^i is constrained to be a coupling constant g. It is then possible to reformulate the theory, where the constraint that the length of ϕ^i $\sqrt{\phi^i \phi^i}$ has to be the coupling g can be replaced by an auxiliary field σ, which serves as a Lagrangian multiplyer field, ensuring said constraint. The superscripts will also be neglected from here on. The new Lagrangian is

$$\mathcal{L}^{4,2} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} \left(\sigma \phi \phi - g^2\right) = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} \sigma \phi \phi - \frac{g^2}{2} \sigma, \tag{3.2}$$

a rescaling of the σ field then swaps the coupling constant over to the interaction term

$$\mathcal{L}^{4,2} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{g_0}{2} \sigma \phi \phi - \frac{1}{2} \sigma. \tag{3.3}$$

The Lagrangian is called $\mathcal{L}^{4,2}$ to stress the connection to ϕ^4 theory in the first number of the superscript. The second number in the superscript denotes the dimension of the theory.

Increasing the dimension to $D = 4$ produces the Lagrangian for massless ϕ^4 theory:

$$\mathcal{L}^{4,4} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{g_0}{2!}\sigma\phi\phi - \frac{1}{2}\sigma^2, \tag{3.4}$$

which for the moment will be called ϕ_4^4 theory. On the other hand one could start with general massless scalar ϕ^4 theory

$$\mathcal{L}^{4} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{\lambda}{4!}\phi^4 \tag{3.5}$$

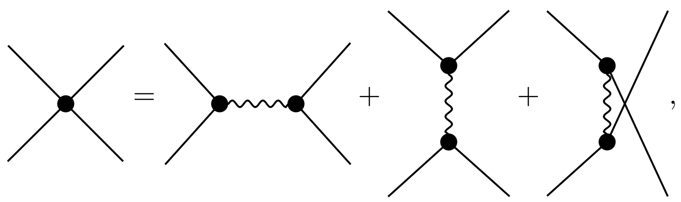
and replace two of the ϕ fields again by a Lagrangian multiplyer σ field

$$\mathcal{L}^{4} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{\lambda}{4!}\phi^4 = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}\left(\frac{g_0}{2}\phi\phi - \sigma\right)^2 \tag{3.6}$$

$$= \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{g_0}{2!}\sigma\phi\phi - \frac{1}{2}\sigma^2 = \mathcal{L}^{4,4} \tag{3.7}$$

where $g_0 = \frac{1}{3}\lambda$. As can be seen, this also leads to (3.4).
While in standard ϕ^4 theory the ϕ propagators are represented by straight lines , a second propagator, the σ propagator represented by a wavy line appears in ϕ_4^4. Both are coupled by the newly introduced core interaction  replacing  .

Since in ϕ_4^4 theory only a 3-point interaction exists, the topology of graphs changes. If we take a look at the 4-point vertex in ϕ^4 theory, all possible connections between incoming and outgoing momenta (or permutations of external half edges) have to be considered in the new formulation, which means the  changes to

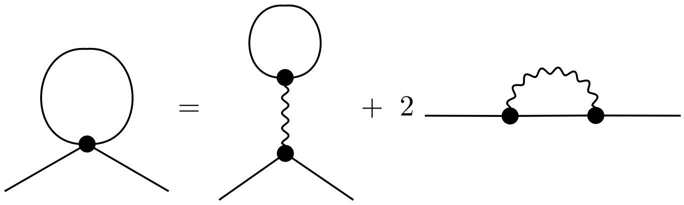


(3.8)

where the diagrams on the right hand side correspond to the s-, t- and u-channel respectively.
Although there are now more graphs, one σ propagator connecting two 3-point vertices replace one 4-point vertex and in principle we get the ϕ^4 graphs back when we shrink all wavy lines. By construction it is now possible to evaluate 4-point graphs in disguise as 3-point graphs. While they are less cumbersome to compute, the disadvantage is that now 1PI, as well as reducible graphs, have to be looked at. Luckily the symmetry factors in both formulations agree order by order, so no extra rules have to be established.

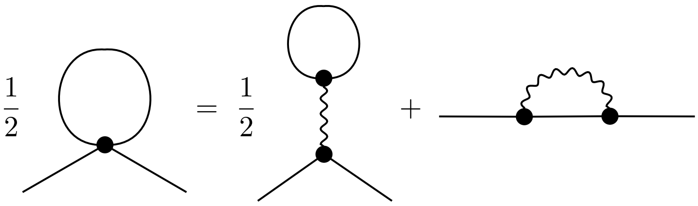
Symmetry Factors

To analyse the relation between symmetry factors of ϕ^4 and $\phi^{4,4}$ theory the procedure will be explained with an one loop example. At every 4-point vertex insert the possible topologies from (3.8) and divide through the highest coefficient. For the propagator it means that the s- and u-channel give contribution to the second term below while the t-channel gives rise to the first term:



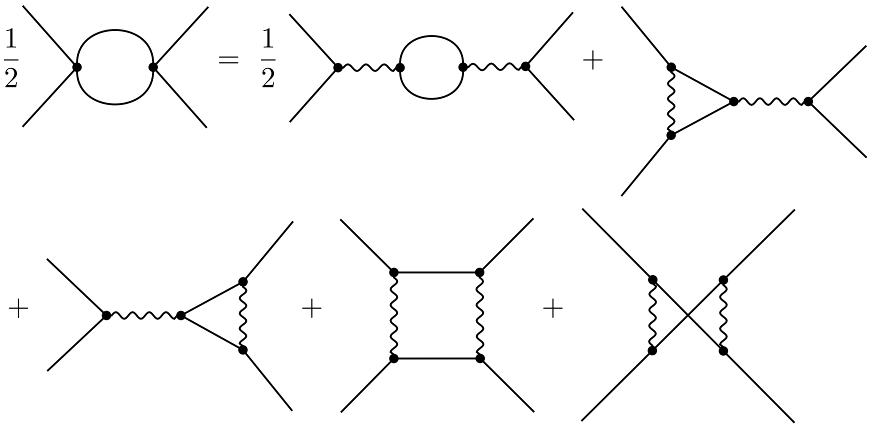
dividing by the highest coefficient yields:

