Multiple polylogarithms and Feynman integrals

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joint work with Francis Brown

arXiv:1302.6215 with M. Lüders, arXiv:1302.7004 and 1405.5640 with L. Adams and S. Weinzierl,

Many thanks to Erik Panzer!

Les Houches, June 2014

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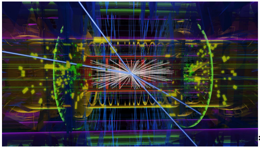
Outline:

- The computational problem: Integration over Feynman parameters
- Multiple polylogarithms in several variables and the program MPL
- Applications: Feynman integrals, hypergeometric functions
- Outlook: Beyond multiple polylogarithms

The computational problem: Integration over Feynman parameters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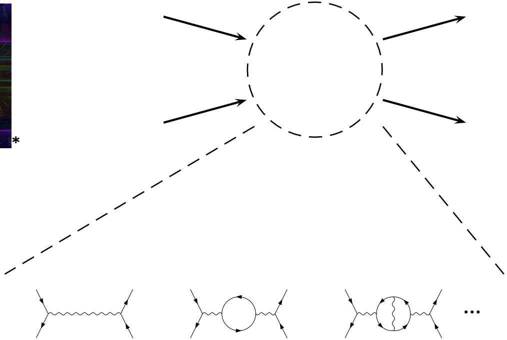


Feynman rules

Feynman integrals $\quad \propto \quad 1 \quad + \quad \int \dots \quad + \quad \int \dots \quad \dots$

(*picture: ATLAS Experiment © 2013 CERN)

Scalar Feynman integrals

For a generic Feynman graph G with N edges and loop-number L (first Betti number) we consider the scalar Feynman integral

$$I(\Lambda) = \int \prod_{i=1}^{L} \frac{d^D k_i}{i\pi^{D/2}} \prod_{j=1}^{N} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}}, \quad N,\ L,\ \nu_j \in \mathbb{Z},\ D \in \mathbb{C},$$

Λ : external parameters, i.e. kinematical invariants and masses m_j; q_j : momenta

Using the "Feynman trick" we can re-write this as

$$I(\Lambda) = \frac{\Gamma\left(\nu - LD/2\right)}{\prod_{j=1}^{N}\Gamma(\nu_j)} \int_0^\infty \dots \int_0^\infty \left(\prod_{i=1}^{N} dx_i x_i^{\nu_i - 1}\right) \delta\left(1 - \sum_{i=1}^{N} x_i\right) \frac{\mathcal{U}^{\nu-(L+1)D/2}}{\left(\mathcal{F}\left(\Lambda\right)\right)^{\nu - LD/2}},$$

$$\text{where } \nu = \textstyle\sum_{j=1}^{N} \nu_j,\ \epsilon = (4-D)/2.$$

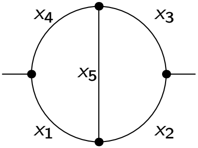
$\mathcal{U}$ and $\mathcal{F}$ are the first and the second Symanzik polynomial.

Symanzik polynomials for a graph G with Feynman parameters x_1, ..., x_N:

$$\mathcal{U} = \sum_{\text{spanning trees } T \text{ of } G} \prod_{\text{edges} \notin T} x_i$$

$$\mathcal{F} = - \sum_{\text{spanning 2-forests } (T_1, T_2)} \left(\prod_{\text{edges} \notin (T_1, T_2)} x_i \right) \left(\sum_{\text{edges} \notin (T_1, T_2)} q_i \right)^2 + \mathcal{U} \sum_{i=1}^{N} x_i m_i^2$$

Example:



$$\mathcal{U} = x_3 x_4 + x_2 x_4 + x_1 x_2 + x_1 x_3 + x_5 (x_1 + x_2 + x_3 + x_4)$$

Symanzik polynomials for a graph G with Feynman parameters x_1, ..., x_N:

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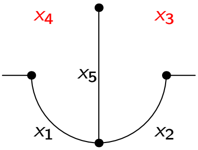
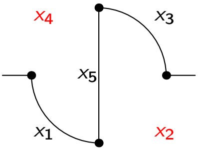


$$\mathcal{U} = x_3 x_4 + x_2 x_4 + x_1 x_2 + x_1 x_3 + x_5 (x_1 + x_2 + x_3 + x_4)$$

Symanzik polynomials for a graph G with Feynman parameters x_1, ..., x_N:

$$\begin{aligned}\mathcal{U} &= \sum_{\text{spanning trees } T \text{ of } G} \; \prod_{\text{edges } \notin T} x_i \\ \mathcal{F} &= - \sum_{\text{spanning 2-forests } (T_1, T_2)} \left(\prod_{\text{edges } \notin (T_1, T_2)} x_i \right) \left(\sum_{\text{edges } \notin (T_1, T_2)} q_i \right)^2 + \mathcal{U} \sum_{i=1}^{N} x_i m_i^2\end{aligned}$$

Example:



$$\mathcal{U} = x_3 x_4 + x_2 x_4 + x_1 x_2 + x_1 x_3 + x_5 (x_1 + x_2 + x_3 + x_4)$$

Assume singularities are taken care of, i.e. ...

- ... the Feynman integral is finite.
- ... by renormalization under the integral (Brown, Kreimer 2011)
- ... by some approach to separate UV and IR singularities, e.g. Panzer (2014).

Computational problem:

Compute a finite integral over Feynman parameters with an integrand of the type:

$$\frac{\left(\prod Q_i\right)\,\text{(multiple) polylogarithms of }\{P_i\}}{\prod P_i}$$

where the P_i and Q_i are polynomials in the Feynman parameters. Usually: Symanzik polynomials

Concept: Try to integrate out all Feynman parameters:

- choose a Feynman parameter x_j in which all P_i are **linear**,
- integrate over x_j by use of an appropriate class of functions, given by iterated integrals

Recent success of this concept in work by E. Panzer, C. Duhr et al, L. Dixon et al, F. Wissbrock et al ...

