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Combinatorial BRST homology and graph differentials

Masterarbeit

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Introduction

In physics, specifically in quantum field theory, the study of Feynman diagrams is a very important topic. Feynman diagrams are essentially just graphs with additional structure, like edge types and edges that are not connected to a vertex, so called exterior legs. Certain theories have a symmetry called *BRST symmetry* that induces a differential on the space of states, the *BRST differential*, such that the states that actually appear in the real world are homology classes of this differential. On the space of these diagrams, a differential can be introduced, the so called *combinatorial BRST differential*. This differential corresponds to the BRST differential, but operates on discrete objects.
The main result of this thesis is a generalization of the chain complex of (certain) Feynman diagrams with the combinatorial BRST differential and the computation of its homology, which is the topic of chapter 1. There are also two other differentials that are part of the combinatorial BRST differential. One of these differentials will always have trivial homology, while the other carries non-trivial information about the graph. This differential is studied in chapter 2. In particular, computing the homology of this differential entails solving an NP-complete problem of graph theory, so computing this homology is NP-hard.
I want to thank Matthias Görg, for spotting a mistake that gave me some new insight while I was fixing it, Maximilian Weber, for criticizing my notation and helping me to work through some details and Yves Radunz for helping me organize thousands of results of my computations by sharing his LaTeX wisdom with me.

1 Chain complexes of Feynman graphs and their generalization

1.1 Marked graphs and their chain complexes

All the graphs in this thesis are undirected and come equipped with a well-order $\leq$ on the vertices, i.e. there exists a bijection $\varphi : \Gamma_{\text{vert}} \to \{0, \cdots, n\}$ for some n such that $v \leq v' \Leftrightarrow \varphi(v) \leq \varphi(v')$, where Γ_{vert} is the set of vertices of Γ. This well-order is usually called a *labeling*, and a *relabeling* of a graph is the same underlying graph with a possibly different well-order on the vertices.

Definition 1.1.1. Let $(M, 0)$ be a pointed set. A *graph with marked subgraphs in M* is a pair (Γ, m), where Γ is a graph and $m : sub(\Gamma) \to M$ is a function, where $sub(\Gamma)$ is the set of subgraphs of Γ. If $m(\gamma) = x$, we say γ *is (x-)marked*, and if $m(\gamma) = 0$ we say γ is *unmarked*. The *skeleton* of (Γ, m) is Γ. A *graph with marked edges* is a graph where the only marked subgraphs are those that consist of two vertices connected by an edge. A *graph with marked vertices* is a graph where the only marked subgraphs are graphs that consist of a single vertex. A *graph with marked cycles* is a graph where the only marked subgraphs are cycles. We will often just write X for a graph with marked subgraphs.

For us, the set of markings M will always be the set $\{0, 1, 2\}$, and a graph with marked subgraphs in $\{0, 1, 2\}$ will just be a graph with marked subgraphs. To change markings on a graph, we will be using the following notation:

Definition 1.1.2. Let (Γ, m) be a graph with marked subgraphs, $x \in \{0, 1, 2\}$, and $S \subseteq sub(\Gamma)$ a set of subgraphs of Γ. Then, let

$$(\Gamma, m)_{S,x} := (\Gamma, m_{S,x}),$$

where

$$m_{S,x}(\gamma) = \begin{cases} x & \text{if } \gamma \in S \\ m(\gamma) & \text{otherwise.} \end{cases}$$

We will also use the notation $X_{S,x}$ in the obvious way. In the case of graphs with marked vertices, which is most of this thesis, if γ is a subgraph of Γ, $X_{\gamma,x}$ will mean $X_{\gamma_{\text{vert}},x}$, where all the vertices of γ become marked with x.

The following notations will be useful:

Definition 1.1.3. Let $(S, <)$ be an ordered set and $s \in S$. Then $S_{<s}$ is the set of elements of S that are smaller than s.

Definition 1.1.4. Let Γ be a graph, $X = (\Gamma, m)$ be a marking. Then $\Gamma^{m\neq 0}_{\text{type}} \subseteq sub(\Gamma)$ are all those subgraphs of Γ that are marked in X and of type $\text{t} \in \{\text{vert}, \text{edge}, \text{cycle}\}$, and $\Gamma^{m=i}_{\text{t}}$ consists of all subgraphs that have a marking i and type t.

We will now define the chain complex we are interested in. We will use many standard definitions and results from homological algebra which can be found in any textbook on the topic, such as [Wei94].

Definition 1.1.5. Let Γ be a graph and define $\mathcal{L}$ to be the free abelian group generated by all the graphs with marked vertices modulo the subgroup that is generated by all graphs with the property that there exist two marked vertices that are adjacent. Let $\mathcal{L}^\Gamma$ be the subgroup of $\mathcal{L}$ generated by all markings that have Γ as their skeleton. If i, j are natural numbers, let $\mathcal{L}^\Gamma_{i,j}$ be the subgroup of $\mathcal{L}^\Gamma$ generated by all graphs that have i 1-marked vertices and j 2-marked vertices. Because in most cases the object of interest will be the group with an arbitrary amount of 1-markings and a fixed amount of 2-markings, we define

$$\mathcal{L}^\Gamma_j := \bigoplus_i \mathcal{L}^\Gamma_{i,j}.$$

By $\Gamma \in \mathcal{L}$ (or any subgroup of $\mathcal{L}$), we mean the completely unmarked graph. If an element X of $\mathcal{L}$ is a sum of graphs that all have i 1-markings, we write $\deg_1 X = i$, and if it is a sum of graphs that have j 2-markings, we write $\deg_2 X = j$. The generators of $\mathcal{L}$, $\mathcal{L}^\Gamma$, etc. are called the *canonical basis*.

Definition 1.1.6. Let $X \in \mathcal{L}^\Gamma$ be an element of the canonical basis. The differentials on $\mathcal{L}^\Gamma_\bullet$ are defined as follows:

$$\begin{aligned} d : \mathcal{L}^\Gamma \to \mathcal{L}^\Gamma, \quad X &\mapsto \sum_{v \in \Gamma^{m=0}_{\mathrm{vert}}} (-1)^{|\Gamma^{m\neq 0}_{\mathrm{vert},<v}|} X_{v,2} \\ \delta : \mathcal{L}^\Gamma \to \mathcal{L}^\Gamma, \quad X &\mapsto (-1)^{|\Gamma^{m\neq 0}_{\mathrm{vert}}|} \sum_{v \in \Gamma^{m=1}_{\mathrm{vert}}} (-1)^{|\Gamma^{m=1}_{\mathrm{vert},>v}|} X_{v,2} \end{aligned}$$

Note that in case $X_{v,2}$ is a marking where two neighbors are marked, $X_{v,2} = 0$.

Proposition 1.1.7. *d and δ are in fact differentials, i.e. $d^2 = 0, \delta^2 = 0$, and $d\delta + \delta d = 0$, so $D := d + \delta$ is also a differential.*

Proof. This proof is essentially the same as corresponding proofs in [Kre13], pages 200 and 201. We will give the proof here again for convenience.
Let X be an element of the canonical basis of $\mathcal{L}^\Gamma$. Then:

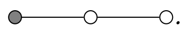
$$\begin{aligned} d^2(X) &= \sum_{v \neq w \in \Gamma^{m=0}_{\mathrm{vert}}} (-1)^{|\Gamma^{m\neq 0}_{\mathrm{vert},<v}| + |\Gamma^{m_{v,2}\neq 0}_{\mathrm{vert},<w}|} X_{\{v,w\},2} = 0 \\ \delta^2(X) &= -\sum_{v \neq w \in \Gamma^{m=1}_{\mathrm{vert}}} (-1)^{|\Gamma^{m=1}_{\mathrm{vert},>v}| + |\Gamma^{m_{v,2}=1}_{\mathrm{vert},>w}|} X_{\{v,w\},2} = 0 \end{aligned}$$

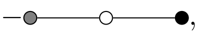
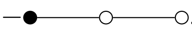
because these sums split into two parts, $v < w$ and $v > w$, which have a relative sign.

$$\begin{aligned} \delta d(X) &= \sum_{v \in \Gamma^{m=0}_{\mathrm{vert}}} (-1)^{|\Gamma^{m\neq 0}_{\mathrm{vert},<v}|} \delta(X_{v,2}) \\ &= -(-1)^{|\Gamma^{m\neq 0}_{\mathrm{vert}}|} \sum_{v \in \Gamma^{m=0}_{\mathrm{vert}}, w \in \Gamma^{m=1}_{\mathrm{vert}}} (-1)^{|\Gamma^{m\neq 0}_{\mathrm{vert},<v}| + |\Gamma^{m=1}_{\mathrm{vert},>w}|} X_{\{v,w\},2} = -\delta d(X) \end{aligned}$$

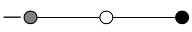
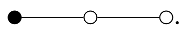
$\square$

We will now give an example. The notation for this and every other image is as follows: Unmarked vertices are drawn as vertices with white center, 1-marked vertices with grey center and 2-marked vertices with black center. In all graphs that are drawn, the ordering is from left to right (if it is relevant), i.e. the leftmost vertex is the first in the order.

Example 1.1.8. *Let* $X :=$ . *Then*

$d(X) = -$ , *and* $\delta(X) = -$ .

Piecing this together gives:

$D(X) = -$  $-$ .

Definition 1.1.9. With these differentials $\mathcal{L}^\Gamma$ can be turned into (3 different) chain complexes. d, δ and D can be restricted to maps

$$d, \delta, D : \mathcal{L}_i^\Gamma \to \mathcal{L}_{i+1}^\Gamma.$$

The resulting chain complexes are called $\mathcal{L}_{d,\bullet}^\Gamma$, $\mathcal{L}_{\delta,\bullet}^\Gamma$ and $\mathcal{L}_{D,\bullet}^\Gamma$ respectively.

We could also define chain complexes $\mathcal{L}_{\partial,\bullet} := \bigoplus_\Gamma \mathcal{L}_{\partial,\bullet}^\Gamma$ for $\partial \in \{d, \delta, D\}$. From a physics perspective, these are more natural objects to study, but because they decompose, all the information is contained in the smaller chain complexes. In any category of R-modules where R is a ring, taking homology commutes with taking direct sums of chain complexes, so $H_i(\mathcal{L}_{\partial,\bullet}) = \bigoplus_\Gamma H_i(\mathcal{L}_{\partial,\bullet}^\Gamma)$. Thus, it suffices to compute $H_i(\mathcal{L}_{\partial,\bullet}^\Gamma)$ for any graph Γ.
There is one property of these chain complexes that we want to mention now. Let the automorphism group of Γ act on the vector space $\mathcal{L}^\Gamma$ in by permuting the markings and leaving the labeling invariant, i.e.

$$\sigma((\Gamma, m)) := (\Gamma, \sigma m)$$

with $(\sigma m)(v) := m(\sigma(v))$. The spaces $\mathcal{L}_{i,j}^\Gamma$ are invariant subspaces, so it is a natural question to ask if this leaves the chain complexes invariant. Clearly, graph automorphisms induce arrows that have the potential to be an isomorphism of chain complexes (because they are isomorphisms), but as it turns out, these arrows do not form a morphism of chain complexes.