Propagator:

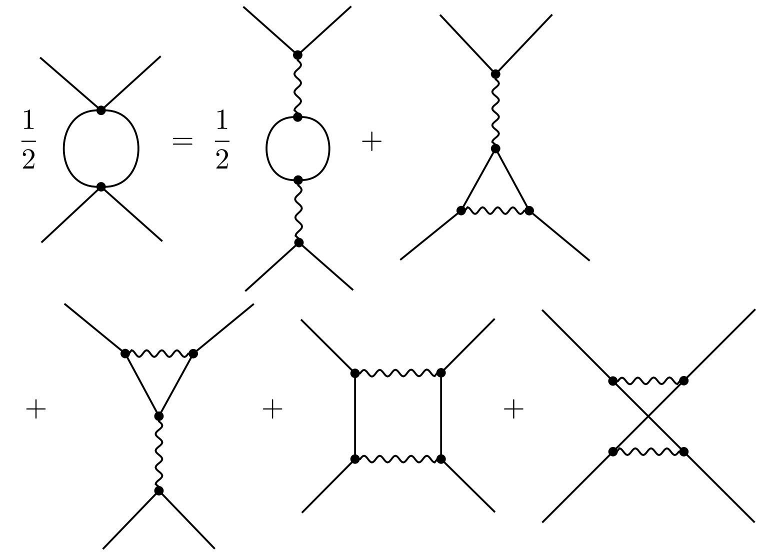


In the same way one gets the vertex expressions:

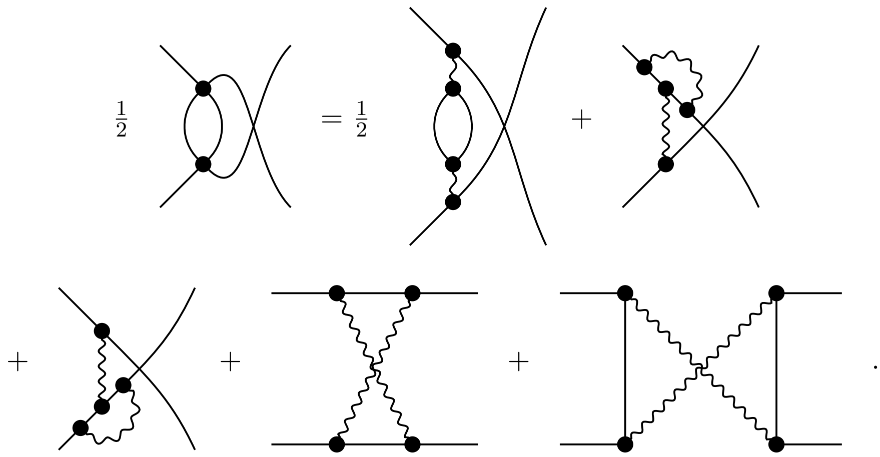
Vertices
s-channel:



t-channel:



u-channel:



It easy easy to see that the appearing box diagrams in the s-, t- and u- channel do not have a residue in the residue set of the theory (more on that in 3.1.1) or in other terms, do not correspond to an interaction term in ϕ_4^4 theory, if we shrink the loop to a point. Such graphs appear in non-self-similar DSEs in the sense of (2.81).

Tower Theories in Higher Dimensions

When the dimension of the Lagrangian $\mathcal{L}^{4,4}$ is not equal to 4, but rather arbitrarily chosen while maintaining renormalizability, this leads to a new Lagrangian which always depends on the core interaction $\frac{g_0}{2!}\sigma\phi\phi$ plus additional terms $F_D(\sigma)$ only depending on σ. Additional terms with a ϕ dependence do not occur, since any monomial in the Lagrangian has to add up to D in their mass dimension to preserve invariance of the action (more on that below) The core interaction then drives the theory through the dimensions and builds a tower of theories, hence the name tower theories.

$$\mathcal{L}^{4,D} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{g_0}{2}\sigma\phi\phi + F_D\left(\sigma\right). \tag{3.9}$$

The additional terms $F_D\left(\sigma\right)$ which are not written explicitly and only depend on field σ and dimensionless coupling constants, have to follow constraints to ensure renormalizability and uniqueness under partial integration. For example the Lagrangians for $\phi^{4,6}$ and $\phi^{4,8}$ are:

$$\mathcal{L}^{4,6} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2}\left(\partial_\mu \sigma\right)^2 + \frac{g_0}{2}\sigma\phi^2 + \frac{g_1}{3!}\sigma^3 \tag{3.10}$$
$$\mathcal{L}^{4,8} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2}\left(\partial_\mu \Box\sigma\right)^2 + \frac{\overline{g}_0}{2}\sigma\phi^2 + \frac{\overline{g}_1}{4!}\sigma^4 + \frac{\overline{g}_2}{3!}\sigma^2\Box\sigma \tag{3.11}$$

and give new terms, respecting uniqueness under partial integration because only $\frac{g_1}{3!}\sigma^2\Box\sigma$ is considered. Although $-\frac{g_1}{3!}\sigma\left(\partial_\mu\sigma\right)^2$ would not invalidate the renormalisability condition, that coupling constants must be dimensionless, but both terms are equivalent under partial integration, therefore only one of the terms may appear.

