Combinatorial Dyson-Schwinger equations and systems I

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To a given QFT is attached a family of graphs.

Feynman graphs

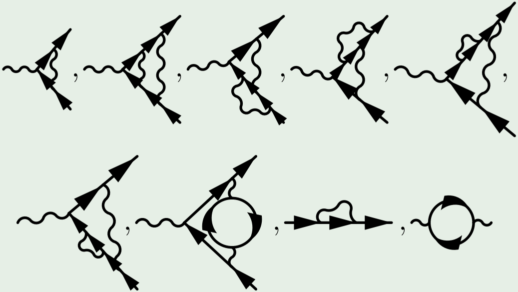
1. A finite number of possible half-edges.
2. A finite number of possible vertices.
3. A finite number of possible external half-edges (external structure).
4. The graph is connected and 1-PI.

To each external structure is associated a formal series in the Feynman graphs.

In QED

1. Half-edges:  (electron),  (photon).
2. Vertices: .
3. External structures: , , .

Examples in QED



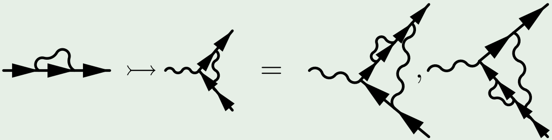
Subgraphs and contraction

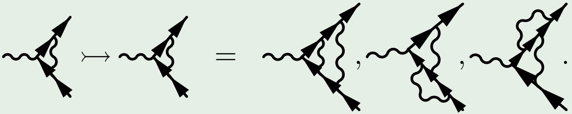
1. A subgraph of a Feynman graph Γ is a subset γ of the set of half-edges Γ such that γ and the vertices of Γ with all half edges in γ is itself a Feynman graph.
2. If Γ is a Feynman graph and $\gamma_1, \ldots, \gamma_k$ are disjoint subgraphs of Γ, $\Gamma/\gamma_1 \ldots \gamma_k$ is the Feynman graph obtained by contraction of $\gamma_1, \ldots, \gamma_k$.

Insertion

Let Γ_1 and Γ_2 be two Feynman graphs. According to the external structure of Γ_1, you can replace a vertex or an edge of Γ_2 by Γ_1 in order to obtain a new Feynman graph.

Examples in QED

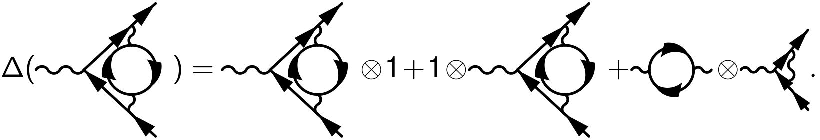




Construction

Let H_{FG} be a free commutative algebra generated by the set of Feynman graphs. It is given a coproduct: for all Feynman graph Γ,

$$\Delta(\Gamma) = \sum_{\gamma_1 \ldots \gamma_k \subseteq \Gamma} \gamma_1 \ldots \gamma_k \otimes \Gamma/\gamma_1 \ldots \gamma_k.$$



The Hopf algebra H_{FG} is graded by the number of loops:

$$|\Gamma| = \sharp E(\Gamma) - \sharp V(\Gamma) + 1.$$

Because of the 1-PI condition, it is connected, that is to say $(H_{FG})_0 = K1_{H_{FG}}$. What is its dual?

Cartier-Quillen-Milnor-Moore theorem

Let H be a cocommutative, graded, connected Hopf algebra over a field of characteristic zero. Then it is the enveloping algebra of its primitive elements.

This theorem can be applied to the graded dual of H_{FG}.

Primitive elements of H_{FG}^*

- Basis of primitive elements: for any Feynman graph Γ,

$$f_\Gamma(\gamma_1 \ldots \gamma_k) = \sharp Aut(\Gamma)\delta_{\gamma_1\ldots\gamma_k,\Gamma}.$$

- The Lie bracket is given by:

$$[f_{\Gamma_1}, f_{\Gamma_2}] = \sum_{\Gamma=\Gamma_1 \rightarrowtail \Gamma_2} f_\Gamma - \sum_{\Gamma=\Gamma_2 \rightarrowtail \Gamma_1} f_\Gamma.$$

We define:

$$f_{\Gamma_1} \circ f_{\Gamma_2} = \sum_{\Gamma = \Gamma_1 \rightarrowtail \Gamma_2} f_\Gamma.$$

The product $\circ$ is not associative, but satisfies:

$$f_1 \circ (f_2 \circ f_3) - (f_1 \circ f_2) \circ f_3 = f_2 \circ (f_1 \circ f_3) - (f_2 \circ f_1) \circ f_3.$$

It is (left) prelie.