Example 1.1.10. *Let* $X := \bullet \quad \circ \quad \bullet$ *and* (12) *be the permutation that exchanges the first and second vertex, i.e.* $(12)X = \circ \quad \bullet \quad \bullet$. *Then* $D(X) = d(X)$ *and it is easy to see that swapping the first two vertices anticommutes with* D*, i.e.* $((12) \circ D)X = -(D \circ (12))X$. *But swapping the first and third vertex clearly commutes with* D.

These sign issues are everything that goes wrong though:

Proposition 1.1.11. *Let* $\partial \in \{d, \delta, D\}$. *There is a natural action of* $\operatorname{Aut}\Gamma$ *on the chain complex* $\mathcal{L}_{\partial,\bullet}^\Gamma \otimes \mathbb{Z}_2$*, i.e. a morphism* $\rho : \operatorname{Aut}\Gamma \to \operatorname{Aut}\mathcal{L}_{\partial,\bullet}^\Gamma \otimes \mathbb{Z}_2$ *such that if* $\alpha \in \operatorname{Aut}\Gamma, X \in \mathcal{L}_{\partial,i}^\Gamma \otimes \mathbb{Z}_2$*, then* $d(\rho(\alpha)(X)) = \rho(\alpha)(d(X))$*, and* α *is an isomorphism of chain complexes.*

Proof. This is obvious from the above. $\square$

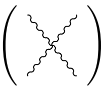
This means that representation theory of $\mathbb{Z}_2$ could be used to study these complexes, but this is out of scope of this thesis. We will later see that $\mathcal{L}^{\Gamma}_{\partial,\bullet} \otimes \mathbb{Z}_2$ is in many other ways better behaved than $\mathcal{L}^{\Gamma}_{\partial,\bullet}$
There are several other general statements that could be proven for all these chain complexes. But as will be shown in this chapter, the homologies of $\mathcal{L}^{\Gamma}_{D,\bullet}$ and $\mathcal{L}^{\Gamma}_{\delta,\bullet}$ do not depend on Γ and are 0 except in degree 0. This means that all these statements become irrelevant for these chain complexes. Because some of these statements are easier stated or proven in the special case of $\mathcal{L}^{\Gamma}_{d,\bullet}$ and because there are some dependencies in these statements that made it difficult to separate these, all such results can be found in chapter 2.

1.2 Connection to Feynman diagrams

The Feynman diagrams under consideration in this thesis consist of gluons and ghosts only, which means that there are two types of edges (ghosts and gluons) and three types of vertices:

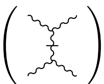
- 3-valent vertices with only gluon edges
- 4-valent vertices with only gluon edges
- 3-valent vertices that have two ghost edges and a gluon edge.

Ghosts will be relevant later in this section. For now, we only care about gluons. Because of the Feynman rules of the 3- and 4-gluon vertices, it is possible to replace a 4-gluon vertex by a sum of 3 3-gluon vertices. This is described in [Kre13] and repeated here. Let Φ denote the Feynman rules of a diagram, then the Feynman rules of the 4-gluon vertex are as follows:



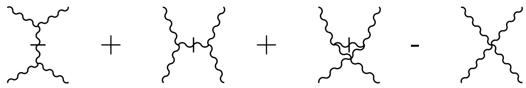
$$\begin{aligned}\Phi\left(\quad\right) &= f^{a_1a_2b}f^{a_3a_4b}(g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3}) \\ &+ f^{a_1a_3b}f^{a_2a_4b}(g^{\mu_1\mu_2}g^{\mu_3\mu_4} - g^{\mu_1\mu_4}g^{\mu_3\mu_2}) \\ &+ f^{a_1a_4b}f^{a_2a_3b}(g^{\mu_1\mu_2}g^{\mu_4\mu_3} - g^{\mu_1\mu_3}g^{\mu_4\mu_2}),\end{aligned}$$

where f^{abc} is defined by the relation $[X^a, X^b] = X^c f^{abc}$ with X^i the generators of the Lie algebra of SU(3), and g is a metric. Note that the second term is the same as the first term, just with 2 and 3 exchanged, and the third term is the same as the second term just with 3 and 4 exchanged. This means it is natural to define a new kind of edge, an edge with a marking, with the following property:



$$\Phi\left(\quad\right) = f^{a_1a_2b}f^{a_3a_4b}(g^{\mu_1\mu_3}g^{\mu_2\mu_4} - g^{\mu_1\mu_4}g^{\mu_2\mu_3})$$

Using this, the following sum of Feynman diagrams lies in the kernel of the map Φ:



As one is mainly interested in the Feynman rules, it is natural to take the quotient by the kernel of Φ. In this quotient, every element can be uniquely represented as a linear combination of diagrams where every vertex is 3-valent and every edge is a gluon edge or a marked gluon edge.

Definition 1.2.1. Let $\mathcal{G}$ be the free abelian group generated by graphs with marked edges with 1- or 3-valent vertices modulo the subgroup that is generated by all graphs with the property that there exist two marked subgraphs that share a vertex or an edge that connects to a 1-valent vertex is marked. If Γ is a graph, let $\mathcal{G}^\Gamma$ be the subgroup generated by all graphs that have Γ as their skeleton, and $\mathcal{G}_i^\Gamma$ be the subgroup of $\mathcal{G}^\Gamma$ that is spanned by all graphs with i double markings.

These graphs correspond to Feynman graphs by replacing every edge by a gluon edge and every marked edge by a marked gluon edge. There are also 4-gluon vertices, but one can write a 4-gluon vertex as the sum of graphs having only 3-gluon vertices as explained previously. Applying this to every 4-gluon vertex leads to a Feynman graph with only 3-vertices with markings, so one can instead consider these graphs. Note, that there cannot be two marked edges that are connected to the same vertex. Also note that exterior legs do not contribute to anything, because they cannot be marked, so edges that connect to 1-valent vertices also cannot be marked.
In physics, there are two ways of filtering different graphs: First, by exterior leg structure, which corresponds to the kind of experiment that might be done (n particles go in, m particles go out, etc.) and second by their *loop order*, i.e. their first Betti number (in regular graph homology). This is motivated as follows: In quantum field theory, Feynman diagrams appear as the result of a Taylor expansion of the so called *partition function*, which might look like this:

$$Z = \sum \hbar^n \alpha_n X_n$$

The α_n are just some coefficients, $\hbar = \frac{h}{2\pi}$, where h is Planck's constant, and X_n is a sum of Feynman diagrams. As it turns out, X_n is the sum of all Feynman graphs in the theory that have loop order n.
Because $\hbar$ is small, Feynman graphs of higher loop order contribute less to the amplitude of some process happening. So, a very natural object of study is the sum of all graphs consisting of with r exterior legs and loop order n. We call this object X_n^r.
As described previously, in the case of gluons (which is the only case that we consider in this thesis) we can write this object as the sum of all graphs that have 3-valent vertices and have gluon or marked gluon edges, where two marked gluon edges never sit right next to each other. This sum can be described using the following operator:

Definition 1.2.2. Let
$$\chi_+ : \mathcal{L}^\Gamma \to \mathcal{L}^\Gamma, X \mapsto \sum_{e \in \Gamma_{\text{edge}}} X_{e,1}.$$
Note that $X_{1,e} = 0$ if X has a marked edge e' such that e and e' connect to the same vertex.

By exponentiating this operator, and applying it to the unmarked graphs, one gets the amplitude we are looking for:

Theorem 1.2.3. *Let $\widetilde{X}_n^r$ be the sum of all 3-valent Feynman graphs that have n as their first Betti number (i.e. n independent cycles), r external edges. Then,*
$$e^{\chi_+} \widetilde{X}_n^r = X_n^r.$$

Proof. See [Kre13], Lemma 4.10. □

The main result of this thesis is that this object is the only non-trivial element of the homology of the chain complex that will now be defined.

Definition 1.2.4. Let $\Gamma \in \mathcal{G}$ be a graph, Γ_{edge} be the set of its edges, and $<$ be a well-ordering of Γ_{edge}. Then its *combinatorial BRST complex* is the chain complex
$$\cdots \longrightarrow \mathcal{G}_i^\Gamma \xrightarrow{S} \mathcal{G}_{i+1}^\Gamma \longrightarrow \cdots,$$
with $S = s + \sigma$ where s and σ are defined on the canonical basis as follows:
$$s : \mathcal{L}^\Gamma \to \mathcal{L}^\Gamma, \quad X \mapsto \sum_{e \in \Gamma_{\text{edge}}^{m=0}} (-1)^{|\Gamma_{\text{edge},<e}^{m\neq 0}|} X_{e,2}$$
$$\sigma : \mathcal{L}^\Gamma \to \mathcal{L}^\Gamma, \quad X \mapsto (-1)^{|\Gamma_{\text{edge}}^{m\neq 0}|} \sum_{e \in \Gamma_{\text{edge}}^{m=1}} (-1)^{|\Gamma_{\text{edge},>e}^{m=1}|} X_{e,2}$$

To see that S is in fact a differential, see Proposition 1.1.7. This differential seems to be related to the differential ∂_E from [CV03], but it is not clear at this point what the connection is.
There is one caveat that has to be addressed here. Feynman diagrams made of gluons are invariant under symmetry, i.e. if α is an automorphism of Γ, the Feynman diagrams of Γ and $\alpha\Gamma$ are considered equal. But the previously defined chain complex is not invariant under this symmetry, because the order of the edges might change signs, see Example 1.1.10 and the following proposition on how this can be fixed.
There is also another kind of marking that behaves in a similar way. Instead of marking edges, now loops will be marked. This corresponds to ghost loops in the theory, and for this, a different set of groups is required:

Definition 1.2.5. Let $\mathcal{C}$ be the free abelian group generated by graphs with marked cycles with 1- or 3-valent vertices modulo the subgroup that is generated by all graphs with the property that two marked cycles share a vertex, and $\mathcal{C}_i^\Gamma$ similarly defined as before. Also, let $<$ be a well-ordering of Γ_{cycle}. Then the *combinatorial ghost cycle complex* of Γ is the chain complex

$$\cdots \longrightarrow \mathcal{C}_i^\Gamma \xrightarrow{T} \mathcal{C}_{i+1}^\Gamma \longrightarrow \cdots,$$

with $T = t + \tau$ where t and τ are defined on the canonical basis as follows:

$$t : \mathcal{L}^\Gamma \to \mathcal{L}^\Gamma, \quad X \mapsto \sum_{c \in \Gamma_{\text{cycle}}^{m=0}} (-1)^{|\Gamma_{\text{cycle},<c}^{m\neq 0}|} X_{c,2}$$

$$\tau : \mathcal{L}^\Gamma \to \mathcal{L}^\Gamma, \quad X \mapsto (-1)^{|\Gamma_{\text{cycle}}^{m\neq 0}|} \sum_{c \in \Gamma_{\text{cycle}}^{m=1}} (-1)^{|\Gamma_{\text{cycle},>c}^{m=1}|} X_{c,2}$$

Just by observation, s and t are very similar to the previously defined d, and the same holds for σ and τ with the previously defined δ. We will now show that the previous chain complex of marked graphs contains all the information that these chain complexes do.