Superficial Degree of Divergence

From the construction prescription of tower theories it is possible to determine the superficial degree of divergence $\omega_D\left(\Gamma\right)$ for a given graph in space-time dimension D by

$$\omega_D\left(\Gamma\right) = D - \sum_{\text{external propagators } p} N_p\ \omega_p + \sum_{\text{vertices } v} M_v\ \omega_v \tag{3.12}$$

with the number of external propagators and vertices N_p and M_v and weights of propagators and vertices ω_p, ω_v respectively. First, we analyse the mass dimension in D space-time dimensions of the involved fields ϕ and σ. From the kinetic terms we see, that ϕ has mass dimension $[\phi] = \frac{D-2}{2}$, while σ has mass dimension $[\sigma] = 2$, giving the weights $\omega_\phi = \frac{D-2}{2}$ and $\omega_\sigma = 2$
Next, we take a look at the appearing interaction terms. The mass dimension $\left[\mathcal{L}^{int}\right]$, belonging to a Lagrangian $\mathcal{L} = \mathcal{L}^{free} + \mathcal{L}^{int}$, of an interaction term is proportional to its derivatives and fields

$$\mathcal{L}^{int} \sim \partial_\mu^d \sigma^a \phi^b \tag{3.13}$$

and therefore dependents on the exponents d,a,b of involved fields and derivatives multiplied by their respective mass dimension. Which in turn means

$$\left[\mathcal{L}^{int}\right] = d + 2a + \frac{D-2}{2}b = D, \tag{3.14}$$

which has to be equal to D to ensure a dimensionless coupling and thus a renormalisable theory. The weight of the vertex in question then is

$$\omega_v = \left[\mathcal{L}_v^{int}\right] - D. \tag{3.15}$$

Does the construction prescription for tower theories indeed give zero vertex weights ω_v to ensure renormalisability? The core interaction $\frac{1}{2}\sigma\phi^2$ has

$$\omega_{-\!\!\prec} = \frac{D-2}{2} 2 + 2 - D = 0, \tag{3.16}$$

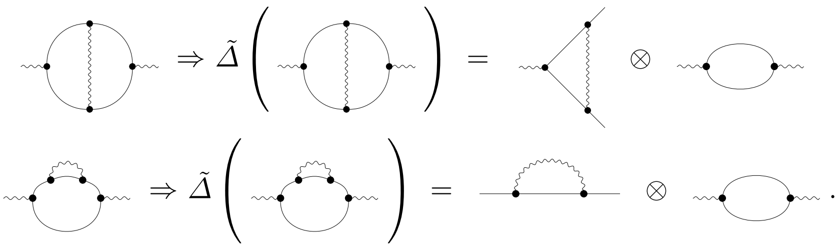
but what about the spectator terms? Since they appear first at a certain space-time dimension $\tilde{D}$, such as the three point or four point vertex in (3.10) and (3.11). Suppose that $D \geq \tilde{D}$, then the weight of a spectator interaction is

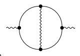
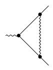
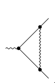
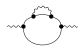
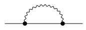
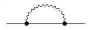
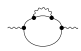
$$\omega_\sigma = \left(D - \tilde{D}\right) + 2\frac{\tilde{D}}{2} - D = 0. \tag{3.17}$$

Therefore all vertices are of zero weight and the superficial degree of divergence in D space-time dimensions of any graph is only depending on its external edges:

$$\omega_D\left(\mathit{\Gamma}\right) = D - \frac{D-2}{2}\mathrm{ext}\left(\text{——}\right) - 2\ \mathrm{ext}\left(\sim\!\sim\right). \tag{3.18}$$

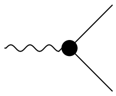
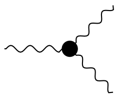
If a graph $\mathit{\Gamma}$ gives $\omega_D\left(\mathit{\Gamma}\right) \geq 0$, it is superficially divergent, but there could be the case that, even if $\mathit{\Gamma}$ is overall convergent ($\omega_D\left(\mathit{\Gamma}\right) < 0$), it has subgraphs γ which still are divergent, so a bit of care has to go into evaluation of graphs. As an example take the corrections to the σ propagator in $D = 2$ dimensions:



In these cases (3.18) leads to ω_2 () $= -2$ which means the graph seems convergent, the subgraph  however has weight ω_2 () $= 0$ and therefore the overall graph is divergent. For the other graph ω_2 () $= -2$, so the graph again seems convergent and again the subgraph  has weight ω_2 () $= 2$ which means  is overall divergent as well. Coming back to the construction of Lagrangians, for the terms in σ this means all terms up to $n = \frac{D}{2}$ have to be considered, possibly containing derivatives. As can be seen above in (3.10) and (3.11) the interaction or spectator terms $\frac{g_1}{3!}\sigma^3$ and respectively $\frac{g_1}{3!}\sigma^2\Box\sigma$ and $\frac{g_2}{4!}\sigma^4$ appear. Diagrammatically both (σ-)3-point terms $\frac{g_1}{3!}\sigma^3$ and $\frac{g_1}{3!}\sigma^2\Box\sigma$ are represented by the same 3-point vertex (or residue)  in their corresponding theory, but are connected to analytic expressions by different Feynman rules due to the different monomials in the Lagrangian $\left(\frac{g_1}{3!}\sigma^3 \text{vs.} \frac{g_1}{3!}\sigma^2\Box\sigma\right)$.

3.1.1 From Towers to DSEs to Hopf Algebra

As discussed above, certain universality classes can be formulated conveniently by Lagrangians defining tower theories in D space-time dimension. Generally any monomial of a given QFT Lagrangian can be represented by a diagrammatic representation also representing a residue of the theory. Canonically every squared field gets assigned a propagator, while monomials of higher order get assigned vertices with external lines representing the order of involved fields, i.e.: in case of $\mathcal{L}^{4,D}$ up to $D = 6$

monomial	diagrammatic rep.	analytic/expression	
$\frac{1}{2}\left(\partial_\mu \phi\right)^2 \longrightarrow$	$\text{———} =$	$\frac{1}{p^2}$	(3.19)
$\frac{1}{2}\left(\sigma\right)^2, \frac{1}{2}\left(\partial_\mu \sigma\right)^2 \longrightarrow$	$\sim\sim\sim =$	$1,\ \frac{1}{p^2}$	(3.20)
$\frac{g_0}{2!}\sigma\phi^2 \longrightarrow$	 $=$	$-\mathrm{i}g_0$	(3.21)
$\frac{g_1}{3!}\sigma^3 \longrightarrow$	 $=$	$-\mathrm{i}g_1.$	(3.22)
			(3.23)

As already mentioned, different monomials can be represented by the same diagrammatic expression. While the diagrammatic expressions look the same, the corresponding Feynman rules are different (see (3.20)) and therefore the residues as well.