In the context of QFT, we shall consider some special infinite sums of Feynman graphs:

Example in QED

$$= \sum_{n\geq 1} x^n \left(\sum_{\gamma \in (n)} s_\gamma \gamma \right).$$

$$= -\sum_{n\geq 1} x^n \left(\sum_{\gamma \in (n)} s_\gamma \gamma \right).$$

Example in QED

 $= -\sum_{n\geq 1} x^n \left(\sum_{\gamma \in X(n)} s_\gamma \gamma \right).$

where the index set $X(n)$ is written as $\gamma \in$  (n)

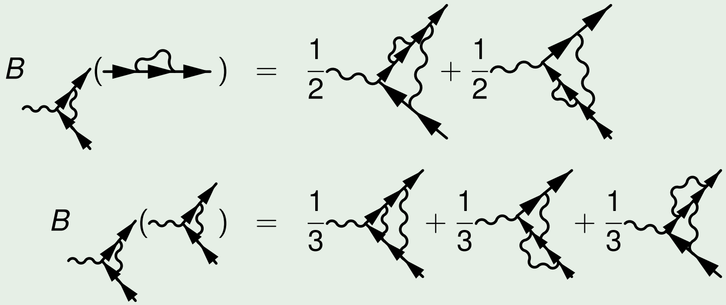
They live in the completion of H_{FG}.

How to describe these formal series?

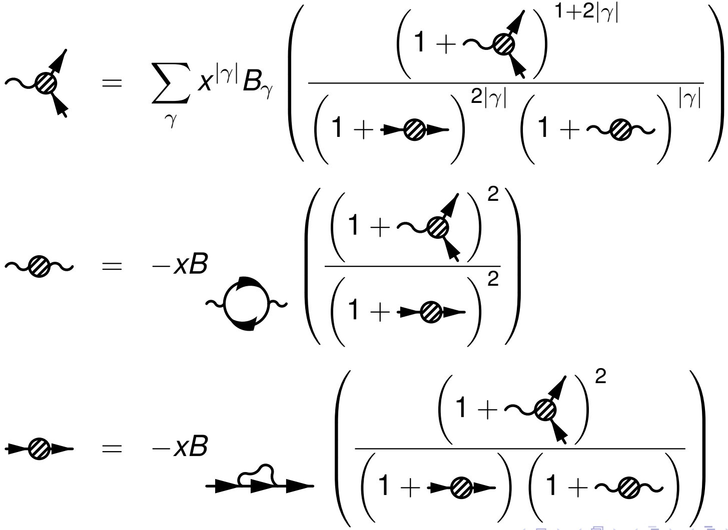
- For any primitive Feynman graph γ, one defines the insertion operator B_γ over H_{FG}. This operator associates to a graph G the sum (with symmetry coefficients) of the insertions of G into γ.
- The propagators then satisfy a system of equations involving the insertion operators, called systems of Dyson-Schwinger equations.

Example

In QED :



In QED:


$=$
$\sum_{\gamma} x^{|\gamma|} B_{\gamma}$
$1+$
$1+2|\gamma|$
$2|\gamma|$
$|\gamma|$
$= \quad -xB$
2
$= \quad -xB$
2

Other example (Bergbauer, Kreimer)

$$X = \sum_{\gamma \text{ primitive}} B_\gamma \left((1+X)^{|\gamma|+1} \right).$$

Question

For a given system of Dyson-Schwinger equations (S), is the subalgebra generated by the homogeneous components of (S) a Hopf subalgebra?

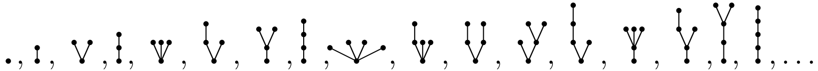
Proposition

The operators B_γ satisfy: for all $x \in H_{FG}$,

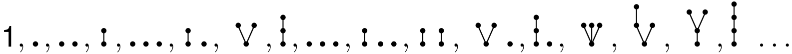
$$\Delta \circ B_\gamma(x) = B_\gamma(x) \otimes 1 + (Id \otimes B_\gamma) \circ \Delta(x).$$

This relation allows to lift any system of Dyson-Schwinger equation to the Hopf algebra of decorated rooted trees.