Iterated integrals

$$
\begin{aligned}
I(t) &= \int_0^t \underbrace{f_w(t^{(w)})dt^{(w)}}_{\omega_w} \ldots \int_0^{t'''} \underbrace{f_2(t'')dt''}_{\omega_2} \int_0^{t''} \underbrace{f_1(t')dt'}_{\omega_1} \\
&\equiv [f_w(t)dt|\ldots|f_2(t)dt|f_1(t)dt] \quad \text{(short-hand notation)}
\end{aligned}
$$

We use the term *iterated integral* for linear combinations of such integrals. The differential one-forms $f_i(t)dt$ belong to a chosen set Ω.

Examples:

- $\Omega_{\text{Polylogs}} = \left\{ \frac{dt}{t}, \frac{dt}{1-t} \right\}$
  - **classical polylogarithms**: $\mathrm{Li}_w(t) = \underbrace{\left[\frac{dt}{t}|\ldots|\frac{dx}{t}|\frac{dx}{1-t}\right]}_{w \text{ times}}$,
  - **multiple polylogarithms in one variable**: $\mathrm{Li}_{n_1 n_2 \ldots}(t) = [\ldots|\underbrace{\frac{dt}{t}|\ldots|\frac{dt}{t}|\frac{dt}{1-t}}_{n_2 \text{ times}}|\underbrace{\frac{dt}{t}|\ldots|\frac{dt}{t}|\frac{dt}{1-t}}_{n_1 \text{ times}}]$

Examples:

- $\Omega_{\mathrm{HPL}} = \left\{ \frac{dt}{t}, \frac{dt}{1-t}, \frac{dt}{1+t} \right\}$

  Harmonic Polylogarithms (Remiddi, Vermaseren 1999),
  (implementation Maitre '05, '07)

- **Two-dimensional Harmonic Polylogarithms** (Gehrmann, Remiddi '01):
  x variable and one additional fixed parameter

- $\Omega_{\mathrm{Cyclotomic}} = \left\{ \frac{dt}{t}, \frac{t^l\, dt}{\phi_k(t)} | k \in \mathbb{N}_+,\, 0 \leq l \leq \varphi(k),\, \phi_k(t) : \text{ cyclotomic polyn.} \right\}$

  Cyclotomic Harmonic Polylogarithms (Ablinger, Blümlein, Schneider '11),
  (implementation Ablinger)

- $\Omega_n^{\mathrm{Hyp}} = \left\{ \frac{dt_n}{t_n}, \frac{dt_n}{t_n - 1}, \ldots, \frac{\left(\prod_{a \leq i \leq n-1} t_i\right) dt_n}{\prod_{a \leq i \leq n} t_i - 1}, 1 \leq a \leq n \right\}$: **Hyperlogarithms**

  Poincare, Kummer 1840, Lappo-Danilevsky 1911 also see Goncharov '01, applications and implementation by Panzer '13, '14

Let Ω_n be the set of differential 1-forms $\frac{df}{f}$
with $f \in \left\{ t_1, \ldots, t_n, \prod_{a \le i \le b} t_i - 1 \right\}$, for $1 \le a \le b \le n$:

$$\Omega_n = \left\{ \frac{dt_1}{t_1}, \ldots, \frac{dt_n}{t_n}, \frac{d\left(\prod_{a \le i \le b} t_i\right)}{\prod_{a \le i \le b} t_i - 1} \text{ where } 1 \le a \le b \le n \right\}$$

Examples:

$$\Omega_1 = \left\{ \frac{dt_1}{t_1}, \frac{dt_1}{t_1 - 1} \right\} \text{ (}\rightarrow\text{ multiple polylogs in one variable)}$$

$$\Omega_2 = \left\{ \frac{dt_1}{t_1}, \frac{dt_2}{t_2}, \frac{dt_1}{t_1 - 1}, \frac{dt_2}{t_2 - 1}, \frac{t_1\, dt_2 + t_2\, dt_1}{t_1 t_2 - 1} \right\}$$

From Ω_n we construct **homotopy invariant** iterated integrals.
Viewed as integrals along paths γ, this means

$$\int_{\gamma_1} \omega_k ... \omega_1 = \int_{\gamma_2} \omega_k ... \omega_1 \textit{ for homotopic paths } \gamma_1, \gamma_2.$$

Problem: Not every sequence of $\omega_i \in \Omega_n$ will provide a homotopy invariant integral.

Theorem (Chen '77) $\Rightarrow$ The integral is homotopy invariant if the sequence (tensor-product) $[\omega_1|...|\omega_m]$ satisfies

$$\sum_{i=1}^{m}[\omega_1|...|\omega_{i-1}|d\omega_i|\omega_{i+1}|...\omega_m] + \sum_{i=1}^{m-1}[\omega_1|...|\omega_{i-1}|\omega_i \wedge \omega_{i+1}|...|\omega_m] = 0.$$

There is an **explicit symbol map** ψ for constructing such homotopy invariant iterated integrals, see CB, Brown '12 (closely related to the "symbol" in Duhr, Gangl, Rhodes '11, Goncharov et al '10).

$\Rightarrow$ Construction provides the **multiple polylogarithms in several variables** $\mathcal{B}(\Omega_n)$.

Some details on the implementation:
$\Omega_n = \Omega_n^{\mathrm{Fiber}} \cup \Omega_n^{\mathrm{Base}}$ where all $f_i \in \Omega_n^{\mathrm{Fiber}}$ depend on the last variable t_n and all $b_i \in \Omega_n^{\mathrm{Base}} \equiv \Omega_{n-1}$ do not.
The bijective *lifting* map $\lambda :\ \Omega_n^{\mathrm{Hyp}} \to \Omega_n^{\mathrm{Fiber}}$ is defined by
$\lambda \frac{\left(\prod_{a \le i \le n-1} t_i\right) dt_n}{\prod_{a \le i \le n} t_i - 1} = \frac{d\left(\prod_{a \le i \le n} t_i\right)}{\prod_{a \le i \le n} t_i - 1}$.
For each pair $f_i,\ f_j \in \Omega_n^{\mathrm{Fiber}}$ we have an explicit relation (due to Arnol'd)

$$f_i \wedge f_j = \sum_k c_k b_k \wedge \alpha_k \text{ with } c_k \in \mathbb{Q},\ b_k \in \Omega_n^{\mathrm{Base}},\ \alpha_k \in \Omega_n^{\mathrm{Fiber}}.$$

W.r.t these relations we define $\rho_i[f_1|...|f_m] = \sum_k c_k b_k \otimes [f_1|...|f_{i-1}|\lambda^{-1}\alpha_k|f_{i+2}|...|f_m]$ where the pair $f_i,\ f_{i+1}$ is replaced by the r.h.s. of their Arnol'd relation.
Let $[a_1|...|a_m]$ be a hyperlogarithm with all $a_i \in \Omega_n^{\mathrm{Hyp}}$. The **symbol map** is recursively computed by ρ:

$$\psi[a_1|...|a_m] = \lambda a_1 \sqcup \psi[a_2|...|a_m] - \sum_{1 \le i < m} \sqcup(\mathrm{id} \otimes \psi)\rho_i[a_1|...|a_m].$$

The procedure of taking **primitives** involves similar steps.