Using the diagrammatic representation of residues above, the set $\mathcal{R} = \left\{ -, \sim, \prec, \prec \right\} = \left\{ \mathcal{R}^{[1]}, \mathcal{R}^{[0]} \right\}$ of vertices $v \in \mathcal{R}^{[0]}$ and propagators $p \in \mathcal{R}^{[1]}$ can be constructed. The residues $\mathcal{R}$ can then be endowed by a Hopf algebra structure as described in 2.4. Further more each residue in $\mathcal{R}$ gives rise to a DSE

$$X^{-} = \mathbb{1} - \sum_k \alpha^k B_+^{k,-}\left(X^{-} \prod_{n_i i \in \mathcal{R}^{[0]}} \left(Q_i\right)^{n_i} \right) \tag{3.24}$$

$$X^{\sim} = \mathbb{1} - \sum_k \alpha^k B_+^{k,\sim}\left(X^{\sim} \prod_{n_i i \in \mathcal{R}^{[0]}} \left(Q_i\right)^{n_i} \right) \tag{3.25}$$

$$X^{\prec} = \mathbb{1} + \sum_k \alpha^k B_+^{k,\prec}\left(X^{\prec} \prod_{n_i i \in \mathcal{R}^{[0]}} \left(Q_i\right)^{n_i} \right) \tag{3.26}$$

$$X^{\prec} = \mathbb{1} + \sum_k \alpha^k B_+^{k,\prec}\left(X^{\prec} \prod_{n_i i \in \mathcal{R}^{[0]}} \left(Q_i\right)^{n_i} \right) \tag{3.27}$$

with invariant charges defined by (2.75) and explicitly written for the three vertices and

$$Q(\alpha) = \frac{X}{X\sqrt{X}} \tag{3.28}$$

$$Q(\alpha) = \frac{X}{(X)^{\frac{3}{2}}}. \tag{3.29}$$

The DSEs (3.24) - (3.27) then generate all elements of the corresponding Hopf algebra of Feynman graphs $H^{4,6}_{FG}$ and can be solved by the ansatz (2.78), which will be sketched below in an example.

example

As an example the DSE

$$X = \mathbb{1} + \alpha B_+ (X Q^2) \tag{3.30}$$

for a single general three point vertex , with a single skeleton graph is considered. In this case the invariant charge Q is simply defined by

$$Q = X. \tag{3.31}$$

The solution to DSE (3.30) is given by ansatz (2.78) and to fourth order yields (replacing by r)

$$X^r = \mathbb{1} + \alpha c_{r,1} + \alpha^2 c_{r,2} + \alpha^3 c_{r,3} + \alpha^4 c_{r,4} + \mathcal{O}(\alpha^5). \tag{3.32}$$

The reduced Green's functions $c_{r,i}$, $i \in \mathbb{N}$ stand for graphs of ith loop order and are computed by subsequent application of the insertion operator B_+. The appearing graphs in can then be represented by rooted trees as defined in section 2.2. Every node in a tree $\bullet$ stands for one insertion of the skeleton graph , therefore the first two trees are

$$c_{,1} = \quad = \bullet \tag{3.33}$$

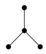
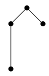
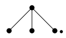
$$c_{,2} = 3 \quad = 3 \tag{3.34}$$

The coefficients in front of the graph or tree come from the possible insertion places of the subdivergencies i.e. can be inserted at every vertex to give thus there are three insertion places and hence the coefficient 3. To fourth order the reduced Green's functions represented by trees therefore yield

$c_1 =$ (3.35)

$c_2 = 3$ (3.36)

$c_3 = 3\Big(3$ $+$  $\Big)$ (3.37)

$c_4 = 18$ $+ 9$  $+ 18$  $+$ . (3.38)

3.1.2 Invariant Charges and the Way to Universality Classes

It will turn out in Chapter 4, that the quotient algebra $\mathcal{H} = H_{FG}/I$, with a Hopf ideal I depending on the invariant charges Q^{v_i} of H_{FG}, gives rise to universality classes by imposing Ward-Takahashi identities on graphs. As an example we take a look at two three point interactions $a =$  and $b =$  which obey Dyson-Schwinger equations

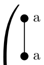
$$X^a = \mathbb{1} + \alpha B_+^a \left(X^a Q^{a2}\right) \tag{3.39}$$

$$X^b = \mathbb{1} + \alpha B_+^b \left(X^b Q^{b2}\right) \tag{3.40}$$

with invariant charges $Q^a = X^a$ and $Q^b = X^b$. The Hopf ideal is then defined by $I = \left\langle Q^a - Q^b \right\rangle$ with elements $i_k = c_{a,k} - c_{b,k}$, $k \in \mathbb{N}$. Since invariant charges are series in polynomials, $c_{a,k}$, $c_{b,k}$ have to agree order by order. As a reminder, for I to be a Hopf ideal the conditions

1. map the co-unit to the kernel $\varepsilon(I) = 0$
2. be closed under co-multiplication $\Delta(I) \subset I \otimes H + H \otimes I$
3. respect the antipode $S(I) \subset I$

have to be fulfilled as defined in 2.8. Does $I = \left\langle Q^a - Q^b \right\rangle$ fulfill these requirements?
1. follows from definition; Q^a and Q^b both start with $\mathbb{1}$ so there is no term proportional to the unit $\mathbb{1}$, which is the only element ε does not map to zero. 3. is respected since the $c_{a,k}$, $c_{b,k}$ respect the antipode. 2. however, is a little more tricky to see and the proof follows below.
It is straight forward to calculate the co-product of I:

$\frac{1}{3}\tilde{\Delta}(i_2) = \tilde{\Delta}\Big($  $-$  $\Big)$ (3.41)

$$= \bullet_a \otimes \bullet_a - \bullet_b \otimes \bullet_b \pm \bullet_a \otimes \bullet_b \tag{3.42}$$

$$= \bullet_a \otimes \left(\bullet_a - \bullet_b\right) - \left(\bullet_a - \bullet_b\right) \otimes \bullet_b \tag{3.43}$$

$$= \bullet_a \otimes i_1 - i_1 \otimes \bullet_b \tag{3.44}$$

$$\subset H \otimes I + I \otimes H. \tag{3.45}$$

So $\Delta\left(i_2\right)$ is closed with respect to the coproduct. Next $\tilde{\Delta}\left(i_3\right)$ needs to be checked.