The Hopf algebra of rooted trees H_R (or Connes-Kreimer Hopf algebra) is the free commutative algebra generated by the set of rooted trees.



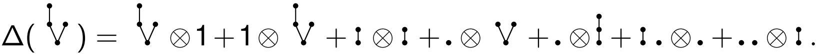
The set of rooted forests is a linear basis of H_R:



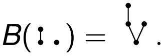
The coproduct is given by admissible cuts:

$$\Delta(t) = \sum_{c \text{ admissible cut}} P^c(t) \otimes R^c(t).$$

cut c									total
Admissible ?	yes	yes	yes	yes	no	yes	yes	no	yes
$W^c(t)$									
$R^c(t)$					$\times$			$\times$	1
$P^c(t)$	1				$\times$			$\times$	



The grafting operator of H_R is the map $B : H_R \longrightarrow H_R$, associating to a forest $t_1 \ldots t_n$ the tree obtained by grafting $t_1, \ldots, t_n$ on a common root. For example:



Proposition

For all $x \in H_R$:

$$\Delta \circ B(x) = B(x) \otimes 1 + (Id \otimes B) \circ \Delta(x).$$

So B is a 1-cocycle of H_R.

Universal property

Let A be a commutative Hopf algebra and let $L : A \longrightarrow A$ be a 1-cocycle of A. Then there exists a unique Hopf algebra morphism $\phi : H_R \longrightarrow A$ with $\phi \circ B = L \circ \phi$.

This will be generalized to the case of several 1-cocycles with the help of decorated rooted trees.

- H_R is graded by the number of vertices and B is homogeneous of degree 1.
- Let $Y = B_\gamma(f(Y))$ be a Dyson-Schwinger equation in a suitable Hopf algebra of Feynman graphs H_{FG}, such that $|\gamma| = 1$.
- There exists a Hopf algebra morphism $\phi : H_R \longrightarrow H_{FG}$, such that $\phi \circ B = B_\gamma \circ \phi$. This morphism is homogeneous of degree 0.
- Let X be the solution of $X = B(f(X))$. Then $\phi(X) = Y$ and for all $n \geq 1$, $\phi(X(n)) = Y(n)$.
- Consequently, if the subalgebra generated by the $X(n)$'s is Hopf, so is the subalgebra generated by the $Y(n)$'s.

Definition

Let $f(h) \in \mathbb{C}[[h]]$.

- The combinatorial Dyson-Schwinger equations associated to $f(h)$ is:
$$X = B(f(X)),$$
where X lives in the completion of H_R.
- This equation has a unique solution $X = \sum X(n)$, with:
$$\left\{ \begin{array}{rcl} X(1) & = & p_0 \bullet, \\ X(n+1) & = & \displaystyle\sum_{k=1}^{n} \sum_{a_1+\ldots+a_k=n} p_k B(X(a_1)\ldots X(a_k)), \end{array} \right.$$
where $f(h) = p_0 + p_1 h + p_2 h^2 + \ldots$

$X(1) = p_0$,

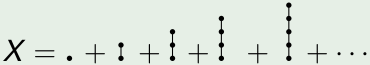
$X(2) = p_0 p_1$,

$X(3) = p_0 p_1^2$ $+ p_0^2 p_2$  ,

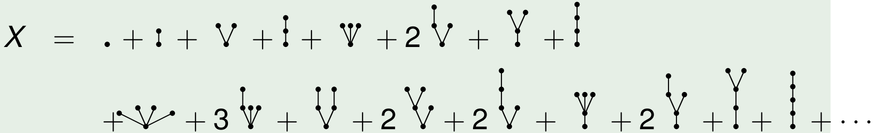
$X(4) = p_0 p_1^3$ $+ p_0^2 p_1 p_2$  $+ 2 p_0^2 p_1 p_2$  $+ p_0^3 p_3$  .

Examples

- If $f(h) = 1 + h$:



- If $f(h) = (1-h)^{-1}$:



Let $f(h) \in \mathbb{C}[[h]]$. The homogeneous components of the unique solution of the combinatorial Dyson-Schwinger equation associated to $f(h)$ generate a subalgebra of H_R denoted by H_f.