Definition 1.2.6. Let Γ be a graph. Then its *line graph* is the graph that has a vertex for every edge in Γ and two vertices are adjacent if the corresponding edges share a vertex.

Proposition 1.2.7. *Let Γ be a graph and Γ' be its line graph. Then for $\partial \in \{d, \delta, D\}$,*

$$\mathcal{G}_{\partial,\bullet}^\Gamma \simeq \mathcal{L}_{\partial,\bullet}^{\Gamma'}.$$

Proof. If e is an edge in Γ, let v_e be the corresponding vertex in Γ'. The isomorphism is given by the map that sends a marked graph (Γ, m) to the marked graph (Γ', m') with $m'(v_e) = m(e)$. This map is clearly bijective, so the induced morphism of groups $\varphi : \mathcal{G}^\Gamma \to \mathcal{L}^{\Gamma'}$ is an isomorphism. By construction of the two differentials, φ clearly commutes with them. □

Proposition 1.2.8. *Let Γ be a graph and Γ' be the graph that has a vertex for every cycle in Γ and an edge between two vertices if and only if the corresponding cycles have a common vertex. Then for $\partial \in \{d, \delta, D\}$,*

$$\mathcal{C}_{\partial,\bullet}^\Gamma \simeq \mathcal{L}_{\partial,\bullet}^{\Gamma'}.$$

Proof. This proof is almost the same as the previous one, if c is a cycle in Γ, the corresponding vertex in Γ' is denoted v_c. Now we define $m'(v_c) := m(c)$ and the rest follows. □

1.3 The homology of $\mathcal{L}^{\Gamma}_{D,\bullet}$

Because δ does in no way interact with the structure of markings (it only marks vertices that have already been marked), one can reduce the chain complex $\mathcal{L}^{\Gamma}_{\delta,\bullet}$ to a very simple chain complex which has the property that it is isomorphic to the reduced simplicial complex of a simplex. This chain complex has trivial homology and thus the homology of $\mathcal{L}^{\Gamma}_{\delta,\bullet}$ can be computed for all Γ. First, we need to define the smaller complex:

Definition 1.3.1. Let $\widetilde{\mathcal{L}}^{\Gamma}_{\delta,\bullet}$ be the subcomplex of $\mathcal{L}^{\Gamma}_{\delta,\bullet}$ that is generated by all markings where every vertex is marked. If Γ is the empty graph, define

$$\widetilde{\mathcal{L}}^{\Gamma}_{\delta,k} := \begin{cases} \mathbb{Z} & \text{if k} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that, if Γ is not discrete, i.e. if Γ has an edge, this complex is 0 everywhere.

Theorem 1.3.2. *Let $\{X_k\}_{k\in I}$ be the canonical basis of $\mathcal{L}^{\Gamma}_{0}$, and Γ_{X_k} be the subgraph of Γ that contains only the vertices that are marked in X_k. Then,*

$$\mathcal{L}^{\Gamma}_{\delta,\bullet} \simeq \bigoplus_{k\in I} \widetilde{\mathcal{L}}^{\Gamma_{X_k}}_{\delta,\bullet}.$$

Proof. Because δ cannot change a 0-marking to some other marking, $\mathcal{L}^{\Gamma}_{\delta,\bullet}$ can be decomposed into the subcomplexes that have markings only where certain vertices are marked. These subcomplexes are by construction isomorphic to $\widetilde{\mathcal{L}}^{\Gamma_k}_{\delta,\bullet}$ for some $k \in I$, and every k appears exactly once in this decomposition. Note that the unmarked graph $\Gamma \in \mathcal{L}^{\Gamma}_{\delta,0}$ is contained in $\widetilde{\mathcal{L}}^{\emptyset}_{\delta,\bullet}$ by construction. Putting everything together yields the following isomorphism of chain complexes

$$\mathcal{L}^{\Gamma}_{\delta,\bullet} \simeq \bigoplus_{k\in I} \widetilde{\mathcal{L}}^{\Gamma_k}_{\delta,\bullet}.$$

□

Notice that the Γ_{X_k} in the previous theorem are discrete graphs because of the requirements put on markings.

Theorem 1.3.3. *Let Γ_n be the discrete graph with $n > 0$ vertices. Then*

$$H_i(\widetilde{\mathcal{L}}^{\Gamma_n}_{\delta,\bullet}) = 0.$$

Proof. We show that the chain complex $\widetilde{\mathcal{L}}^{\Gamma_n}_{\delta,\bullet}$ is isomorphic to the augmented simplicial complex of the $n-1$ simplex. Let $\Delta_{n-1} = [e_1, \cdots, e_n]$ be the $n-1$ simplex embedded in $\mathbb{R}^n$, v_k the k-th vertex of Γ_n, and consider the map φ that sends an element X of the canonical basis of $\mathcal{L}^{\Gamma}_{i,j}$ to $[\{e_k : v_k \text{ is not } 1-\text{marked in } X\}]$, where $[\cdot]$ denotes the convex hull. Note that for the fully marked graph, φ would evaluate to zero in the ordinary simplicial chain complex of Δ_{n-1}, but we want φ to be an isomorphism, so we consider the augmented chain complex of Δ_{n-1}, which has the empty face of the simplex as generator in degree -1 and $\partial([e_i]) = \emptyset \neq 0$. Now, let X be a fixed element of the canonical basis of $\mathcal{L}_{\delta,j}^{\Gamma_n}$, and $\varphi(X) = [e_{a_1}, \cdots, e_{a_j}]$. Then,

$$\partial(\varphi(X)) = \sum_{k=1}^{n}(-1)^k[e_{a_1}, \cdots, \widehat{e_{a_k}}, \cdots, e_{a_j}] = \varphi\left((-1)^n\sum_{k=1}^{n}(-1)^{n-k}(X)_{v_{a_k},2}\right) = \varphi(\delta(X)),$$

so φ induces a morphism of chain complexes which is a bijection on the generators, so φ is an isomorphism. But Δ_{n-1} is contractible, so all its reduced homologies vanish, see for example [Hat02], page 111. □

Corollary 1.3.4. *For every graph* Γ*, we have*

$$H_i(\mathcal{L}_{\delta,\bullet}^{\Gamma}) = \begin{cases}\mathbb{Z} & \textit{for } i = 0\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. We only have to show that

$$H_i(\widetilde{\mathcal{L}}_{\delta,\bullet}^{\Gamma_0}) = \begin{cases}\mathbb{Z} & \text{for } i = 0\\ 0 & \text{otherwise,}\end{cases}$$

but this is obvious from the definition. □

Now, we have everything required to compute the homology of $\mathcal{L}_{D,\bullet}^{\Gamma}$. It is possible to do this directly, but the next theorem is more general and gives us the desired result right away.

Theorem 1.3.5. *Let $\mathcal{C}_i$ be $\mathbb{N}$-graded modules over a fixed ring and $d_i, \delta_i : \mathcal{C}_i \to \mathcal{C}_{i+1}$ such that $d^2 = \delta^2 = 0$ and $d\delta + \delta d = 0$, i.e. such that d, δ and $D := d + \delta$ are differentials that turn $\mathcal{C}_\bullet$ into chain complexes and $\deg d_i = 0, \deg \delta_i = -1$. Assume that $H_i(\mathcal{C}_{\delta,\bullet}) = 0$ for every $i \geq 0$, and that $H_0(\mathcal{C}_{\delta,\bullet})$ is of pure degree 0. Then $H_i(\mathcal{C}_{D,\bullet}) \simeq H_i(\mathcal{C}_{\delta,\bullet})$ as ungraded modules.*

Proof. For any $X \in \mathcal{C}_i$, let X_k be the part of pure degree k. We construct a map $(H_i(\mathcal{C}_{\delta,\bullet}))_0 \to H_i(\mathcal{C}_{D,\bullet})$ as follows: Take any $X_0 \in \ker \delta_i$ of degree 0, then $\delta(X_0) = 0$. Now, because d and δ anticommute, $\delta d(X_0) = 0$, so there exists an X_1 of degree 1 such that $\delta(X_1) = d(X_0)$ (because $H_{i+1}(\mathcal{C}_{\delta,\bullet}) = 0$). This can be continued inductively to construct an element X with $D(X) = 0$ which means we can construct a map $\widetilde{\varphi}_i$: $(\ker \delta_i)_0 \to \ker D_i$ such that $\widetilde{\varphi}_i(X)_0 = X$, and composing with the natural projection gives a map $\varphi_i : (\ker \delta_i)_0 \to H_i(\mathcal{C}_{D,\bullet})$. This map is surjective, because given $X \in H_i(\mathcal{C}_{D,\bullet})$, we can first subtract $\varphi_i(X_0)$ to get $X_0 = 0$, so it suffices to show that it is surjective on all elements with $X_0 = 0$. Now, we show that for all $X \in \ker D$, $X_0 \in \operatorname{im} \delta_{i-1}$ implies that $X \in \operatorname{im} D_{i-1}$, from which follows the surjectivity of this map and also $(\operatorname{im} \delta_{i-1})_0 \subseteq \ker \varphi_i$.
Let $X \in \ker D$, i.e. $d(X_k) = -\delta(X_{k+1})$ and such that $X_0 = \delta(Y_1)$. We want to construct

an element $Y = \sum Y_k$ such that $D(Y) = X$. Choose $Y_0 = 0$, then $X_0 = \delta(Y_1) = d(Y_0) + \delta(Y_1)$. Now assume that $X_n = d(Y_n) + \delta(Y_{n+1})$, then

$$\delta(X_{n+1}) = -d(X_n) = -dd(Y_n) - d\delta(Y_{n+1}) = \delta d(Y_{n+1}),$$

so $\delta(X_{n+1} - d(Y_{n+1})) = 0$ and thus there exists a Y_{n+2} such that $\delta(Y_{n+2}) = X_{n+1} - d(Y_{n+1})$, or $X_{n+1} = d(Y_{n+1}) + \delta(Y_{n+2})$. This inductively chosen Y satisfies $D(Y) = X$. Next, we claim that $(\operatorname{im}\delta_{i-1})_0 = \ker\varphi_i$. One inclusion was already shown. For the other inclusion, let $X_0 \in \ker\varphi_i$, then $X_0 = d_{i-1}(Y_0) + \delta_{i-1}(Y_1)$. In the case that $i = 0$, we have $X_0 = 0$, and in all other cases, $\delta(X_0) = 0$ so $X_0 = \delta_{i-1}(Z_1)$ because $H_i(\mathcal{C}_{\delta,\bullet}) = 0$. This means that φ_i induces an isomorphism $(H_i(\mathcal{C}_{D,\bullet}))_0 \simeq H_i(\mathcal{C}_{\delta,\bullet})$, and because all the homologies are concentrated in degree 0, the claim follows. $\square$

Corollary 1.3.6. *Let X be an element of the canonical basis and*

$$\chi(X) := \sum_{v\in\Gamma^{m=0}_{vert}} X_{v,1}.$$

Then, if Γ is any graph, $H_0(\mathcal{L}^{\Gamma}_{D,\bullet}) = \langle e^{\chi}\Gamma_{S,1}\rangle$ and all the other homologies vanish.