$$\frac{1}{3}\tilde{\Delta}\left(i_3\right)=\tilde{\Delta}\left(3\left\{\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}-\begin{smallmatrix}\bullet_b\\ \vert\\ \bullet_b\\ \vert\\ \bullet_b\end{smallmatrix}\right\}+\begin{smallmatrix} & \bullet_a & \\ / & & \backslash\\ \bullet_a & & \bullet_a\end{smallmatrix}-\begin{smallmatrix} & \bullet_b & \\ / & & \backslash\\ \bullet_b & & \bullet_b\end{smallmatrix}\right) \tag{3.46}$$

$$=3\left\{\bullet_a\otimes\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}+\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}\otimes\bullet_a-\left(\bullet_b\otimes\begin{smallmatrix}\bullet_b\\ \vert\\ \bullet_b\end{smallmatrix}+\begin{smallmatrix}\bullet_b\\ \vert\\ \bullet_b\end{smallmatrix}\otimes\bullet_b\right)\right\} \tag{3.47}$$

$$+2\bullet_a\otimes\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}+\left(\bullet_a\right)^2\otimes\bullet_a-\left(2\bullet_b+\otimes\begin{smallmatrix}\bullet_b\\ \vert\\ \bullet_b\end{smallmatrix}+\left(\bullet_b\right)^2\otimes\bullet_b\right) \tag{3.48}$$

$$\pm\left\{3\left(\bullet_b\otimes\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}+\begin{smallmatrix}\bullet_b\\ \vert\\ \bullet_b\end{smallmatrix}\otimes\bullet_a\right)+2\bullet_b\otimes\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}+\left(\bullet_b\right)^2\otimes\bullet_a+2\bullet_a\bullet_b\otimes\left(\bullet_a-\bullet_b\right)\right\} \tag{3.49}$$

$$=3\left(\left\{\bullet_a-\bullet_b\right\}\otimes\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}+\left\{\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}-\begin{smallmatrix}\bullet_b\\ \vert\\ \bullet_b\end{smallmatrix}\right\}\otimes\bullet_a-\left(\bullet_b\otimes\left\{\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}-\begin{smallmatrix}\bullet_b\\ \vert\\ \bullet_b\end{smallmatrix}\right\}+\begin{smallmatrix}\bullet_b\\ \vert\\ \bullet_b\end{smallmatrix}\otimes\left\{\bullet_a-\bullet_b\right\}\right)\right) \tag{3.50}$$

$$+2\left\{\bullet_a-\bullet_b\right\}\otimes\begin{smallmatrix}\bullet_1\\ \vert\\ \bullet_1\end{smallmatrix}+\left\{\bullet_a-\bullet_b\right\}^2\otimes\bullet_a \tag{3.51}$$

$$-\left(2\bullet_a\otimes\left\{\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}-\begin{smallmatrix}\bullet_b\\ \vert\\ \bullet_b\end{smallmatrix}\right\}+\left\{\bullet_a-\bullet_b\right\}^2\otimes\bullet_b+2\bullet_a\bullet_b\otimes\left\{\bullet_a-\bullet_b\right\}\right) \tag{3.52}$$

$$=3\left(i_1\otimes\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}+i_2\otimes\bullet_a-\left(\bullet_b\otimes i_2-\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}\otimes i_1\right)\right) \tag{3.53}$$

$$+2i_1\otimes\begin{smallmatrix}\bullet_a\\ \vert\\ \bullet_a\end{smallmatrix}+\left(i_1\right)^2\otimes\bullet_a+\left(2\bullet_b\otimes i_2+\left(i_2\right)^2\otimes\bullet_b-2\bullet_a\bullet_b\otimes i_1\right) \tag{3.54}$$

$$\subset\ I\otimes H+H\otimes I \tag{3.55}$$

this tells us, that $I=\left\langle Q^a-Q^b\right\rangle$ is a Hopf ideal indeed.

CHAPTER 4

Proofs

4.1 Renormalisation Functions and Tower Theories at the Wilson-Fisher Fixed Point

At a fixed point where the renormalisation β-functions vanish, (2.90) simplifies to

$$\left[\frac{N}{2}\gamma\left(g,L\right)-\frac{\partial}{\partial L}\right]G^{r_N}=0.$$

The simplification of (2.90) is not the only positive feature following from a vanishing β-function. As Kißler showed in [KiSS] a vanishing β-function entails momentum renormalisation scheme independence, which permits setting invariant charges $Q^{v_i}=\mathbb{1}$ at a fixed point. This characteristic can be used to analyse the anomalous dimension $\eta=\gamma^{—}$ of the tower theory in question and deduct, that it is an invariant in all dimensions.

4.1.1 Hopf Ideals at the Wilson-Fisher Fixed Point

In the following it is assumed, that we are already at a fixed point, where the $\beta_i(g^*)$-functions of couplings $g^*=(g_1^*,g_2^*,\ldots)$ are equal to zero $\beta_i(g^*)=0$. Since every coupling corresponds to an invariant charge in the Hopf algebra, these can be taken equal as well $Q=Q^{v_1}=Q^{v_2}=\ldots$. To incorporate this into a Hopf algebra, an ideal I of relations between the different Q^{v_k} is defined via $I=\langle Q^{v_1}-Q^{v_2},\ldots,Q^{v_{n-1}}-Q^{v_n},Q^{v_n}-Q^{v_1}\rangle$, so that in the quotient Hopf algebra $\mathcal{H}=H/I$ the desired relations are respected.

Ideals and Invariant Charges

A way to see that arbitrary invariant charges give Hopf ideals is given in the following short analysis using Sweedler's notation $\Delta(X)=X'\otimes X''$, closely following the arguments of [Pri18]: Firstly, recall that the co-unit ε is an algebra morphism on H and thus its kernel $\ker\varepsilon$ generates an ideal $I\subset H$. To promote the ideal I to an Hopf ideal it also has to satisfy $\Delta(I)\subset I\otimes H+H\otimes I$, giving it the co-ideal property and therefore creating an biideal and letting it fulfill $S(I)\subset I$ finally promotes I to an Hopf ideal. In summary any Hopf ideal needs to satisfy the following conditions:

1. $\varepsilon(I)=0$
2. $\Delta(I)\subset I\otimes H+H\otimes I$
3. $S(I)\subset I$.