H_f is not always a Hopf subalgebra

For example, for $f(h) = 1 + h + h^2 + 2h^3 + \cdots$, then:

$X = . +$  $+$  $+$  $+ 2$  $+ 2$  $+$  $+$  $+ \cdots$

So:

$$\begin{aligned} \Delta(X(4)) &= X(4) \otimes 1 + 1 \otimes X(4) + (10X(1)^2 + 3X(2)) \otimes X(2) \\ &+ (X(1)^3 + 2X(1)X(2) + X(3)) \otimes X(1) \end{aligned}$$

$+X(1) \otimes (8$  $+ 5$  $)$.

If $f(0) = 0$, the unique solution of $X = B(f(X))$ is 0. From now, up to a normalization we shall assume that $f(0) = 1$.

Theorem

Let $f(h) \in \mathbb{C}[[h]]$, with $f(0) = 1$. The following assertions are equivalent:

1. H_f is a Hopf subalgebra of H_R.
2. There exists $(\alpha, \beta) \in \mathbb{C}^2$ such that $(1 - \alpha\beta h)f'(h) = \alpha f(h)$.
3. There exists $(\alpha, \beta) \in \mathbb{C}^2$ such that $f(h) = 1$ if $\alpha = 0$ or $f(h) = e^{\alpha h}$ if $\beta = 0$ or $f(h) = (1 - \alpha\beta h)^{-\frac{1}{\beta}}$ if $\alpha\beta \neq 0$.

$1 \Longrightarrow 2$. We put $f(h) = 1 + p_1 h + p_2 h^2 + \cdots$. Then $X(1) = \cdot\,\cdot$.
Let us write:

$$\Delta(X(n+1)) = X(n+1) \otimes 1 + 1 \otimes X(n+1) + X(1) \otimes Y(n) + \ldots.$$

1. By definition of the coproduct, $Y(n)$ is obtained by cutting a leaf in all possible ways in $X(n+1)$. So it is a linear span of trees of degree n.
2. As H_f is a Hopf subalgebra, $Y(n)$ belongs to H_f.

Hence, there exists a scalar λ_n such that $Y(n) = \lambda_n X_n$.

lemma

Let us write:

$$X = \sum_t a_t t.$$

For any rooted tree t:

$$\lambda_{|t|} a_t = \sum_{t'} n(t, t') a_{t'},$$

where $n(t, t')$ is the number of leaves of t' such that the cut of this leaf gives t.

We here assume that f is not constant. We can prove that $p_1 \neq 0$.

For t the ladder $(B)^n(1)$, we obtain:

$$p_1^{n-1}\lambda_n = 2(n-1)p_1^{n-2}p_2 + p_1^n.$$

Hence:

$$\lambda_n = 2\frac{p_2}{p_1}(n-1) + p_1.$$

We put $\alpha = p_1$ and $\beta = 2\frac{p_2}{p_1^2} - 1$, then:

$$\lambda_n = \alpha(1 + (n-1)(1+\beta)).$$

For t the corolla $B(\bullet^{n-1})$, we obtain:

$$\lambda_n p_{n-1} = np_n + (n-1)p_{n-1}p_1.$$

Hence:

$$\alpha(1+(n-1)\beta)p_{n-1} = np_n.$$

Summing:

$$(1-\alpha\beta h)f'(h) = \alpha f(h).$$

$X(1) = \bullet,$

$X(2) = \alpha$,

$X(3) = \alpha^2 \Big(\frac{(1+\beta)}{2}$  $+$ $\Big),$

$X(4) = \alpha^3 \Big(\frac{(1+2\beta)(1+\beta)}{6}$  $+ (1+\beta)$  $+ \frac{(1+\beta)}{2}$  $+$ $\Big),$

$X(5) = \alpha^4 \Big(\frac{(1+3\beta)(1+2\beta)(1+\beta)}{24}$  $+ \frac{(1+2\beta)(1+\beta)}{2}$ 

$+ \frac{(1+\beta)^2}{2}$  $+ (1+\beta)$  $+ \frac{(1+2\beta)(1+\beta)}{6}$ 

$+ \frac{(1+\beta)}{2}$  $+ (1+\beta)$  $+ \frac{(1+\beta)}{2}$  $+$ $\Big).$

Particular cases

- If $(\alpha, \beta) = (1, -1)$, $f = 1 + h$ and $X(n) = (B)^n(1)$ for all n.
- If $(\alpha, \beta) = (1, 1)$, $f = (1 - h)^{-1}$ and:

$$X(n) = \sum_{|t|=n} \sharp\{\text{embeddings of } t \text{ in the plane}\} t.$$