Properties of $\mathcal{B}(\Omega_n)$ (Brown '05):

- They are well-defined functions of n variables, corresponding to end-points of paths.
- On these functions, functional relations are algebraic identities.
- They can be decomposed to an explicit basis.
- $\mathcal{B}(\Omega_n)$ is closed under taking primitives.
- Let $\mathcal{Z}$ be the $\mathbb{Q}$-vector space of multiple zeta values. The limits at 0 and 1 of functions in $\mathcal{B}(\Omega_n)$ are $\mathcal{Z}$-linear combinations of elements in $\mathcal{B}(\Omega_{n-1})$.

Consequence: We can integrate over these functions from 0 to 1.

Integration strategy for a Feynman parameter x_j :

In Feynman parameters: $\int_0^\infty dx_m \dots \int_0^\infty dx_j \frac{(\prod Q_i) I(\{P_i\})}{\prod P_i}$ with all P_i **linear in** x_j;

In cubical coordinates: $\int_0^\infty dx_1 \dots \int_0^1 dt_n \sum_j f_j \beta_j, \quad \beta \in \mathcal{B}(\Omega_n)$,

f_j having denominators in $\left\{ t_1, \dots, t_n, \prod_{a \le i \le b} t_i - 1 \right\}$,

integrate here over t_n (i.e. over the x_j dependence)

In Feynman parameters: $\int_0^\infty dx_m \dots \int_0^\infty dx_{j+1} \frac{(\prod Q_i') I(\{P_i'\})}{\prod P_i'}$

We can continue if there is a **next Feynman parameter** x_{j+1} in which all polynomials of the **new set** $\{P_i'\}$ are **linear**. When is this the case?

Which are the new polynomials P_i'?

Example:

Start with the set of polynomials $\{P_1, P_2\}$: $P_1 = A_1 x_j + B_1$, $P_2 = A_2 x_j + B_2$,

$$\int_0^\infty \frac{1}{P_1 P_2} dx_j = \int_0^\infty \frac{1}{(A_1 x_j + B_1)(A_2 x_j + B_2)} dx_j$$

$$= \int_0^\infty \frac{A_1}{(A_1 B_2 - B_1 A_2)(A_1 x_j + B_1)} dx_j - \int_0^\infty \frac{A_2}{(A_1 B_2 - B_1 A_2)(A_2 x_j + B_2)} dx_j$$

$$= \frac{\ln A_1 - \ln A_2 - \ln B_1 + \ln B_2}{A_1 B_2 - B_1 A_2}$$

New set: $\{A_1, B_1, A_2, B_2, A_1 B_2 - B_1 A_2\}$

Linear reducibility

Linear reduction algorithm (Brown '08)

- If the polynomials $S = \{P_1, \ldots, P_m\}$ are linear in a Feynman parameter x_{r_1}, consider:
$$P_i = A_i x_{r_1} + B_i,\ A_i = \frac{\partial P_i}{x_{r_1}},\ h_i = B_i|_{x_{r_1}=0}$$
- $S_{(r_1)} =$ irreducible factors of $\{A_i\}_{1\le i\le n}$, $\{B_i\}_{1\le i\le n}$, $\left\{B_i A_j - A_i B_j\right\}_{1\le i<j\le n}$
- iterate for a sequence $(x_{r_1}, x_{r_2}, \ldots, x_{r_n}) \Rightarrow S_{(r_1)}, S_{(r_1, r_2)}, \ldots, S_{(r_1, \ldots, r_n)}$
- take intersections like: $S_{[r_1, r_2]} = S_{(r_1, r_2)} \cap S_{(r_2, r_1)}, \ldots$

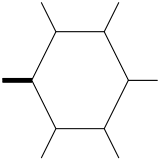
$$x_{r_1}, x_{r_2}, \ldots, x_{r_n} \quad \Rightarrow \quad S_{(r_1)}, S_{[r_1, r_2]}, \ldots, S_{[r_1, \ldots, r_n]}$$

- $S = \{P_1, \ldots, P_m\}$ is **linearly reducible** if for all $1 \le k \le n$ every polynomial in $S_{[r_1, \ldots, r_k]}$ is linear in $x_{r_{k+1}}$.,
- If $S = \{\mathcal{U}_G, \mathcal{F}_G\}$ is linearly reducible we call the **Feynman graph** G linearly reducible.

Some linearly reducible (massless) Feynman graphs :

- all vacuum graphs with vertex width 3 $\Rightarrow$ corresponding propagator-type graphs (Brown '09)
- all two-loop graphs with four on-shell legs (and many with three- and four loops) (CB, Lueders, '13)
- all minors of linearly reducible graphs (Brown '09, CB, Lueders, '13)
- all propagator-type graphs with ≤ 4 loops (Panzer '13)
- all graphs with three off-shell legs and ≤ 3 loops (Panzer '14)
- all graphs with vertex width 3 with three off-shell legs (Panzer PhD thesis)
- all ladder-shaped graphs with four off-shell legs (Panzer PhD thesis)

Applications
1) Parametric Feynman integrals:



As an example consider the **one-loop hexagon** integral in $D = 6$ dimensions with on-shell conditions $p_1^2 = m^2$, $p_i^2 = 0$, $i = 2, \ldots, 6$ to the external momenta:

$$I = \int_{x_i \geq 0} \prod_{i=1}^{6} dx_i \delta\left(1 - x_6\right) \frac{2}{\mathcal{F}^3},$$

$$\mathcal{F} = \sum_{i,\, j=0,\, i<j} x_i x_j (-s_{ij}^2), \text{ where } s_{ij} = \sum_{k=i}^{j-1} p_k.$$