Secondly, how does the invariant charge Q fair in this regard? With minimal changes, namely adding $-\mathbb{1}$ one gets an Hopf ideal $I=\langle Q_i-\mathbb{1}\rangle$

Proposition 4.1 *An invariant charge Q permit for a Hopf ideal $I = \langle Q - \mathbb{1} \rangle$.*

Proof 4.2 *1.: From the definition*

$$Q^{v_i} = \frac{X^{v_i}}{\prod_{p_i \text{incedent to } v_i} \sqrt{X^{p_i}}} \tag{4.1}$$

with residues $v_i \in \mathcal{R}^{[0]}$ and $p_j \in \mathcal{R}^{[1]}$ and DSEs X^r. It is clear, that Q (superscripts will be neglected from here on when unambiguous) is a series of polynomials in $\alpha(g)$ with coefficients in H. The only constant term $\mathbb{1}$ of said series produces $\varepsilon(Q) \neq 0$, thus $\varepsilon(Q - \mathbb{1}) = 0$.
2. is fulfilled due to the closed form of the co-product on Q implied in [Yea08]:

$$\Delta(Q) = \sum_{k \geq 0} Q^{k+1} \otimes Q|_k - \mathbb{1} \otimes \mathbb{1} \subset I \otimes H + H \otimes I \tag{4.2}$$

with $Q|_k$ being the kth order monomial in Q.
3. is fulfilled since 2 is fulfilled. If one considers the antipode in a slightly different form, this is easy to see:

$$S(I) = S(Q-1) = \sum_{k \geq 0} S(Q')Q'' - 1 \tag{4.3}$$
$$= m(S \otimes id)\Delta(Q - \mathbb{1}) = (S \star id)(Q - \mathbb{1}) = 0 \tag{4.4}$$
$$\subset I \tag{4.5}$$

where $(S \star id)(x) = \varepsilon(x) = 0$ for $x \in I$ by definition of the antipode and the convolution product is used.□

Remark 4.3 ***Sums of Hopf Ideals*** *Due to the linearity of ε,Δ,S, sums of Hopf ideals are again Hopf ideals. Especially noteworthy in this regard is an Hopf ideal consisting of differences of invariant charges*

$$I = \langle Q^{v_1} - Q^{v_2}, \ldots, Q^{v_{n-1}} - Q^{v_n}, Q^{v_n} - Q^{v_1} \rangle . \tag{4.6}$$

4.1.2 Renormalisation Group Functions

As mentioned above, at a fixed point $\beta_i(g^*) = 0$ from which follows $Q^{v_i} = \mathbb{1}$. This in turn simplifies all DSEs to be linear, the vertex DSEs solely depend on vertex corrections while the propagator DSEs depend only on propagator corrections.
The following derivation appears here for the first time. It gives a proof, that the anomalous scaling dimension γ—of the renormalisation group is independent of the space-time dimension D in a given tower theory and thus defines a universality class.

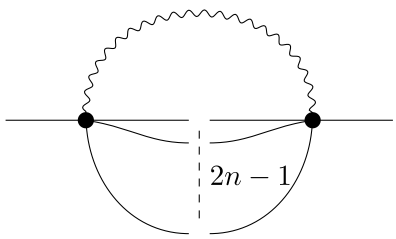
Proposition 4.4 *For a tower theory at a fixed point, with a core interaction $\frac{g_0}{(2n)!}\sigma\,(\phi\phi)^n$ connecting auxiliary fields σ with physical fields ϕ, there exists a unique anomalous dimension γ—, independent of the space-time dimension, defining a universality class.*

Proof 4.5 *The direct space-time dimension dependence comes from the invariant charges Q^{v_i} in the DSE. Since at the WF-FP $Q^{v_i} = 1, \forall i$ the direct dimension dependence has gone. However there still can be an indirect dimension dependence from the skeleton graphs, which shall be analysed now.*
A direct, way to see that a scaling solution exists, follows directly from the RGE (2.90) for a propagator Green's function G^p with vanishing β-function:

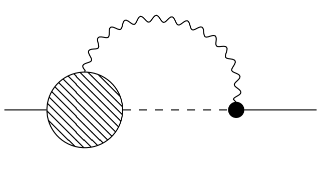
$$\left[\gamma^p(\alpha) - \frac{\partial}{\partial L}\right] G^p(\alpha, L, R) = 0 \tag{4.7}$$

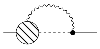
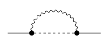
$$\Rightarrow G^p = \exp\left(\gamma^p L\right). \tag{4.8}$$

Here the question, if G— is space-time dependent enters again. Since there is only one vertex connecting ϕ and σ fields in the theory, and there are no interactions between ϕ fields alone. The ϕ propagator skeleton therefore has a banana graph shape with $2n-1$ ϕ edges and one σ edge



Note that G—is independent of other vertices than the core interactionand σ-propagator corrections, due to its linear DSE. Since there are no pure ϕ interactions, connections only between ϕ edges can not appear. Any insertion of a number of vertices on the σ edge would either be a subdivergence or destroy the external leg structure. An insertion of a number of vertices on or between the ϕ edges would either be a subdivergence of a 2-point up to $4n-2$-point ϕ function or would destroy the external edge structure as well. A closer look at the ϕ-propagator skeletons reveals, that there is only one skeleton graph. Since the number of ϕ edges apparently does not play a role, they will be replaced by a single dashed line for cleaner looks. New skeletons therefore can only occur from inserting another vertex linking the σ and ϕ edges, which is the same as inserting an  *vertex next to a vertex. Consider one of the vertices of the first skeleton graph*



 *can either be a 1PI graph which leads to a subdivergence and not a new skeleton graph, or*  *is not 1PI, but then it is a combination of a propagator correction and a vertex correction, thus only adding a subdivergence. Therefore we know the only skeleton is*  *and hence*

$G^{—}$ is explicitly independent of space-time dimension D!
So applying (4.7) to $G^{—}$ yields the desired solution

$$\gamma^{—} = \partial_L G^{—}. \tag{4.9}$$

Uniqueness follows directly from (4.7): assume there exist $\gamma_1, \gamma_2,\ \gamma_1 \neq \gamma_2$ and $G_1^{—} = G^{—}(\gamma_1)\,,\ G_2^{—} = G^{—}(\gamma_2)$ such that $G_1^{—} = G_2^{—}$, then from (4.7) follows $\gamma_1 = \gamma_2$, which is a contradiction to the assumption and thus proofs uniqueness. $\square$

To conclude, we have seen, that a fixed tower theory gives the same anomalous dimension $\gamma^{—}$ at the a fixed point independent of its space-time dimension D and thus is indeed a formulation of a universality class.