Proof. It suffices to show that $D(e^{\chi}\Gamma) = 0$. This was proven in [Kre13] on page 202 and is repeated here for convenience. First, note that

$$e^{\chi}\Gamma = \sum_{n\in\mathbb{N}} \sum_{v_1<\cdots<v_n\in\Gamma^{m=0}_{vert}} X_{\{v_1,\cdots,v_n\},1}.$$

Then:

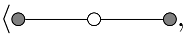
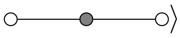
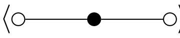
$$\begin{aligned}
\delta(e^{\chi}\Gamma) &= \sum_{n\in\mathbb{N}} \sum_{v_1<\cdots<v_n\in\Gamma^{m=0}_{vert}} \delta(X_{\{v_1,\cdots,v_n\},1}) \\
&= \sum_{n>0} \sum_{v_1<\cdots<v_n\in\Gamma^{m=0}_{vert}} (-1)^n \sum_{i=1}^{n} (-1)^{n-i} (X_{\{v_1,\cdots,v_n\},1})_{v_i,2} \\
&= \sum_{n>0} \sum_{v_1<\cdots<v_n\in\Gamma^{m=0}_{vert}} \sum_{i=1}^{n} (-1)^{i} (X_{\{v_1,\cdots,v_n\},1})_{v_i,2} \\
d(e^{\chi}\Gamma) &= \sum_{n\in\mathbb{N}} \sum_{v_1<\cdots<v_n\in\Gamma^{m=0}_{vert}} d(X_{\{v_1,\cdots,v_n\},1}) \\
&= \sum_{n\in\mathbb{N}} \sum_{v_1<\cdots<v_n<v_{n+1}\in\Gamma^{m=0}_{vert}} \sum_{i=1}^{n+1} (-1)^{i-1} (X_{\{v_1,\cdots,v_n,v_{n+1}\},1})_{v_i,2}
\end{aligned}$$

Shifting n by one reveals that these two expressions are equal up to a relative sign. $\square$

Note that the map χ from the previous corollary corresponds to the map χ_+ from Definition 1.2.2.

2 The chain complex $\mathcal{L}^{\Gamma}_{d,\bullet}$

It is not true, that $H_i(\mathcal{L}^{\Gamma}_{d,\bullet}) = 0$ in general. In fact, this chain complex is rarely zero and its homology is related to computationally hard problems in graph theory. For example, let Γ be the tree with a root and two leaves. Then the marking with the root 2-marked lies in the kernel of d, but not in the image. There is also a homology class with the root 1-marked, and one with both leaves singly marked, so

$H_0(\mathcal{L}^{\Gamma}_{d,\bullet}) \simeq \langle$ ,  $\rangle$, $\quad H_1(\mathcal{L}^{\Gamma}_{d,\bullet}) \simeq \langle$  $\rangle$.

There are two ways of approaching the differential d, because of a symmetry in graph theory relating independent sets with cliques.

2.1 General properties

This section is about properties of $\mathcal{L}^{\Gamma}_{d,\bullet}$ and its complementary partner $\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}$ that have a good formulation in both cases. We first need some graph-theoretic definitions.

Definition 2.1.1. Let Γ be a graph. An *independent set* is a subset S of the set of vertices of Γ such that for all $x, y \in S$, x and y are not connected by an edge. An independent set is called *maximal* if there is no independent set S' such that $S \subsetneq S'$.

Definition 2.1.2. If Γ is a graph, then let $\bar{\Gamma}$ be the graph that has the same vertices as Γ, but an edge between two vertices v and v' if and only if there is no edge between them in Γ. $\bar{\Gamma}$ is called the *complement graph* of Γ.

Independent sets in a graph correspond to cliques in the complement.

Definition 2.1.3. Let Γ be a graph. Then a *clique* in Γ is a subset of the set of vertices such that every pair of vertices in this subset has an edge between them. Similarly to independent sets, a clique C is called *maximal* if there is no clique C' such that $C \subsetneq C'$.

Definition 2.1.4. Let $\bar{\mathcal{L}}^{\Gamma}$ be the abelian group generated by markings that are contained in one clique and $\bar{\mathcal{L}}^{\Gamma}_i$ as usual, and let

$$\bar{d} : \bar{\mathcal{L}}^{\Gamma} \to \bar{\mathcal{L}}^{\Gamma}, \quad X \mapsto \sum_{v \in \Gamma^{m=0}_{\mathrm{vert}}} (-1)^{|\Gamma^{m \neq 0}_{\mathrm{vert},<v}|} X_{v,2},$$

where $X_{v,2}$ is interpreted to be 0 if it is not in $\bar{\mathcal{L}}^{\Gamma}$.

Proposition 2.1.5. *For any graph Γ,*

$$\bar{\mathcal{L}}^{\bar{\Gamma}}_{\bar{d},\bullet} \simeq \mathcal{L}^{\Gamma}_{d,\bullet}.$$

Proof. This is obvious from the definitions. $\square$

This chain complex gives us the advantage of switching the point of view, which is very helpful in some cases. Also, because in small graphs, humans have an easier time picking up cliques instead of independent sets, it is usually somewhat easier to compute the homology of $\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}$ by hand.

Definition 2.1.6. Let Γ, Γ' be graphs, then a *morphism* $\varphi : \Gamma \to \Gamma'$ is a map $\varphi_{\text{vert}} : \Gamma_{\text{vert}} \to \Gamma'_{\text{vert}}$ together with a map $\varphi_{\text{edge}} : \Gamma_{\text{edge}} \to \Gamma'_{\text{edge}}$ (usually both written as φ) such that if $v, w \in \Gamma_{\text{vert}}$ are connected by an edge e, $\varphi(v)$ and $\varphi(w)$ are connected by $\varphi(e)$. A morphism of graphs is an *embedding* if it is injective on vertices and edges. An embedding is called *full* if it induces bijections between the edges connecting v and w and the edges connecting $\varphi(v)$ and $\varphi(w)$.

Proposition 2.1.7. *Let $\varphi : \Gamma' \to \Gamma$ be an embedding of graphs and $|\Gamma_{\text{vert}}| = |\Gamma'_{\text{vert}}|$. Then there exist morphisms of chain complexes*

$$\mathcal{L}^{\varphi,1}_{d,\bullet} : \mathcal{L}^{\Gamma'}_{d,\bullet} \to \mathcal{L}^{\Gamma}_{d,\bullet}$$
$$\bar{\mathcal{L}}^{\varphi,1}_{\bar{d},\bullet} : \bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet} \to \bar{\mathcal{L}}^{\Gamma'}_{\bar{d},\bullet}$$

that turn $\mathcal{L}^{-}_{d,\bullet}$ and $\bar{\mathcal{L}}^{-}_{\bar{d},\bullet}$ into covariant and contravariant functors respectively. $\mathcal{L}^{\varphi,1}_{d,\bullet}$ and $\bar{\mathcal{L}}^{\varphi,1}_{\bar{d},\bullet}$ are epimorphisms.

Proof. On the canonical basis, we define

$$\mathcal{L}^{\varphi,1}_{d,\bullet}((\Gamma', m)) := \begin{cases} (\Gamma, m) & \text{if } (\Gamma, m) \in \mathcal{L}^{\Gamma}_{d,\bullet} \\ 0 & \text{otherwise.} \end{cases}$$

This is clearly a morphism of chain complexes and this construction is functorial. The reason that $|\Gamma_{\text{vert}}| = |\Gamma'_{\text{vert}}|$ is required is that otherwise, there might be more allowed markings in Γ than in Γ' which would make d not commute with $\mathcal{L}^{\varphi,1}_{d,\bullet}$. $\bar{\mathcal{L}}^{\varphi,1}_{\bar{d},\bullet}$ is defined similarly, but because dualizing graphs inverts the direction of φ (if they have the same amount of vertices), i.e. there is an embedding $\bar{\varphi} : \bar{\Gamma} \to \bar{\Gamma}'$, this functor is contravariant. That both maps are epimorphisms is clear from the construction. $\square$

Proposition 2.1.8. *Let $\varphi : \Gamma' \to \Gamma$ be a full embedding. Then there exist morphisms of chain complexes*

$$\mathcal{L}^{\varphi,2}_{d,\bullet} : \mathcal{L}^{\Gamma}_{d,\bullet} \to \mathcal{L}^{\Gamma'}_{d,\bullet}$$
$$\bar{\mathcal{L}}^{\varphi,2}_{\bar{d},\bullet} : \bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet} \to \bar{\mathcal{L}}^{\Gamma'}_{\bar{d},\bullet}$$

that turn $\mathcal{L}^{-}_{d,\bullet}$ and $\bar{\mathcal{L}}^{-}_{\bar{d},\bullet}$ into contravariant functors. $\mathcal{L}^{\varphi,2}_{d,\bullet}$ and $\bar{\mathcal{L}}^{\varphi,2}_{\bar{d},\bullet}$ are epimorphisms.

Proof. Let $\mathcal{L}^{\varphi,2}_{d,\bullet} : \mathcal{L}^{\Gamma} \to \mathcal{L}^{\Gamma'}$ be the map that sends every marking of Γ where every vertex outside of Γ' is unmarked to the corresponding marking of Γ', and everything else to 0. Clearly, $\mathcal{L}^{\varphi,2}_{d,\bullet}(X)$ has the same number of markings as X, so $\mathcal{L}^{\varphi,2}_{d,\bullet}$ induces maps

$\mathcal{L}^{\Gamma}_{d,\bullet} \to \mathcal{L}^{\Gamma'}_{d,\bullet}$. To distinguish the differentials, we call them d_Γ and $d_{\Gamma'}$ for the rest of this proof. Then:

$$\begin{aligned}\mathcal{L}^{\varphi,2}_{d,\bullet}(d_\Gamma(X)) &= \sum_{v\in\Gamma_{\mathrm{vert}}} (-1)^{|\Gamma^{m\neq 0}_{\mathrm{vert},<v}|}\mathcal{L}^{\varphi,2}_{d,\bullet}(X_{v,2}) \\ &= \sum_{v\in\Gamma'_{\mathrm{vert}}} (-1)^{|\Gamma'^{m'\neq 0}_{\mathrm{vert},<v}|}(\mathcal{L}^{\varphi,2}_{d,\bullet}(X))_{v,2} = d_{\Gamma'}(\mathcal{L}^{\varphi,2}_{d,\bullet}(X)).\end{aligned}$$

The signs agree because all the marked vertices lie in Γ' already. Fullness of the embedding is required for changing the summation, if the embedding was not full there could be more summands on the right than on the left. That $\mathcal{L}^{\varphi,2}_{d,\bullet}$ is an epimorphism also follows from the fullness.
The exact same construction works also for $\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}$. □

Corollary 2.1.9. *There is a contravariant functor $\bar{\mathcal{L}}^{-}_{\bar{d},\bullet}$ from the category of graphs with embeddings to the category of chain complexes of abelian groups.*

Proof. Every embedding of a graph can be decomposed into a full embedding and an embedding between two graphs that have the same number of vertices. This can be used to combine the two contravariant functors from the previous propositions to a single functor. □