Del-Duca, Duhr and Smirnov (2011) computed the integral, after a simplification to

$$I = \frac{1}{s_{14}^2 s_{25}^2 s_{36}^2} \int_{x_i \geq 0} \frac{\prod_{i=1}^3 dx_i}{(u_2 + x_1 + x_2)(u_3 x_1 + u_1 x_3 + x_2)(u_4 x_1 x_2 + x_2 + x_1 x_3 + x_3)}$$

using cross-ratios

$$u_1 = \frac{s_{26}^2 s_{35}^2}{s_{25}^2 s_{36}^2},\ u_2 = \frac{s_{13}^2 s_{46}^2}{s_{36}^2 s_{14}^2},\ u_3 = \frac{s_{15}^2 s_{24}^2}{s_{14}^2 s_{25}^2},\ u_4 = \frac{s_{12}^2 s_{36}^2}{s_{13}^2 s_{26}^2}.$$

We introduce new variables u, v, x, y by

$u_1 = \frac{1}{1+y}$, $u_2 = \frac{1+v}{1+v-u}$, $u_3 = \frac{(1-u)(-y-x)}{(1+y)(-1+u-v)}$, $u_4 = \frac{1+v-x}{1+v}$.

With this choice, the limit of each u_i at a tangential base-point corresponding to the ordering $(x_2, x_3, x_1, u, v, x, y)$ is 1.

$\Rightarrow$ We can integrate out x_2, x_3, x_1.

Our result agrees with the program by Panzer (2014).

2) Expansion of generalized hypergeometric functions

- **Gaussian hypergeometric function:**

  ${}_2F_1(a,\, b;\, c;\, z) = \sum_{m\geq 0} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}$

  for $|z| < 1$ or $|z| = 1$ and $\mathrm{Re}(c-a-b) > 0$; with Pochhammer-symbol $(x)_y = \frac{\Gamma(x+y)}{\Gamma(x)}$

- **Generalized hypergeometric functions:**

  ${}_pF_q(a_1,\, ...,\, a_p;\, b_1,\, ...,\, b_q;\, z) = \sum_{m\geq 0} \frac{\prod_{i=1}^{p}(a_i)_m}{\prod_{j=1}^{q}(b_j)_m} \frac{z^m}{m!}$

  for $q \geq p$ or $q = p-1$ and ($|z| < 1$ or $|z| = 1$ and $\mathrm{Re}\left(\sum_{i=1}^{p-1} b_i - \sum_{i=1}^{p} a_i\right) > 0$

- **Appell functions**:In

  ${}_2F_1(a,\, b;\, c;\, x) \cdot {}_2F_1(a',\, b';\, c';\, y) = \sum_{m\geq 0}\sum_{n\geq 0} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_m (c')_n} \frac{x^m y^n}{m!n!}$

  replace terms like $(a)_m(a')_n$ by $(a)_{m+n}$ to obtain

  $F_1(a;\, b,\, b';\, c;\, x,\, y) = \sum_{m\geq 0}\sum_{n\geq 0} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m!n!},\ |x|,\, |y| < 1,$

  $F_2(a;\, b,\, b';\, c,\, c';\, x,\, y) = \sum_{m\geq 0}\sum_{n\geq 0} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n} \frac{x^m y^n}{m!n!},\ |x| + |y| < 1,$

  $F_3(a,\, a';\, b,\, b';\, c;\, x,\, y) = \sum_{m\geq 0}\sum_{n\geq 0} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}} \frac{x^m y^n}{m!n!},\ |x|,\, |y| < 1,$

  $F_4(a;\, b;\, c,\, c';\, x,\, y) = \sum_{m\geq 0}\sum_{n\geq 0} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n} \frac{x^m y^n}{m!n!},\ |x|^{\frac{1}{2}},\, |y|^{\frac{1}{2}} < 1,$

- Horn functions, Lauricella functions, Kampé de Feriét functions, ...

Hypergeometric-functions-approach:

- **Step 1:** Express the Feynman integral by hypergeometric functions, e.g. using the Mellin-Barnes approach.

  $\Rightarrow$ The hypergeometric functions depend on the regularization parameter ϵ.

  E.g. in ${}_pF_q(a_1, ..., a_p, b_1, ..., b_q; z)$ the a_i and b_i are of the form

$$\lambda_j + \epsilon\rho_j$$

  - massless case: all λ_j are integers
  - massive case: some λ_j are half-integers

- **Step 2:** Use differential properties to reduce hyp. fct. by lowering a_i and b_i by integers (e.g using `HYPERDIRE` by Bytev, Kalmykov, Kniehl, Moch).

- **Step 3:** Expansion of the hypergeometric functions at $\epsilon = 0$.

Solutions to the expansion problem:

- Moch, Uwer, Weinzierl '02: Use of nested sums as

$$Z(n;\, m_1,\, ...,\, m_k;\, x_1,\, ...,\, x_k) = \sum_{n \geq i_1 > \ldots > i_k > 0} \frac{x_1^{i_1}}{i_1^{m_1}} \cdots \frac{x_k^{i_k}}{i_k^{m_k}}$$

  with $Z(\infty;\, m_1,\, ...,\, m_k;\, x_1,\, ...,\, x_k) = Li_{m_k,\, ...,\, m_1}(x_k,\, ...,\, x_1)$

  for the expansion of four types of sums called A, B, C, D. (programs: `xsummer` (Moch, Uwer '05), `nestedsums` (Weinzierl '02))

  Examples: The generalized hypergeometric functions ${}_pF_{p-1}$ are covered by type A: $\sum_{i=1}^{n} \frac{x^i}{(i+c)^m} \frac{\Gamma(i+a_1+b_1\epsilon)}{\Gamma(i+c_1+d_1\epsilon)} \cdots \frac{\Gamma(i+a_k+b_k\epsilon)}{\Gamma(i+c_k+d_k\epsilon)} Z(i+o-1, m_1, \ldots, m_l, x_1, \ldots, x_l),\ a_j, c_j, o \in \mathbb{Z};\ c \in \mathbb{N}$
  The Appell function F_2 requires the combination of all four algorithms A, B, C, D.