Remark 4.6 *There is a question which arises from the external field propagator $\sim\!\sim\!\sim$. Its momentum dependence $\left(p^2\right)^{-\frac{D-4}{2}}$ changes in different space-time dimensions, which may lead to unequal anomalous dimensions $\gamma_D^{—} \neq \gamma_{D'}^{—}$. These unequalities depend on the fixed point in question. Unfortunately the analysis of differences between fixed points and their relation to anomalous dimensions are beyond the scope of this work.*

CHAPTER 5

Conclusions

The question whether or not a universality class can be formulated as a tower theory was approached via the correspondence of Feynman graphs and Hopf algebra. This approach has the advantage of working with algebraic objects, which represent complicated analytical expressions, rather than having to make sense of the analytic expressions in the first place. The connection between the statistical physics notion of critical exponents, which define universality classes, and their connection to quantum field theory was presented. This lead to the realisation that, in our approach, a single parameter, the renormalisation group equation anomalous dimension is sufficiently describing a universality class.
By use of the groundwork laid by D. Kreimer and collaborators, it was possible to find relations, which incorporate certain crucial aspects into the Hopf algebra, such as that at a fixed point invariant charges become constant and thus it is possible to use a quotient Hopf algebra linearising the resulting Dyson-Schwinger equations. Here with this approach it was possible for the first time to show that the core interaction of a tower theory defines the renormalisation group anomalous dimension and hence a universality class. As mentioned, the specifics of the fixed point at which the universality class arises were not explicitly taken into account, which lead to an open question. This question and the subtleties how different fixed points play a role in the formulation of universality classes therefore is still to be conquered.
As already mentioned in chapter 2.7, the main use of universality classes is in the theory of critical exponents in statistical physics. It tells us, that whether one knows the specifics of a theory or not, as long as only the critical exponents are of interest it is possible to use any theory lying in the same universality class to compute them. Which can lead to great computational simplification.
Useful examples are often found in different magnetic media or even the superfluid phase of Helium. When endowed with a $O(N)$ symmetry, the thoroughly used example of ϕ^4 tower theories includes the $O(N)$ non-linear sigma model as the two dimensional formulation $\phi^{4,2}$ [Gra], the Ising model in $D = 3$ dimensions and $N = 1$, the Heisenberg model in the case of $D = 3$, $N = 3$ [Sta99; Wil74]. Other uses in physics in more complex environments can be found in [Gra17c] for the Landau-Ginzburg theory or in [Gra18] for QED-Gross-Neveu theory and their respective universality classes.
Furthermore in J. Gracey suggested in [Gra17b], that it is possible to construct different universality classes based on a common underlying theory (ϕ^4 in this regard) by changing the number of derivatives in the kinetic term while keeping the core interaction fixed. This approach could be promising in endeavours to connect different conformal field theories and categorise them.

Bibliography

[Ber05] BERGBAUER, CHRISTOPH and DIRK KREIMER: 'Hopf algebras in renormalization theory: Locality and Dyson-Schwinger equations from Hochschild cohomology'. *arXiv:hep-th/0506190* (June 22, 2005), vol. (cit. on pp. 1, 17).

[Bro01] BROADHURST, D. J. and D. KREIMER: 'Exact solutions of Dyson-Schwinger equations for iterated one-loop integrals and propagator-coupling duality'. *Nuclear Physics B* (Apr. 2001), vol. 600(2): pp. 403–422 (cit. on p. 1).

[Bro11] BROWN, FRANCIS and DIRK KREIMER: 'Angles, scales and parametric renormalization'. *arXiv:1112.1180 [hep-th]* (Dec. 6, 2011), vol. (cit. on p. 1).

[Cal70] CALLAN, CURTIS G.: 'Broken Scale Invariance in Scalar Field Theory'. *Physical Review D* (Oct. 15, 1970), vol. 2(8): pp. 1541–1547 (cit. on p. 18).

[Con98] CONNES, ALAIN and DIRK KREIMER: 'Hopf Algebras, Renormalization and Noncommutative Geometry'. *Communications in Mathematical Physics* (Dec. 1, 1998), vol. 199(1): pp. 203–242 (cit. on p. 10).

[Con99] CONNES, ALAIN and DIRK KREIMER: 'Renormalization in quantum field theory and the Riemann-Hilbert problem'. *Journal of High Energy Physics* (Sept. 22, 1999), vol. 1999(9): pp. 024–024 (cit. on p. 12).

[Con00] CONNES, ALAIN and DIRK KREIMER: 'Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem'. *Communications in Mathematical Physics* (Mar. 1, 2000), vol. 210(1): pp. 249–273 (cit. on p. 1).

[Con01] CONNES, ALAIN and DIRK KREIMER: 'Renormalization in quantum field theory and the Riemann-Hilbert problem II: the \$\beta\$-function, diffeomorphisms and the renormalization group'. *Communications in Mathematical Physics* (Jan. 2001), vol. 216(1): pp. 215–241 (cit. on p. 1).

[Del85] DEL GIUDICE, E., S. DOGLIA, M. MILANI, and G. VITIELLO: 'A quantum field theoretical approach to the collective behaviour of biological systems'. *Nuclear Physics B* (Jan. 1, 1985), vol. 251: pp. 375–400 (cit. on p. 1).

[Ell17] ELLIS, JOSHUA P.: 'Ti k Z-Feynman: Feynman diagrams with Ti k Z'. *Computer Physics Communications* (Jan. 2017), vol. 210: pp. 103–123 (cit. on p. 1).