We will now prove that the homologies of $\mathcal{L}^{\Gamma}_{d,\bullet}$ are free, which will be used many times in the rest of this thesis. We need the following lemma:

Lemma 2.1.10. *Let Γ_n be the graph with n vertices and no edges. Then*

$$H_i(\mathcal{L}^{\Gamma_n}_{d,\bullet}) = 0.$$

Proof. We prove this by induction on n. For $n = 1$ the claim is trivially verified. Now, let $\varphi : \Gamma_{n-1} \to \Gamma_n$ be the (full) embedding of the first $n-1$ vertices, and consider the epimorphism $\mathcal{L}^{\varphi,2}_{d,\bullet} : \mathcal{L}^{\Gamma_n}_{d,\bullet} \to \mathcal{L}^{\Gamma_{n-1}}_{d,\bullet}$, which gives us a short exact sequence of chain complexes

$$0 \to \ker \mathcal{L}^{\varphi,2}_{d,\bullet} \to \mathcal{L}^{\Gamma_n}_{d,\bullet} \to \mathcal{L}^{\Gamma_{n-1}}_{d,\bullet} \to 0$$

which induces a long exact sequence

$$\cdots \to H_i(\ker \mathcal{L}^{\varphi,2}_{d,\bullet}) \to H_i(\mathcal{L}^{\Gamma_n}_{d,\bullet}) \to H_i(\mathcal{L}^{\Gamma_{n-1}}_{d,\bullet}) \to H_{i+1}(\ker \mathcal{L}^{\varphi,2}_{d,\bullet}) \to \cdots$$

By induction hypothesis, $H_i(\mathcal{L}^{\Gamma_{n-1}}_{d,\bullet}) = 0$, so

$$H_i(\mathcal{L}^{\Gamma_n}_{d,\bullet}) \simeq H_i(\ker \mathcal{L}^{\varphi,2}_{d,\bullet}).$$

But $\ker \mathcal{L}^{\varphi,2}_{d,\bullet} \simeq \mathcal{L}^{\Gamma_{n-1}}_{d,\bullet}[-1]$ (where $[-1]$ means a shift by -1), because all markings that have the n-th vertex marked lie in $\ker \mathcal{L}^{\varphi,2}_{d,\bullet}$, so the claim follows by induction. □

Theorem 2.1.11. *$H_i(\mathcal{L}^{\Gamma}_{d,\bullet})$ is free.*

Proof. By [Lan02], I Theorem 8.4, every finitely generated torsion-free abelian group is free, so because $\ker d_i$ is free, it suffices to show that if $X \notin \operatorname{im} d_{i-1}$, we have $kX \notin \operatorname{im} d_{i-1}$ for all $k \in \mathbb{Z}$. We use the embedding $\varphi : \Gamma_n \to \Gamma$ for some n. Then there is a short exact sequence

$$0 \to \ker \mathcal{L}^{\varphi}_{d,\bullet} \to \mathcal{L}^{\Gamma_n}_{d,\bullet} \xrightarrow{\mathcal{L}^{\varphi}_{d,\bullet}} \mathcal{L}^{\Gamma}_{d,\bullet} \to 0$$

which induces a long exact sequence in homology:

$$\cdots \to H_i(\ker \mathcal{L}^{\varphi}_{d,\bullet}) \to H_i(\mathcal{L}^{\Gamma_n}_{d,\bullet}) \to H_i(\mathcal{L}^{\Gamma}_{d,\bullet}) \to H_{i+1}(\ker \mathcal{L}^{\varphi}_{d,\bullet}) \to \cdots$$

By the previous proposition, $H_i(\mathcal{L}^{\Gamma_n}_{d,\bullet}) = 0$, so there are isomorphisms

$$H_i(\mathcal{L}^{\Gamma}_{d,\bullet}) \simeq H_{i+1}(\ker \mathcal{L}^{\varphi}_{d,\bullet}).$$

Now, $\mathcal{L}^{\Gamma_n}_{d,\bullet} \simeq \ker \mathcal{L}^{\varphi}_{d,\bullet} \oplus \mathcal{C}_\bullet$ for some chain complex $\mathcal{C}_\bullet$ because the differential leaves the property of being in the kernel of $\mathcal{L}^{\varphi}_{d,\bullet}$ invariant. So if $X \in \ker \mathcal{L}^{\varphi}_{d,i}$ with $kX = d(Y)$, by the previous proposition $X = d(Z)$ in $\mathcal{L}^{\Gamma_n}_{d,\bullet}$. But then Z also lives in $\ker \mathcal{L}^{\varphi}_{d,\bullet}$. □

Next, we turn to the general issue of different labelings. Even though the chain complexes for different labelings might not be isomorphic, their homologies are:

Theorem 2.1.12. *Let Γ and Γ' be two isomorphic graphs where the isomorphism does not necessarily preserve the ordering on the vertices. Then*

$$H_i(\mathcal{L}^{\Gamma}_{d,\bullet}) \simeq H_i(\mathcal{L}^{\Gamma'}_{d,\bullet}).$$

Proof. We have $\mathcal{L}^{\Gamma}_{d,\bullet} \otimes \mathbb{Z}_2 \simeq \mathcal{L}^{\Gamma'}_{d,\bullet} \otimes \mathbb{Z}_2$, so their homologies are also the same. All of the homology groups are free, and

$$\operatorname{rk} H_i(\mathcal{L}^{\Gamma}_{d,\bullet}) = \dim H_i(\mathcal{L}^{\Gamma}_{d,\bullet}) \otimes \mathbb{Z}_2 = \dim H_i(\mathcal{L}^{\Gamma'}_{d,\bullet}) \otimes \mathbb{Z}_2 = \operatorname{rk} H_i(\mathcal{L}^{\Gamma'}_{d,\bullet}).$$

□

Because of this theorem, we will not worry too much about any particular labelings. In case a special choice of labeling is required, this choice will be made, and because of the previous result this choice of labeling will not impact the generality of the result on the level of the homologies.

Proposition 2.1.13. *Let $\mathcal{C}_\bullet$ be a chain complex of vector spaces with finitely many non-zero terms. Then*

$$\sum_i (-1)^i \dim \mathcal{C}_i = \sum_i (-1)^i \dim H_i(\mathcal{C}_\bullet).$$

Proof. See [Lan02], XX Theorem 3.1. □

Using this, there is one result on the homologies of $\mathcal{L}^{\Gamma}_{d,\bullet}$ that one gets for free:

Theorem 2.1.14. *Let Γ be any graph, then*

$$\sum_i (-1)^i \operatorname{rk} H_i(\mathcal{L}^{\Gamma}_{d,\bullet}) = 1.$$

Proof. By the previous proposition, we have:

$$\sum_i (-1)^i \dim H_i(\mathcal{L}^{\Gamma}_{d,\bullet} \otimes \mathbb{Z}_2) = \sum_i (-1)^i \dim \mathcal{L}^{\Gamma}_i \otimes \mathbb{Z}_2 = \sum_i (-1)^i \dim H_i(\mathcal{L}^{\Gamma}_{D,\bullet} \otimes \mathbb{Z}_2) = 1,$$

because the homology of $\mathcal{L}^{\Gamma}_{D,\bullet}$ is $\mathbb{Z}$ in degree 0 and 0 otherwise. Because $\operatorname{rk} H_i(\mathcal{L}^{\Gamma}_{d,\bullet}) = \dim H_i(\mathcal{L}^{\Gamma}_{d,\bullet})$, the claim follows. □

There is a connection of the mathematically more natural chain complex that is generated by all the markings that have no 1-markings. This chain complex is computationally a lot easier because the vector space of all possible markings is $2^{|\Gamma|}$ dimensional, instead of $3^{|\Gamma|}$ dimensional, but to compensate for this, one has to find all independent sets of the graph, which is also a hard problem (see [BM11]).

Theorem 2.1.15. *Let $\mathcal{L}^{\Gamma}_{d,1,\bullet}$ be the subcomplex of $\mathcal{L}^{\Gamma}_{d,\bullet}$ that has no 1-markings and if $\gamma \subseteq \Gamma$ is a subgraph, let Γ_γ be the full subgraph of Γ that contains every vertex that is not also in γ or adjacent to a vertex of γ. Then,*

$$\mathcal{L}^{\Gamma}_{d,\bullet} \otimes \mathbb{Z}_2 = \bigoplus_{\gamma \subseteq \Gamma} \mathcal{L}^{\Gamma_\gamma}_{d,1,\bullet} \otimes \mathbb{Z}_2,$$

where the sum ranges over all independent sets γ.

Proof. If X is a marking in $\mathcal{L}^{\Gamma_\gamma}_{d,1,\bullet}$, we map it to the marking of Γ where every 2-marking is the same as X, and every vertex of γ is marked. This means we have an injective map $\mathcal{L}^{\Gamma_\gamma}_{d,1,\bullet} \to \mathcal{L}^{\Gamma}_{d,\bullet}$, and the collection of these maps induces a map $\bigoplus_{\gamma \subseteq \Gamma} \mathcal{L}^{\Gamma_\gamma}_{d,1,\bullet} \to \mathcal{L}^{\Gamma}_{d,\bullet}$. This map is injective because every component is injective and if there were two markings of different Γ_γ and $\Gamma_{\gamma'}$, then they do not agree on the single marked vertices. Also, this map is clearly surjective, because every element of the canonical basis is in its image. The differentials agree by definition. □

We will now give interpretations to some homologies.

Proposition 2.1.16.

$$H_0(\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}) \simeq \langle \Gamma_{C,1} \rangle_{C \in I},$$

where I ranges over all maximal cliques of Γ.

Proof. Clearly, if C is a maximal clique, $\bar{d}(\Gamma_{C,1}) = 0$. Let $0 \neq X \in \mathcal{L}^{\Gamma}_0$ with $\bar{d}(X) = 0$, then we have to show that X corresponds to a sum of maximal cliques. Let Y be an element of the canonical basis and consider the map φ given by

$$Y \mapsto \varphi(Y) = \sum_{v \in \Gamma^{m=2}_{\mathrm{vert}}} (-1)^{|\Gamma^{m\neq 0}_{\mathrm{vert},<v}|} Y_{v,0}.$$

Then:

$$\varphi(d(Y)) = \varphi\left(\sum_{v\in\Gamma_{\text{vert}}} (-1)^{|\Gamma^{m\neq 0}_{\text{vert},<v}|} Y_{v,2}\right) = \sum_{v\in\Gamma_{\text{vert}}} (-1)^{|\Gamma^{m\neq 0}_{\text{vert},<v}|}\varphi(Y_{v,2})$$

$\varphi(Y_{v,2}) = 0$ if v cannot be marked and $(-1)^{|\Gamma^{m\neq 0}_{\text{vert},<v}|}Y$ otherwise. Thus, if A is the linear map that multiplies Y with the number of vertices of Y that can be marked,

$$0 = \varphi(d(Y)) = AY.$$

This can be extended linearly, so

$$0 = \varphi(d(X)) = AX,$$

but the kernel of A is generated by all the maximal cliques. $\square$

Because of this, computing the homology of $\mathcal{L}^{\Gamma}_{d,\bullet}$ is NP-hard.