- Huber, Maitre '05: Combination of nested sums with an **integral-approach** for ${}_2F_1$. (programs: `HypExp`, `HypExp2`)

Expansion by use of integral representations:

$$ {}_2F_1(a,\ b;\ c;\ z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt $$

for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $|\arg(1-z)|\pi$

Example:

$$ {}_2F_1(1,\ 1+\epsilon;\ 3+\epsilon;\ z) = \frac{\Gamma(3+\epsilon)}{\Gamma(\epsilon+1)} \int_0^1 \frac{t^{\epsilon}(1-t)}{1-tz} dt $$

$$ = \int_0^1 \frac{2(t-1)}{tz-1} dt + \epsilon \int_0^1 \frac{(3+2\ln t)(t-1)}{tz-1} dt + \epsilon^2 \int_0^1 \frac{(1+3\ln t+\ln^2 t)(t-1)}{tz-1} dt $$

$$ +\epsilon^3 \int_0^1 \frac{(9\ln t+2\ln^2 t+6)(t-1)\ln t}{6(tz-1)} dt + \mathcal{O}(\epsilon^4) $$

$$ = \frac{1}{z^2}(2z + 2(1-z)\ln(1-z)) + \epsilon \frac{1}{z^2}\left(z + 3(1-z)\ln(1-z) + 2(1-z)\mathrm{Li}_2(z)\right) $$

$$ +\epsilon^2 \frac{1}{z^2}(1-z)\left(\ln(1-z) + 3\mathrm{Li}_2(z) - 2\mathrm{Li}_3(z)\right) $$

$$ +\epsilon^3 \frac{1}{z^2}(1-z)\left(\mathrm{Li}_2(z) + 3\mathrm{Li}_3(z) + 2\mathrm{Li}_4(z)\right) + \mathcal{O}(\epsilon^4) $$ (integr. with MPL, checked with HypExp)

Integral representations of **generalized hypergeometric functions**:

$$
{}_pF_q(a_1,\ldots;\, b_1,\ldots;\, z)= \frac{\Gamma(b_q)}{\Gamma(a_p)\Gamma(b_q-a_p)}\int_0^1 t^{a_p-1}(1-t)^{b_q-a_p-1}\,{}_{p-1}F_{q-1}(a_1,\ldots;\, b_1,\ldots;\, zt)dt
$$

for $\mathrm{Re}(b_q) > \mathrm{Re}(a_p) > 0$ and ($p \leq q$ or $p = q+1$ and $|\arg(1-z)| < \pi$)

Appell functions:

$$
F_1(a;\, b,\, b';\, c;\, x,\, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 \frac{u^{a-1}(1-u)^{c-a-1}}{(1-ux)^b(1-uy)^{b'}}du,\ \mathrm{Re}(c) > \mathrm{Re}(a) > 0,
$$

$$
F_2(a;\, b,\, b';\, c,\, c';\, x,\, y) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')}\int_0^1\int_0^1 u^{b-1}v^{b'-1}\frac{(1-u)^{c-b'-1}(1-v)^{c'-b'-1}}{(1-ux-vy)^a}du\,dv,
$$

$$
F_3(a,\, a';\, b,\, b';\, c';\, x,\, y) = \frac{\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}\int\int_{u,v\geq 0,\ u+v\leq 1} u^{b-1}v^{b'-1}\frac{(1-u-v)^{c-b-b'}}{(1-ux)^a(1-vy)^{a'}}dudv,
$$

$$
F_4(a;\, b;\, c,\, c';\, x(1-y),\, y(1-x)) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c'-b)} \times\int_0^1\int_0^1 u^{a-1}v^{b-1}\frac{(1-u)^{c-a-1}(1-v)^{c'-b-1}}{(1-ux)^b(1-vy)^a}\left(1-\frac{uvxy}{(1-ux)(1-vy)}\right)^{c+c'-a-b-1}du\,dv
$$

Example for ${}_{p+1}F_p$:

$${}_3F_2(2,\, 1+\epsilon,\, 1+\epsilon;\, 3+\epsilon,\, 2+\epsilon;\, z) = \frac{\Gamma(3+\epsilon)\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)^2} \int_0^1 \int_0^1 \frac{x(1-x)^{\epsilon} y^{\epsilon}}{(1-xyz)^{1+\epsilon}} dx\, dy$$

$$= -\int_0^1 \int_0^1 \frac{2x}{2xy-1} dx\, dy + \epsilon \int_0^1 \int_0^1 \frac{x(2\ln(1-xyz)-2\ln(1-x)-2\ln y-5}{xyz-1} dx\, dy$$

$$-\epsilon^2 \int_0^1 \int_0^1 \frac{x}{xyz-1} \left(\ln^2(1-xyz) + \ln^2(1-x) + \ln^2(y) + 4 + 5\ln(y) + 2\ln(1-x)\ln(y)\right.$$

$$\left. +5\ln(1-x) - 2\ln(1-xyz)\ln(1-x) - 2\ln(1-xyz)\ln(y) - 5\ln(1-xyz)\right) dx\, dy$$

$$+\mathcal{O}(\epsilon^3)$$

$$= \frac{2}{z^2}\left(z + (1-z)\ln(1-z)\right) + \epsilon \frac{1}{z^2}\left(5z + 7(1-z)\ln(1-z) + 2\mathrm{Li}_2(z) - 4z\mathrm{Li}_2(z)\right)$$

$$+\epsilon^2 \frac{1}{z^2}\left(4z + 9(1-z)\ln(1-z) + (7-12z)\mathrm{Li}_2(z) - (2-6z)\mathrm{Li}_3(z)\right) + \mathcal{O}(\epsilon^3)$$

(integrated with `MPL`, checked with `HypExp`)

Example for Appell F_1:

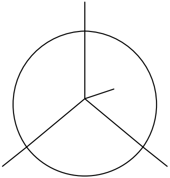
$$
\begin{aligned}
F_1(a;\, b_1,\, b_2;\, c;\, x,\, y) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tx)^{-b_1}(1-ty)^{-b_2}\, dt \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t_3^{a-1}(1-t_3)^{c-a-1}(1-t_1 t_2 t_3)^{-b_1}(1-t_2 t_3)^{-b_2}
\end{aligned}
$$

is in the appropriate form after introducing the variables $t_1 = x/y,\ t_2 = y,\ t_3 = t$.
As an example we compute

$$
\begin{aligned}
F_1(a;\, b_1,\, b_2;\, c;\, x,\, y) &= \frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)} \int_0^1 \frac{(1-z_3)^{\epsilon}}{(1-z_1 z_2 z_3)(1-z_2 z_3)} dz_3 \\
&= \frac{1}{x-y}\left(\ln(1-y) - \ln(1-x)\right) \\
&\quad + \frac{\epsilon}{x-y}\left(\ln(1-y) - \ln(1-x) + \frac{1}{2}\ln(1-y)^2 - \frac{1}{2}\ln(1-x)^2\right. \\
&\quad \left.\mathrm{Li}_2(x) + \mathrm{Li}_2(y)\right) + \mathcal{O}\left(\epsilon^2\right)
\end{aligned}
$$

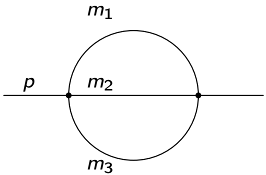
Outlook: Beyond multiple polylogarithms

A forbidden minor: K_4 with four on-shell legs (CB, Lüders '13)



- J. Henn, A. Smirnov, V. Smirnov 2013, using the differential equations approach: Evaluation of the K_4 up to functions of weight six in the ϵ-expansion in terms of harmonic polylogarithms,
- E. Panzer 2014: a change of variables linearizing the polynomials at the critical step $\Rightarrow$ integration over Feynman parameters $\Rightarrow$ evaluation in terms of hyperlogarithms to any order

The massive two-loop sunrise integral (finite in $D = 2$ dimensions)



$$S\left(p^2,\, m_1,\, m_2,\, m_3\right) = \int_\sigma \frac{x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2}{\mathcal{F}}$$

with $\sigma = \{[x_1 : x_2 : x_3] \in \mathbb{P}^2 | x_i \geq 0,\ i = 1,\, 2,\, 3\}$

and the Second Symanzik polynomial:

$$\mathcal{F} = -x_1 x_2 x_3 p^2 + (x_1 x_2 + x_2 x_3 + x_1 x_3)(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2)$$

linearly irreducible, defining an elliptic curve

- Case $m_1 = m_2 = m_3$:
  - Broadhurst, Fleischer, Tarasov (1993): **second order differential equation**
  - Groote, Pivovarov (2000), Laporta, Remiddi (2004): **elliptic integrals**
  - Bloch, Vanhove (2013): **elliptic dilogarithm**
- Case of arbitrary masses:
  - Berends, Buza, Böhm, Scharf (1994): **Lauricella functions**
  - Müller-Stach, Weinzierl, Zayadeh (2012): **second order differential equation**
  - Adams, CB, Weinzierl (2013): **elliptic integrals**
  - Adams, CB, Weinzierl (2014): **elliptic dilogarithm**

Adams, CB, Weinzierl (2014):
With functions

$$\mathrm{ELi}_{n;\,m}(x;\,y;\,q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk},$$

$$E_{2;\,0}(x;\,y;\,q) = \frac{1}{i}\left(\frac{1}{2}\mathrm{Li}_2(x) - \frac{1}{2}\mathrm{Li}_2(x^{-1}) + \mathrm{ELi}_{2;\,0}(x;\,y;\,q) - \mathrm{ELi}_{2;\,0}(x^{-1};\,y;\,q)\right)$$

the result for arbitrary masses takes the form

$$S\left(p^2,\,m_1,\,m_2,\,m_3\right) = \frac{\psi(q)}{\pi}\sum_{i=1}^{3}\mathrm{ELi}_{2;\,0}(w_i(q);\,-1;\,-q)$$

while for equal masses

$$S\left(p^2,\,m\right) = 3\frac{\psi(q)}{\pi}\mathrm{ELi}_{2;\,0}(r_3;\,-1;\,-q), \text{ with } r_3 = e^{\frac{2\pi i}{q}}.$$

Here $\psi(q)$ solves the homogeneous differential equation (complete elliptic integral), and w_i are functions of q, m_1, m_2, m_3 determined by transformations on intersection points of the elliptic curve with σ.

More details: see talk by Luise Adams

Conclusions:

- Multiple polylogarithms in several variables are homotopy invariant iterated integrals with particularly good properties. They are useful for the computation of Feynman integrals by integrating over Feynman parameters.

- The expansion of (generalized) hypergeometric functions is a further application of the integration program for multiple polylogarithms. The approach of expansion via integral represantions may extend the existing approaches.
- The two-loop sunrise integral is an example for a case beyond multiple polylogarithms. For arbitrary particle masses, the integral can be expressed by integrals over elliptic integrals or - more interestin

Basic definitions:

Riemann zeta function: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$,

Multiple zeta values: $\zeta(s_1, \ldots, s_k) \sum_{n_1 > n_2 > \ldots > n_k \geq 1} \frac{1}{n_1^{s_1} \ldots n_k^{s_k}}$ for $s_2, \ldots, s_k > 0$; $s_1 \geq 2$

Expansion of the logarithm: $-\ln(1+z) = \sum_{n=1}^{\infty} \frac{(-z)^n}{n}$

Multiple polylogarithms:
$\mathrm{Li}_{(s_1, \ldots, s_k)}(z_1, \ldots, z_k) = \sum_{n_1 > n_2 > \ldots > n_k \geq 1} \frac{z_1^{n_1} \ldots z_k^{n_k}}{n_1^{s_1} \ldots n_k^{s_k}}_{n_1}$, $s_i \geq 1$, $|z_i| < 1$

Euler's Gamma-function: $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ for $\mathrm{Re}(x) > 0$

Multiple polylogarithms in several variables

Let

- k be a field (either $\mathbb{R}$ or $\mathbb{C}$),
- M a smooth manifold over k,
- $\gamma : [0, 1] \to M$ a smooth path on M,
- ω_1, ..., ω_n smooth differential 1-forms on M,
- $\gamma^\star(\omega_i) = f_i(t)dt$ the pull-back of ω_i to $[0, 1]$

Def.: The *iterated integral* of ω_1, ..., ω_n along γ is

$$\int_\gamma \omega_n ... \omega_1 = \int_{0 \leq t_1 \leq ... \leq t_n \leq 1} f_n(t_n)dt_n ... f_1(t_1)dt_1.$$

We use the term *iterated integral* for k-linear combinations of such integrals.