[Fis74] FISHER, MICHAEL E.: 'The renormalization group in the theory of critical behavior'. *Reviews of Modern Physics* (Oct. 1, 1974), vol. 46(4): pp. 597–616 (cit. on p. 20).

[Foi10] FOISSY, LOÏC: 'Systems of Dyson-Schwinger equations'. *arXiv:0909.0358 [math]* (Mar. 2, 2010), vol. (cit. on p. 7).

[Gra] Gracey, J A: ‘ -functions in higher dimensional eld theories’. (), vol.: p. 8 (cit. on pp. 1, 25, 27, 43).

[Gra18] Gracey, J. A.: ‘Fermion bilinear operator critical exponents at O (1 / N 2) in the QED-Gross-Neveu universality class’. *Physical Review D* (Oct. 16, 2018), vol. 98(8) (cit. on pp. 27, 43).

[Gra17a] Gracey, J. A.: ‘Renormalization of scalar field theories in rational spacetime dimensions’. *arXiv:1703.09685 [hep-th]* (Mar. 28, 2017), vol. (cit. on p. 1).

[Gra17b] Gracey, J. A. and R. M. Simms: ‘Higher dimensional higher derivative $\phi4$*theory*’. *Physical Review D* (July 26, 2017), vol. 96(2) (cit. on pp. 1, 25, 43).

[Gra17c] Gracey, J. A. and R. M. Simms: ‘Six dimensional Landau-Ginzburg-Wilson theory’. *Physical Review D* (Jan. 31, 2017), vol. 95(2): p. 025029 (cit. on p. 43).

[Kas95] Kassel, Christian: *Quantum groups*. Softcover reprint of the hardcover 1st ed. 1995. Graduate texts in mathematics 155. OCLC: 246648976. New York, NY: Springer-Science [u.a.], 1995. 531 pp. (cit. on p. 4).

[KiSS] KiSSler, Henry: ‘Systems of linear DysonSchwinger equations’. (), vol. (cit. on p. 39).

[Kre02] Kreimer, D.: ‘Combinatorics of (perturbative) quantum field theory’. *Physics Reports*. Renormalization group theory in the new millennium. IV (June 1, 2002), vol. 363(4): pp. 387–424 (cit. on p. 7).

[Kre06a] Kreimer, Dirk: ‘Anatomy of a gauge theory’. *Annals of Physics* (Dec. 2006), vol. 321(12): pp. 2757–2781 (cit. on p. 1).

[Kre06b] Kreimer, Dirk: ‘Dyson Schwinger Equations: From Hopf algebras to Number Theory’. *arXiv:hep-th/0609004* (Sept. 1, 2006), vol. (cit. on p. 1).

[Kre03] Kreimer, Dirk: ‘New mathematical structures in renormalizable quantum field theories’. *Annals of Physics* (Jan. 2003), vol. 303(1): pp. 179–202 (cit. on p. 1).

[Kre97] Kreimer, Dirk: ‘On the Hopf algebra structure of perturbative quantum field theories’. *arXiv:q-alg/9707029* (July 23, 1997), vol. (cit. on p. 1).

[Kre] Kreimer, Dirk: ‘Renormalization & Renormalization Group’ (cit. on p. 10).

[Kre09] Kreimer, Dirk and Walter D. van Suijlekom: ‘Recursive relations in the core Hopf algebra’. *Nuclear Physics B* (Oct. 2009), vol. 820(3): pp. 682–693 (cit. on p. 1).

[Kre06c] Kreimer, Dirk and Karen Yeats: ‘An \'Etude in non-linear Dyson–Schwinger Equations’. *Nuclear Physics B - Proceedings Supplements* (Oct. 2006), vol. 160: pp. 116–121 (cit. on p. 1).

[Pes16] Peskin, Michael Edward and Daniel V. Schroeder: *An introduction to quantum field theory: student economy edition*. OCLC: 927402141. Boulder, Colorado: Westview Press, 2016. 842 pp. (cit. on p. 20).

[Pri18] Prinz, David: ‘Algebraic Structures in the Coupling of Gravity to Gauge Theories’. *arXiv:1812.09919 [gr-qc, physics:hep-th, physics:math-ph]* (Dec. 24, 2018), vol. (cit. on p. 39).

[Sta99] Stanley, H Eugene: 'Scaling, universality, and renormalization: Three pillars of modern critical phenomena'. *Rev. Mod. Phys.* (1999), vol. 71(2): p. 9 (cit. on pp. 20, 22, 43).

[Sym70] Symanzik, K.: 'Small distance behaviour in field theory and power counting'. *Communications in Mathematical Physics* (Sept. 1970), vol. 18(3): pp. 227–246 (cit. on p. 18).

[Ton] Tong, David: 'University of Cambridge Part III Mathematical Tripos' (cit. on p. 20).

[Wal74] Walecka, J. D: 'A theory of highly condensed matter'. *Annals of Physics* (Apr. 1, 1974), vol. 83(2): pp. 491–529 (cit. on p. 1).

[Wei73] Weinberg, Steven: 'New Approach to the Renormalization Group'. *Physical Review D* (Nov. 15, 1973), vol. 8(10): pp. 3497–3509 (cit. on p. 18).

[Wil74] Wilson, K: 'The renormalization group and the expansion'. *Physics Reports* (Aug. 1974), vol. 12(2): pp. 75–199 (cit. on pp. 18, 43).

[Yea17] Yeats, Karen: *A Combinatorial Perspective on Quantum Field Theory*. Vol. 15. SpringerBriefs in Mathematical Physics. Cham: Springer International Publishing, 2017 (cit. on p. 1).

[Yea08] Yeats, Karen: 'Growth estimates for Dyson-Schwinger equations'. *arXiv:0810.2249 [math-ph]* (Oct. 13, 2008), vol. (cit. on p. 40).

[Zin96] Zinn-Justin, Jean: *Quantum field theory and critical phenomena*. 3rd ed. International series of monographs on physics 92. Oxford: Clarendon Press, 1996. 1008 pp. (cit. on p. 1).

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