Theorem 2.1.17. *Let k a fixed positive integer. Then the problem of answering the question "Does a graph Γ have a clique of size k?" is NP-complete.*

Corollary 2.1.18. *The problem of finding all maximal cliques of a graph is NP-hard.*

This can be found in [BM11].
Let c be any clique, then we define an equivalence relation $\sim_c$ on the set of maximal cliques of Γ by letting $C \sim_c C'$ if $C \cap C' \supsetneq c$ and completing this relation to an equivalence relation.

Proposition 2.1.19. *Let Γ be a graph and consider the chain complex $\bar{\mathcal{L}}^{\Gamma}_{\bar{d},1} \otimes \mathbb{Z}_2$. Then*

$$V := \left\{ \sum_{C'\sim_c C}\ \sum_{w\in C'-c} (\Gamma_{c,1})_{w,2} \,\middle|\, \begin{array}{l} c \text{ is a non-maximal clique and} \\ C \supset c \text{ is a maximal clique} \end{array} \right\} \subseteq \ker \bar{d}_1$$

and

$$W := \left\{ \bar{d}_0(\Gamma_{c,1}) : c \text{ is a non-maximal clique} \right\} \subseteq \operatorname{im} \bar{d}_0$$

are bases of the respective vector spaces.

Proof. To show that $V \subseteq \ker \bar{d}_1$, consider

$$d_1\left(\sum_{C'\sim_c C}\ \sum_{w\in C'-c} (\Gamma_{c,1})_{w,2}\right) = \sum_{C'\sim_c C}\ \sum_{w\in C'-c}\ \sum_{x\in\Gamma} (\Gamma_{c,1})_{\{w,x\},2} = 0$$

because every pair $\{w, x\}$ appears twice in that sum.
Now, let $\langle\cdot,\cdot\rangle$ be the scalar product on $\mathcal{L}^{\Gamma} \otimes \mathbb{Z}_2$ such that the canonical basis is orthonormal and let $0 \neq X \in \langle V\rangle^{\perp}$. Fix a non-maximal clique c and a vertex v such that $\langle X, (\Gamma_{\gamma,1})_{v,2}\rangle \neq 0$. Let C be a maximal clique that contains c and v and let C' be any

clique such that $C \cap C' \supseteq c \cup \{v\}$. Now let $w \in C'$, then $0 = \langle d_1(X), (\Gamma_{c,1})_{\{v,w\},2}\rangle$. For this to be true, $(\Gamma_{c,1})_{\{v,w\},2}$ has to appear twice in the sum and for this to be possible, $\langle X, (\Gamma_{c,1})_{w,2}\rangle \neq 0$. By the same argument, if C'' is a clique such that $C' \cap C'' \supseteq c \cup \{w\}$, for arbitrary elements x in $C'' - c$, $\langle X, (\Gamma_{c,1})_{x,2}\rangle \neq 0$, so for arbitrary $C' \sim_c C$, if $w \in C' - c$ we have $\langle X, (\Gamma_{c,1})_{w,2}\rangle \neq 0$. But this means that

$$\left\langle X, \sum_{C' \sim_c C} \sum_{w \in C' - c} (\Gamma_{c,1})_{w,2} \right\rangle = 1,$$

a contradiction. So, $\ker d_1 = \langle V \rangle$.
That $\langle W \rangle = \operatorname{im} \bar{d}_0$ is obvious by definition. That these generators are linearly independent is clear from the fact that every such element has a specific clique that is 1-marked in every summand. □

Corollary 2.1.20. *Let N_c the number of equivalence classes of $\sim_c$. Then,*

(i) $\operatorname{rk} H_1(\bar{\mathcal{L}}^\Gamma_{\bar{d},\bullet}) = \sum_{c \text{ clique}} N_c - 1$

(ii) $\operatorname{rk} H_0(\bar{\mathcal{L}}^\Gamma_{\bar{d},\bullet}) - 1 \leq \operatorname{rk} H_1(\bar{\mathcal{L}}^\Gamma_{\bar{d},\bullet})$

Proof. The first statement follows because of the dimension of V is $\sum_{c \text{ clique}} N_c$ and the dimension of W is $\sum_{c \text{ clique}} 1$. For connected graphs, the second statement follows from the fact that for every maximal clique there exists another maximal clique such that they have non-trivial intersection c. But then $N_c \geq 2$. Thus, there exist at least $n-1$ cliques c with $N_c \geq 2$, where $n = \operatorname{rk} H_0(\bar{\mathcal{L}}^\Gamma_{\bar{d},\bullet})$. Because we always have $N_c \geq 1$, the claim follows for connected graphs. For unconnected graphs, the statement follows now with 2.4.4. □

Proposition 2.1.21. *Let n be the size of the biggest clique of Γ. Then*

$$H_n(\bar{\mathcal{L}}^\Gamma_{\bar{d},\bullet} \otimes \mathbb{Z}_2) = \langle \Gamma_{C,2} \rangle_{C \in I} / V,$$

where I ranges over all maximal cliques of Γ (i.e. all cliques of size n) and

$$V = \left\langle \left\{ \sum_{C \supsetneq C'} \Gamma_{S,2} : C' \textit{ is a clique with } n-1 \textit{ elements} \right\} \right\rangle.$$

Proof. Clearly, the kernel of d_n is $\langle \Gamma_{C,2} \rangle_{C \in I}$. For any clique C' with $|C'| = n-1$,

$$d(\Gamma_{C',2}) = \sum_{C \supsetneq C'} \Gamma_{C,2}.$$

□

Example 2.1.22. *Let Γ be the 5-gon graph. It has 2 as the size of its biggest clique and* $\dim H_2(\mathcal{L}^\Gamma_{d,\bullet} \otimes \mathbb{Z}_2) = 1$.

Of course all of these formulas for the homologies can be stated for $\mathcal{L}^\Gamma_{d,\bullet}$ by exchanging the word clique for independent set.

2.2 Computational aspects

The groups $\mathcal{L}^\Gamma$ can be described in a way that is suitable for computations: If Γ has n vertices and if A_Γ is the adjacency matrix of Γ, A_Γ induces a bilinear form b on $\mathbb{Z}^n$ in the usual way (where n is the number of vertices of Γ). Every $v \in \mathbb{Z}^n$ can be regarded as a function $\Gamma_{\text{vert}} \to \mathbb{Z}$, so if every coefficient of v is $0, 1$ or 2, v induces a graph marking. To decide whether a marking is allowed in $\mathcal{L}^\Gamma$, the only question that remains is if no two adjacent vertices are both marked, i.e. if $v_i \cdot v_j = 0$ for all pairs (i, j) that are neighbors, i.e. where $a_{ij} = 1$. This is equivalent to asking if $a_{ij} v_i v_j = 0$ for all pairs i, j, and because all these terms are positive, this is equivalent to asking if

$$\sum_{i,j} a_{ij} v_i v_j = b(v, v) = 0.$$

One can then implement the differential on this basis and use programs such as Sage to compute homologies.

2.3 Independent set graph complex

We will now prove results that follow more natural for the independent set graph complex.

Lemma 2.3.1. *Let $\Gamma = \Gamma^1 \cup \Gamma^2$, and let the numbering be such that all vertices in Γ^1 are smaller than every vertex in Γ^2. Let $X = (\Gamma^1, m_1)$ and $Y = (\Gamma^2, m_2)$ be elements of the canonical bases in the chain complexes of Γ^1 and Γ^2 respectively, of arbitrary degrees. Let $X \cup Y := (\Gamma, m_1 \cup m_2)$ with*

$$m_1 \cup m_2(\gamma) := \begin{cases} m_1(\gamma) & \text{if } \gamma \in sub(\Gamma^1) \\ m_2(\gamma) & \text{if } \gamma \in sub(\Gamma^2). \end{cases}$$

Then:

$$d(X \cup Y) = d(X) \cup Y + (-1)^{|(\Gamma^1)^{m_1 \neq 0}_{\text{vert}}|} X \cup d(Y)$$

Proof.

$$\begin{aligned}
d(X \cup Y) = & \sum_{v \in (\Gamma^1 \cup \Gamma^2)^{m=0}_{\text{vert}}} (-1)^{|\Gamma^{m\neq 0}_{\text{vert},<v}|} (X \cup Y)_{v,2} \\
= & \sum_{v \in (\Gamma^1)^{m=0}_{\text{vert}}} (-1)^{|\Gamma^{m\neq 0}_{\text{vert},<v}|} (X \cup Y)_{v,2} + \sum_{v \in (\Gamma^2)^{m=0}_{\text{vert}}} (-1)^{|\Gamma^{m\neq 0}_{\text{vert},<v}|} (X \cup Y)_{v,2} \\
= & \sum_{v \in (\Gamma^1)^{m_1=0}_{\text{vert}}} (-1)^{|(\Gamma^1)^{m_1\neq 0}_{\text{vert},<v}|} X_{v,2} \cup Y + \sum_{v \in (\Gamma^2)^{m_2=0}_{\text{vert}}} (-1)^{|(\Gamma^1)^{m_1\neq 0}_{\text{vert},<v}| + |(\Gamma^2)^{m_2\neq 0}_{\text{vert},<v}|} X \cup Y_{v,2} \\
= & \, d(X) \cup Y + (-1)^{|(\Gamma^1)^{m_1\neq 0}_{\text{vert}}|} X \cup \sum_{v \in (\Gamma^2)^{m_2=0}_{\text{vert}}} (-1)^{|(\Gamma^2)^{m_2\neq 0}_{\text{vert},<v}|} Y_{v,2} \\
= & \, d(X) \cup Y + (-1)^{|(\Gamma^1)^{m_1\neq 0}_{\text{vert}}|} X \cup d(Y)
\end{aligned}$$

$\square$

Corollary 2.3.2. *Let $\Gamma = \Gamma^1 \cup \Gamma^2$, and $\mathcal{L}_{i,j}^{\Gamma^1,\Gamma^2} := \mathcal{L}_{d,i}^{\Gamma^1} \otimes \mathcal{L}_{d,j}^{\Gamma^2}$. Then $\mathcal{L}_{d,\bullet}^{\Gamma}$ is isomorphic to the associated total complex of the double complex having the $\mathcal{L}_{i,j}^{\Gamma^1,\Gamma^2}$ as objects, and as differentials the maps $d \otimes \mathrm{id}$ and $s \otimes d$, where*

$$s(X) := (-1)^{|(\Gamma^1)_{\mathrm{vert}}^{m_1 \neq 0}|} X$$

This means we can apply the Künneth formula:

Theorem 2.3.3. *If R is a PID and $A_\bullet$ and $B_\bullet$ are chain complexes of free R-modules, there is a short exact sequence*

$$0 \to \bigoplus_{i+j=k} H_i(A_\bullet) \otimes_R H_j(B_\bullet) \to H_k((A \otimes_R B)_\bullet) \to \bigoplus_{i+j=k} \mathrm{Tor}_R^1(H_i(A_\bullet), H_{j-1}(B_\bullet)) \to 0$$