From this Ω_n we want to construct iterated integrals which are *homotopy invariant*, i.e.

$$\int_{\gamma_1} \omega_n ... \omega_1 = \int_{\gamma_2} \omega_n ... \omega_1 \; \textit{for homotopic paths } \gamma_1, \gamma_2.$$

Consider tensor products $\omega_1 \otimes ... \otimes \omega_m \equiv [\omega_1|...|\omega_m]$ over $\mathbb{Q}$.

Define an operator D by

$$D\left([\omega_1|...|\omega_m]\right) = \sum_{i=1}^{m} [\omega_1|...|\omega_{i-1}|d\omega_i|\omega_{i+1}|...\omega_m] + \sum_{i=1}^{m-1} [\omega_1|...|\omega_{i-1}|\omega_i \wedge \omega_{i+1}|...|\omega_m].$$

Def.: A $\mathbb{Q}-$linear combination of tensor products

$$\xi = \sum_{l=0}^{m} \sum_{i_1, \dots, i_l} c_{i_1, \dots, i_l} [\omega_{i_1}|...|\omega_{i_l}], \; c_{i_1, \dots, i_l} \in \mathbb{Q}$$

is called *integrable word* if

$$D(\xi) = 0.$$

Consider the integration map

$$\sum_{l=0}^{m} \sum_{i_1, \dots, i_l} c_{i_1, \dots, i_l} [\omega_{i_1} | \dots | \omega_{i_l}] \mapsto \sum_{l=0}^{m} \sum_{i_1, \dots, i_l} c_{i_1, \dots, i_l} \int_\gamma \omega_{i_1} \dots \omega_{i_l}$$

Theorem (Chen '77): Under certain conditions on Ω this map is an isomorphism from *integrable words* to *homotopy invariant iterated integrals*.

Our class of homotopy invariant functions:

- Construct the integrable words of 1-forms in Ω_n.

  (for an explicit construction see CB, Brown '12
  and cf. Duhr, Gangl, Rhodes '11, Goncharov et al '10)

- By the integration map obtain the set of **multiple polylogarithms in several variables** $\mathcal{B}(\Omega_n)$.

Integration procedure for a Feynman parameter x_j :

- Given: Integrand $\frac{\sum \{Q\} \cdot I(\{P\})}{\{P\}}$ with Q, P polynomials in Feynman parameters, all P linear in x_j and $I(\{P\})$ iterated integrals with differential forms $\frac{dP}{P}$
- Let $\{P\} = \{A(x_j)\} \cup \{B\}$ where all $A(x_j)$ depend on x_j and all B do not. By a reverse shuffle we factor $I(\{P\}) = I'(\{B\}) \cdot I''\left(\{A(x_j)\}\right)$.
- Factor out "trailing zeroes": $I''\left(\{A(x_j)\}\right) = \sum \ln^k(x_j) \cdot I'''\left(\{A(x_j)\}\right)$ such that no $I'''\left(\{A(x_j)\}\right)$ begins with $\frac{dx_j}{x_j}$
- For n polynomials in $\{A(x_j)\}$ introduce n cubical coordinates $t_1, \ldots, t_n$ as rational functions in the x_i such that:
  - each form is replaced by $\omega \in \Omega_n^{\mathrm{Hyp}}$ and forms independent of x_j
  - each point where all $0 \leq t_i \leq 1$ corresponds to a point where all $x_i \geq 0$.
- Integration $\int_0^1 dt_n \ldots$: a) Primitives by concatenation and "symbol map" $\Rightarrow$ iterated integrals in $\mathcal{B}(\Omega)$. b) Limits at $t_n = 0$ and $t_n = 1$.
- Back to Feynman parameters, introducing integration constants due to different vanishing conditions of the $x-$ and $t-$integrals.

A well known **functional equation** is the five-term-relation:

$$-\mathrm{Li}_2\left(\frac{1-y}{1-\frac{1}{x}}\right)-\mathrm{Li}_2\left(\frac{1-x}{1-\frac{1}{y}}\right)+\mathrm{Li}_2(xy)-\mathrm{Li}_2(x)-\mathrm{Li}_2(y)=\frac{1}{2}\ln^2(1-x)+\frac{1}{2}\ln^2(1-y)$$

Writing each function as iterated integral on the total space (using ψ), the relation becomes obvious:

$$\mathrm{Li}_2\left(\frac{1-y}{1-\frac{1}{x}}\right)=\left[\frac{dx}{x}+\frac{dx}{1-x}-\frac{dy}{1-y}\Big|\frac{xdy+ydx}{1-xy}\right]-\left[\frac{dx}{1-x}\Big|\frac{dy}{1-y}\right]-\left[\frac{dx}{x}+\frac{dx}{1-x}\Big|\frac{dx}{1-x}\right]$$

$$\mathrm{Li}_2\left(\frac{1-x}{1-\frac{1}{y}}\right)=\left[\frac{dy}{y}+\frac{dy}{1-y}-\frac{dx}{1-x}\Big|\frac{xdy+ydx}{1-xy}\right]+\left[\frac{dx}{1-x}\Big|\frac{dy}{1-y}\right]-\left[\frac{dy}{y}+\frac{dy}{1-y}\Big|\frac{dy}{1-y}\right]$$

$$\mathrm{Li}_2(xy)=\left[\frac{dx}{x}+\frac{dy}{y}\Big|\frac{xdy+ydx}{1-xy}\right],\ \mathrm{Li}_2(x)=\left[\frac{dx}{x}\Big|\frac{dx}{1-x}\right],\ \mathrm{Li}_2(y)=\left[\frac{dy}{y}\Big|\frac{dy}{1-y}\right]$$

Example 1: Vacuum graphs with $\nu = 2L$ and $D = 4$:

$$I_G = \int_{x_j \geq 0} \left(\prod_{i=1}^{N} dx_i x_i^{\nu_i - 1} \right) \delta \left(1 - \sum_{i=1}^{N} x_i \right) \frac{1}{\mathcal{U}_G^2}$$

Example 2: Sunrise graph with $\nu = L + 1$ and $D = 2$:

$$I_G(\Lambda_G) = \int_{x_j \geq 0} \left(\prod_{i=1}^{N} dx_i x_i^{\nu_i - 1} \right) \delta \left(1 - \sum_{i=1}^{N} x_i \right) \frac{1}{\mathcal{F}_G \left(\Lambda_G \right)}$$

From this Ω_n we want to construct iterated integrals which are *homotopy invariant*.