Proof. This can be found for example in [Hat02], pg. 274 Thm 3B.5. $\square$

In our case, $R = \mathbb{Z}$. Next, we could show that $\mathrm{Tor}_{\mathbb{Z}}(H_i(A_\bullet), H_{j-1}(B_\bullet)) = 0$, which would imply that the chain complexes are quasi-isomorphic. We will not use this fact, and instead prove the special case of the Künneth formula that we will use:

Theorem 2.3.4. *If R is a PID and $A_\bullet$ and $B_\bullet$ are chain complexes of free R-modules, such that all the homologies of $A_\bullet$ are free, there is an isomorphism*

$$\bigoplus_{i+j=k} H_i(A_\bullet) \otimes_R H_j(B_\bullet) \simeq H_k((A \otimes_R B)_\bullet)$$

Proof. This proof is essentially the one that was just quoted, but slightly simplified for our special case. First, assume that the complex A has a trivial differential, so $H_k(A_\bullet) = A_k$ and $H_\bullet(A_\bullet) = A_\bullet$ as chain complexes. Then, the decomposition

$$(A \otimes_R B)_k = \bigoplus_i A_i \otimes B_{k-i}$$

carries over to a direct sum decomposition of chain complexes:

$$(A \otimes_R B)_\bullet = \bigoplus_i A_i \otimes_R (B[-i])_\bullet,$$

where the notation $[-i]$ denotes a degree shift, and the tensor product of a chain complex with a module is component-wise. This means that homology can be taken for every component of the direct sum independently:

$$H_k((A \otimes_R B)_\bullet) = \bigoplus_i H_k(A_i \otimes_R (B[-i])_\bullet)$$

Because A_i is free for every i, $A_i \simeq \bigoplus_{j=1}^{\dim A_i} R$, and so

$$H_k(A_i \otimes_R (B[-i])_\bullet) \simeq H_k(\bigoplus_{j=1}^{\dim A_i} R \otimes_R (B[-i])_\bullet) \simeq \bigoplus_{j=1}^{\dim A_i} H_k((B[-i])_\bullet)$$

$$\simeq \bigoplus_{j=1}^{\dim A_i} R \otimes_R H_{k-i}(B_\bullet) \simeq A_i \otimes_R H_{k-i}(B_\bullet)$$

Next, assume that A has an arbitrary differential, and consider the differential d_A of A as a morphism of chain complexes $d_A : A_\bullet \to (A[-1])_\bullet$. Let $K_\bullet := \ker d_A$ and $I_\bullet := \operatorname{im} d_A$ be the kernel and image of this morphism respectively, considered as chain complexes, and note that both of these are free again, because they are sub complexes of the free complexes $A_\bullet$ and $(A[-1])_\bullet$. We now have a short exact sequence

$$0 \to K_\bullet \to A_\bullet \to I_\bullet \to 0$$

which splits, because $I_\bullet$ is free. Because of the splitting, if we tensor this complex with $B_\bullet$, we get another short exact sequence of chain complexes, which contains the chain complex we are interested in. Taking homology gives us a long exact sequence:

$$\cdots \to H_k((K \otimes B)_\bullet) \to H_k((A \otimes B)_\bullet) \to H_k((I \otimes B)_\bullet) \to H_{k+1}((K \otimes B)_\bullet) \to \cdots$$

Because K and I have trivial differentials, we can apply the first part of the proof, which results in the following exact sequence:

$$\cdots \to \bigoplus_{i+j=k} H_i(K_\bullet) \otimes_R H_j(B_\bullet) \to H_k((A \otimes B)_\bullet)$$

$$\to \bigoplus_{i+j=k} H_i(I_\bullet) \otimes_R H_j(B_\bullet) \to \bigoplus_{i+j=k+1} H_i(K_\bullet) \otimes_R H_j(B_\bullet) \to \cdots$$

Applying the usual decomposition of a long exact sequence into short exact sequences yields sequences,

$$0 \to \operatorname{im} \to H_k((A \otimes B)_\bullet) \to \ker \to 0$$

where

$$\operatorname{im} = \bigoplus_{i+j=k} K_i \otimes_R H_j(B_\bullet) / \bigoplus_{i+j=k-1} I_i \otimes_R H_j(B_\bullet).$$

Because $H_i(A_\bullet) = K_i/I_{i-1}$, we have $\operatorname{im} = \bigoplus_{i+j=k} H_i(A_\bullet) \otimes_R H_j(B_\bullet)$, so it suffices to show that $\ker = 0$, or equivalently, that the map

$$\varphi : H_k((A \otimes B)_\bullet) \to \bigoplus_{i+j=k} H_i(I_\bullet) \otimes_R H_j(B_\bullet) = H_k((A \otimes B)_\bullet) \to H_k((I \otimes B)_\bullet)$$

is the zero map. But this map also appears in another short exact sequence. By definition, there is a short exact sequence:

$$0 \to I_i \to K_i \to H_i(A_\bullet) \to 0$$

Tensoring with $H_j(B_\bullet)$ yields:

$$0 \to I_i \otimes H_j(B_\bullet) \to K_i \otimes H_j(B_\bullet) \to H_i(A_\bullet) \otimes H_j(B_\bullet) \to 0$$

There is a zero on the left because $H_j(B_\bullet)$ is free. Summing over i and j gives a part of the long exact sequence, and 0 is precisely where the image would appear. $\square$

Corollary 2.3.5. *If* $\Gamma = \Gamma_1 \cup \Gamma_2$, *we have*

$$H_k(\mathcal{L}^{\Gamma}_{d,\bullet}) = \bigoplus_{i+j=k} H_j(\mathcal{L}^{\Gamma_1}_{d,\bullet}) \otimes H_j(\mathcal{L}^{\Gamma_2}_{d,\bullet})$$

This means the homologies of non-connected graphs can be reduced to homologies of connected graphs.

Example 2.3.6. $H_\bullet(\mathcal{L}^{\Gamma}_{d,\bullet}) \simeq H_\bullet(\mathcal{L}^{\Gamma\cup *}_{d,\bullet})$, *where* $*$ *is the graph with a single vertex*

Proof. We clearly have

$$H_0(\mathcal{L}^{*}_{d,\bullet}) = \langle 1 \rangle, H_1(\mathcal{L}^{*}_{d,\bullet}) = 0$$

with all the higher homologies vanishing. Applying the Künneth formula gives the result. □

There is another way of reducing the problem of finding these graph homologies to simpler graphs. This involves using the fact that replacing a single vertex by a complete graph ought not change the corresponding chain complex much and should just duplicate things.

Proposition 2.3.7. *Let* Γ *have a complete subgraph* γ *such that every vertex of* γ *is connected to the same set of vertices in* Γ *and such that whenever* $v < w < x \in \Gamma_{\text{vert}}$ *with* $v, x \in \gamma$, $w \in \gamma$. *Let* Γ/γ *be the quotient graph where* v *is the single vertex that resulted from the contraction of* γ. *Then there is a short exact sequence of chain complexes:*

$$0 \to \ker \varphi \to \mathcal{L}^{\Gamma}_{d,\bullet} \otimes \mathbb{Q} \xrightarrow{\varphi} \mathcal{L}^{\Gamma/\gamma}_{d,\bullet} \otimes \mathbb{Q} \to 0$$

Proof. Let v_γ denote the vertex of Γ/γ that corresponds to γ and let X be an element of the canonical basis. Define $\varphi(X)$ to be the identity on markings outside of γ, and if γ has a marking, $\varphi(X)$ will have the same marking at v_γ and be multiplied by $\frac{1}{|\gamma|}$. This is well-defined by the assumptions on γ and a morphism of chain complexes, because

$$\begin{aligned}
\varphi(d(X)) &= \sum_{v \in \Gamma^{m=0}_{\text{vert}}, v \notin \gamma} (-1)^{|\Gamma^{m\neq 0}_{\text{vert},<v}|} \varphi(X_{v,2}) + \sum_{v \in \Gamma^{m=0}_{\text{vert}}, v \in \gamma} (-1)^{|\Gamma^{m\neq 0}_{\text{vert},<v}|} \varphi(X_{v,2}) \\
&= \sum_{v \in \Gamma^{m=0}_{\text{vert}}, v \notin \gamma} (-1)^{|\Gamma^{m\neq 0}_{\text{vert},<v}|} \varphi(X)_{v,2} + \frac{1}{|\gamma|} \sum_{v \in \Gamma^{m=0}_{\text{vert}}, v \in \gamma} (-1)^{|\Gamma^{m\neq 0}_{\text{vert},<v}|} \varphi(X)_{v_\gamma,2} \\
&= \sum_{v \in \Gamma^{m=0}_{\text{vert}}, v \notin \gamma} (-1)^{|\Gamma^{m\neq 0}_{\text{vert},<v}|} \varphi(X)_{v,2} + (-1)^{|\Gamma^{m\neq 0}_{\text{vert},<v_\gamma}|} \varphi(X)_{v_\gamma,2} \\
&= d(\varphi(X)).
\end{aligned}$$

That this map is an epimorphism is trivial. □

Proposition 2.3.8. *Let* Γ, γ *and* φ *as in the previous proposition and let* Γ_γ *be as in Theorem 2.1.15. Then,*

$$\ker \varphi_\bullet = (\mathcal{L}^{\Gamma_\gamma}_{d,\bullet} \otimes \mathbb{Q})^{\binom{|\gamma|}{2}}$$

Proof. By using the canonical basis, it is clear that $\ker\varphi$ is generated by markings of the form $X - X'$ where X and X' are marked at precisely the same places outside of γ, and have the same kind of marking in γ at different vertices. There are exactly $\binom{|\gamma|}{2}$ ways to select 2 vertices of γ and because none of these generators interact in any way, it is a direct sum of $\binom{|\gamma|}{2}$ chain complexes. Because there is always a vertex of γ marked, no neighbor of γ can be marked, so all the information is contained in the subgraph Γ_γ. Note here that if Γ_γ is the empty graph, $\mathcal{L}^{\Gamma_\gamma}_{d,0} = 1$. $\square$

Note that this means there is a long exact sequence in the homologies (using that homology commutes with direct sum and tensor product):

$$0 \to H_0(\mathcal{L}^{\Gamma_\gamma}_{d,\bullet} \otimes \mathbb{Q})^{\binom{|\gamma|}{2}} \to H_0(\mathcal{L}^{\Gamma}_{d,\bullet} \otimes \mathbb{Q}) \to H_0(\mathcal{L}^{\Gamma/\gamma}_{d,\bullet} \otimes \mathbb{Q}) \to H_1(\mathcal{L}^{\Gamma_\gamma}_{d,\bullet} \otimes \mathbb{Q})^{\binom{|\gamma|}{2}} \to \cdots$$

Because $\dim_{\mathbb{Q}} H_i(\mathcal{L}^{\Gamma}_{d,\bullet} \otimes \mathbb{Q}) = \operatorname{rk} H_i(\mathcal{L}^{\Gamma}_{d,\bullet}) = \dim_{\mathbb{Z}_2} H_i(\mathcal{L}^{\Gamma}_{d,\bullet} \otimes \mathbb{Z}_2)$, and because all of them have the same basis, one might be able to reduce the computation of $H_i(\mathcal{L}^{\Gamma}_{d,\bullet})$. As the complexity of these computations grows exponentially in the number of vertices (at the worst case, see the discussion before Theorem 2.1.15), this could be used to reduce computation times for certain graphs considerably.