Def.: Smooth paths γ_1, γ_2 on M are *homotopic* if their end-points coincide (i.e. $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$) and γ_1 can be continuously transformed into γ_2.

Def.: An iterated integral is called *homotopy invariant* if it satisfies

$$\int_{\gamma_1} \omega_n ... \omega_1 = \int_{\gamma_2} \omega_n ... \omega_1$$

for homotopic paths γ_1, γ_2.

By such integrals we obtain function of variables given only by the end-points of paths.

For e an edge of G consider the **deletion** ($G\backslash e$) and **contraction** ($G//e$) of e

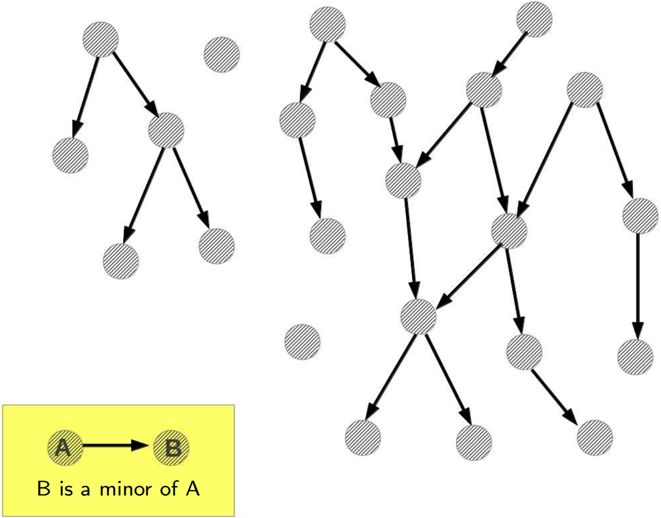
The deletion and contraction of different edges is **commutative**.

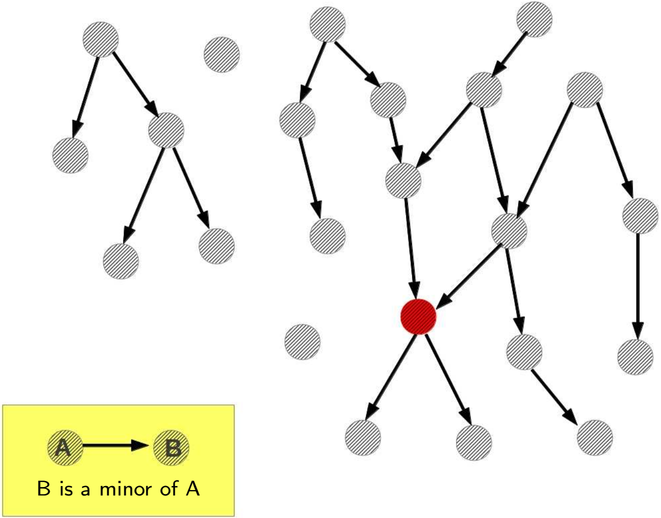
$\Rightarrow$If C, D are disjoint sets of edges of G then $G\backslash D//C$ is a unique graph.

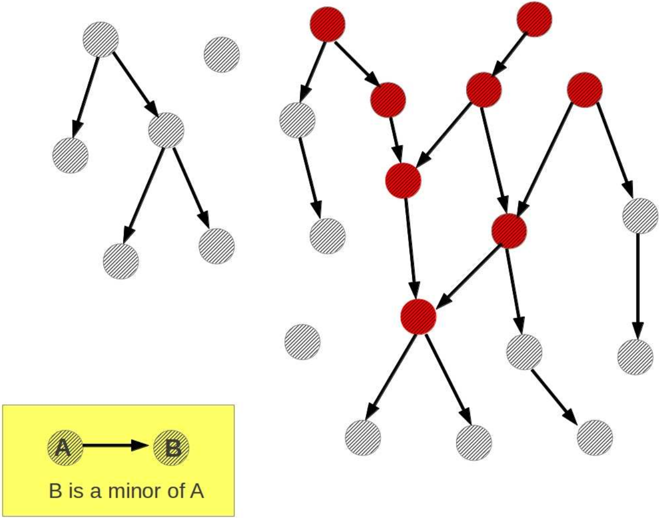
Any such graph is called **minor** of G.

Def.: A set $\mathcal{G}$ of graphs is called **minor-closed** if for each $G \in \mathcal{G}$ all minors belong to $\mathcal{G}$ as well.

Example: The set of all planar graphs is minor-closed.

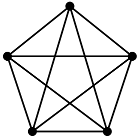

A
B
B is a minor of A

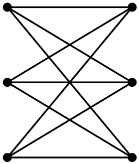

A
B
B is a minor of A


A
B
B is a minor of A

Theorem (Robertson and Seymour): Any minor-closed set of graphs can be defined by a finite set of graphs which are **not** in the set (so-called *forbidden minors*).

Example:
The set of planar graphs is the set of all graphs which have neither K_5 nor $K_{3,\,3}$ as a minor. (Wagner's theorem)





Theorem (Brown '09, CB and Lüders '13)

The set of linearly reducible Feynman graphs is minor-closed.

$\Rightarrow$ Search for the forbidden minors!

Case study by M. Lüders:

- Let Λ be the set of massless Feynman graphs with four on-shell legs. (On-shell condition: $p_i^2 = 0$, $i = 1, \ldots, 4$)
- At two loops all graphs are linearly reducible.
- First forbidden minors at three loops.