2.4 Clique graph complex

If a graph is a disjoint union if two graphs, we also have a decomposition of the chain complex $\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}$, which is slightly more subtle than in the previous case. First, we state a criterion for the complement graph to be disconnected.

Lemma 2.4.1. *Let Γ be a graph. Then $\bar{\Gamma}$ is disconnected if and only if there exist two induced subgraphs Γ_1, Γ_2 of Γ such that every vertex is contained in one of these subgraphs and every vertex of Γ_1 is connected to every vertex of Γ_2 by an edge.*

Proof. This is clear from the definitions. $\square$

Proposition 2.4.2. *Let $\Gamma = \Gamma_1 \cup \Gamma_2$ with all vertices in Γ_1 smaller than every vertex in Γ_2. Then*

$$\bar{\mathcal{L}}^{\Gamma}_{\bar{d},k} \simeq \bar{\mathcal{L}}^{\Gamma_1}_{\bar{d},k} \oplus \bar{\mathcal{L}}^{\Gamma_2}_{\bar{d},k} \text{ for } k \geq 1 \text{ and}$$
$$\bar{\mathcal{L}}^{\Gamma}_{\bar{d},0} \oplus \mathbb{Z} \simeq \bar{\mathcal{L}}^{\Gamma_1}_{\bar{d},0} \oplus \bar{\mathcal{L}}^{\Gamma_2}_{\bar{d},0}.$$

Proof. This is obvious, because every marking lies entirely in one of the two Γ_i, except the completely unmarked graph. But in degree $(0,0)$ (no markings at all), there is only one marking for Γ and two for Γ_1 and Γ_2 respectively, so one has to add $\mathbb{Z}$ to get an isomorphism. $\square$

Corollary 2.4.3. *Let $\Gamma = \Gamma_1 \cup \Gamma_2$ with all vertices in Γ_1 smaller than every vertex in Γ_2 and $k \geq 2$. Then:*

$$\dim H_k(\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}) = \dim H_k(\bar{\mathcal{L}}^{\Gamma_1}_{\bar{d},\bullet}) + \dim H_k(\bar{\mathcal{L}}^{\Gamma_2}_{\bar{d},\bullet})$$

As a consequence of the issues at $k = 0$, the homology is not always a direct sum of the homologies (which would be impossible anyways, because the alternating sum of the betti numbers always has to be 1).

Proposition 2.4.4. *Let $\Gamma = \Gamma_1 \cup \Gamma_2$ with all vertices in Γ_1 smaller than every vertex in Γ_2. Then:*

$$\operatorname{rk} H_0(\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}) = \operatorname{rk} H_0(\bar{\mathcal{L}}^{\Gamma_1}_{\bar{d},\bullet}) + \operatorname{rk} H_0(\bar{\mathcal{L}}^{\Gamma_2}_{\bar{d},\bullet})$$

$$\operatorname{rk} H_1(\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}) = \operatorname{rk} H_1(\bar{\mathcal{L}}^{\Gamma_1}_{\bar{d},\bullet}) + \operatorname{rk} H_1(\bar{\mathcal{L}}^{\Gamma_2}_{\bar{d},\bullet}) + 1$$

Proof. The spaces $\bar{\mathcal{L}}^{\Gamma}_{\bar{d},0}$ and $\bar{\mathcal{L}}^{\Gamma_1}_{\bar{d},0} \oplus \bar{\mathcal{L}}^{\Gamma_2}_{\bar{d},0}$ differ by one generator only: the first one has a single generator with no markings, the second space has two. None of these are in the kernel of $\bar{d}$, so

$$H_0(\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}) = H_0(\bar{\mathcal{L}}^{\Gamma_1}_{\bar{d},\bullet}) \oplus H_0(\bar{\mathcal{L}}^{\Gamma_2}_{\bar{d},\bullet}).$$

The kernel of $\bar{d}_1$ is also the same in both cases, but because the images of the two generators with no markings are linearly independent from the rest and add to the image of generator with no markings of the other case, the dimensions of the two split up homologies add to one less than the homology of Γ. □

Corollary 2.4.5. *Let v be a vertex of Γ that is adjacent to every other vertex, and Γ' be the full subgraph of Γ that contains every vertex except v. Then:*

$$\begin{aligned}
\operatorname{rk} H_0(\mathcal{L}^{\Gamma}_{d,\bullet}) &= \operatorname{rk} H_0(\mathcal{L}^{\Gamma'}_{d,\bullet}) + 1 \\
\operatorname{rk} H_1(\mathcal{L}^{\Gamma}_{d,\bullet}) &= \operatorname{rk} H_1(\mathcal{L}^{\Gamma'}_{d,\bullet}) + 1 \\
\operatorname{rk} H_k(\mathcal{L}^{\Gamma}_{d,\bullet}) &= \operatorname{rk} H_k(\mathcal{L}^{\Gamma'}_{d,\bullet}), \ \textit{for } k > 1.
\end{aligned}$$

Proof. Apply the previous proposition to $\bar{\Gamma} = \bar{\Gamma'} \cup \{v\}$. □

Definition 2.4.6. A *leaf vertex* of a graph Γ is a vertex v that has only a single edge.

Proposition 2.4.7. *Let Γ be a connected graph with $|\Gamma_{\text{vert}}| \geq 3$ and v be a leaf vertex of Γ. Let Γ' be $\Gamma - \{v\}$. Then:*

$$H_i(\mathcal{L}^{\Gamma}_{d,\bullet}) \simeq \begin{cases} H_i(\mathcal{L}^{\Gamma'}_{d,\bullet}) \oplus \mathbb{Z} & \textit{if } i < 2 \\ H_i(\mathcal{L}^{\Gamma'}_{d,\bullet}) & \textit{if } i \geq 2 \end{cases}$$

Proof. It again suffices to show this only after tensoring with $\mathbb{Z}_2$. Let Γ_v be the graph with v removed and v' the vertex v is adjacent to. Then,

$$\begin{aligned}
\bar{\mathcal{L}}^{\Gamma}_{\bar{d},0} &= \bar{\mathcal{L}}^{\Gamma_v}_{\bar{d},0} \oplus \langle \Gamma_{v,1}, \Gamma_{\{v,v'\},1} \rangle, \\
\bar{\mathcal{L}}^{\Gamma}_{\bar{d},1} &= \bar{\mathcal{L}}^{\Gamma_v}_{\bar{d},1} \oplus \langle \Gamma_{v,2}, (\Gamma_{v,1})_{v',2}, (\Gamma_{v',1})_{v,2} \rangle, \\
\bar{\mathcal{L}}^{\Gamma}_{\bar{d},2} &= \bar{\mathcal{L}}^{\Gamma_v}_{\bar{d},2} \oplus \langle \Gamma_{\{v,v'\},2} \rangle.
\end{aligned}$$

All of the new elements except $\Gamma_{v,1}$ and $\Gamma_{v,2}$ are clearly in the kernel of $\bar{d}$. Because v is not connected to any other vertex in Γ_v, $d(\Gamma_{v,1}) = (\Gamma_{v,1})_{v',2}$ and $d(\Gamma_{v,2}) = \Gamma_{\{v,v'\},2}$, so

$$H_0(\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}) = H_0(\bar{\mathcal{L}}^{\Gamma_v}_{\bar{d},\bullet}) \oplus \langle \Gamma_{\{v,v'\},1} \rangle$$
$$H_2(\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}) = H_2(\bar{\mathcal{L}}^{\Gamma_v}_{\bar{d},\bullet}).$$

But because the alternating sum of the ranks of the homologies has to be 1,

$$H_1(\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}) = H_1(\bar{\mathcal{L}}^{\Gamma_v}_{\bar{d},\bullet}) \oplus \langle (\Gamma_{v',1})^{v,2} \rangle.$$

□

Corollary 2.4.8. *Let Γ be a tree with n vertices. Then*

$$\operatorname{rk} H_0(\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}) = n-1, \operatorname{rk} H_1(\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}) = n-2$$

and all the other homologies vanish.

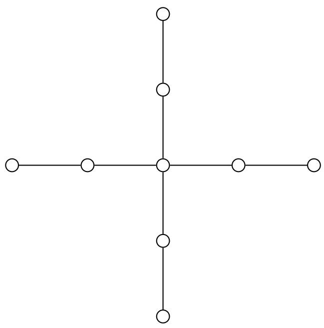
Proof. Note that there is only one tree with 2 vertices, and all trees with 2 or less vertices satisfy this claim. Now the rest follows from the previous corollary. □

2.5 Open problems

Conjecture 2.5.1. *Let Γ be a connected graph and $n := \dim H_0(\mathcal{L}^{\Gamma}_{d,\bullet})$. Then*

$$\dim H_1(\mathcal{L}^{\Gamma}_{d,\bullet}) \leq 2n.$$

Example 2.5.2. *The following graph has a sharp equality for the previous conjecture with $\operatorname{rk} H_0(\mathcal{L}^{\Gamma}_{d,\bullet}) = 16$ and $\operatorname{rk} H_1(\mathcal{L}^{\Gamma}_{d,\bullet}) = 32$.*

$\Gamma :=$ 

This graph also shows that this inequality does not hold for unconnected graphs: Placing two of these graphs next to each other results in a graph Γ' with $\operatorname{rk} H_0(\mathcal{L}^{\Gamma'}_{d,\bullet}) = 256$ and $\operatorname{rk} H_1(\mathcal{L}^{\Gamma'}_{d,\bullet}) = 1024$.

This conjecture is motivated by analyzing the results of over 80000 computations and the statement is true for all graphs with 8 vertices or less.

Proposition 2.5.3. *Let* Γ *be a graph and* v, v' *two leaf vertices of* Γ *that are connected to the same vertex* w*. Let* Γ' *be* $\Gamma - \{v'\}$*. Then:*

$$H_0(\mathcal{L}^{\Gamma}_{d,\bullet}) \simeq H_0(\mathcal{L}^{\Gamma'}_{d,\bullet})$$

Proof. It suffices to give a bijection between the maximal independent sets corresponding to the homology classes. Given a maximal independent set X of Γ, one gets a maximal independent set of Γ' by removing v' if it is contained in X. This map is a bijection because for maximal independent sets in Γ, either v and v' are part of the independent set, or w is part of the independent set. □

Conjecture 2.5.4. *Let* Γ, Γ', v *and* v' *be as in the previous proposition. Then:*

$$H_k(\mathcal{L}^{\Gamma}_{d,\bullet}) \simeq H_k(\mathcal{L}^{\Gamma'}_{d,\bullet})$$

for all $k \geq 0$

There are several things that are left to study in relation to this chain complex. For example, the complex that has no 1-markings could be studied and might give new insights. Also, questions of what one can say about a graph if the homology of $\mathcal{L}^{\Gamma}_{d,\bullet}$ or $\bar{\mathcal{L}}^{\Gamma}_{\bar{d},\bullet}$ is given are unanswered.

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Selbstständigkeitserklärung

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