- Si $(\alpha, \beta) = (1, 0)$, $f = e^h$ and:

$$X(n) = \sum_{|t|=n} \frac{1}{\sharp\{\text{symmetries of } t\}} t.$$

(Left) prelie algebra

A prelie algebra $\mathfrak{g}$ is a vector space with a linear product $\circ$ such that for all $x, y, z \in \mathfrak{g}$:

$$x \circ (y \circ z) - (x \circ y) \circ z = y \circ (x \circ z) - (y \circ x) \circ z.$$

Associated Lie bracket

If $\circ$ is a prelie product on $\mathfrak{g}$, its antisymmetrization is a Lie bracket.

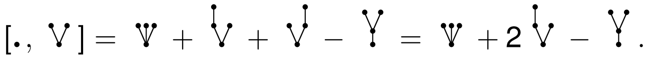
Primitive elements of the dual of H_R

For any rooted tree t let us define:

$$f_t : \left\{ \begin{array}{rcl} H_R & \longrightarrow & \mathbb{C} \\ F & \longrightarrow & s_t \delta_{F,t}. \end{array} \right.$$

The family (f_t) is a basis of the primitive elements of H_R^*. The Lie bracket is given by:

$$[f_{t_1}, f_{t_2}] = \sum_{t' = t_1 \rightarrowtail t_2} f_{t'} - \sum_{t' = t_2 \rightarrowtail t_1} f_{t'}.$$



We define:

$$f_{t_1} \circ f_{t_2} = \sum_{t'=t_1 \rightarrowtail t_2} f_{t'}.$$

This product is prelie.

Theorem (Chapoton-Livernet)

As a prelie algebra, $Prim(H_R^*)$ is freely generated by $f_\bullet$.

Faà di Bruno prelie algebra

$\mathfrak{g}_{FdB}$ has a basis $(e_i)_{i\geq 1}$, and the prelie product is defined by:

$$e_i \circ e_j = (j+\lambda)e_{i+j}.$$

For all $i, j, k \geq 1$:

$$e_i \circ (e_j \circ e_k) - (e_i \circ e_j) \circ e_k = k(k+\lambda)e_{i+j+k}.$$

Theorem

- If $\beta \neq -1$ and $\alpha = 1$,

$$\Delta(X) = X \otimes 1 + \sum_{j=1}^{\infty} (1 + \lambda X)^{1+\frac{j}{\lambda}} \otimes X(j),$$

  with $\lambda = \dfrac{-1}{1+\beta}$.
- If $\beta = -1$ and $\alpha = 1$,

$$\Delta(X) = 1 \otimes X + X \otimes 1 + X \otimes X.$$

Corollary

Il $\alpha \neq 0$, the prelie algebra of the primitive elements of the dual of the Hopf algebra generated by the $X(i)$'s has a basis $(e_i)_{i \geq 1}$.

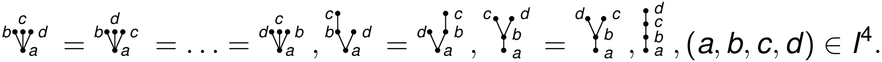
- If $\beta \neq -1$, $e_i \circ e_j = (\lambda + j)e_{i+j}$ (Faà di Bruno case).
- If $\beta = -1$, $e_i \circ e_j = e_{i+j}$ (symmetric case).

In QFT, generally Dyson-Schwinger equations involve several 1-cocycles, for example [Bergbauer-Kreimer]:

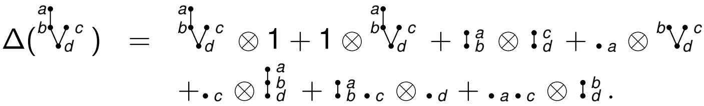
$$X = \sum_{n=1}^{\infty} B_n((1+X)^{n+1}),$$

where B_n is the insertion operator into a primitive Feynman graph with n loops.

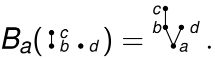
Let I be a set. Set of rooted trees decorated by I:



The Connes-Kreimer construction is extended to obtain the Hopf algebra H_R^I.



For all $d \in I$, there is a grafting operator $B_d : H_R^I \longrightarrow H_R^I$. For example, if $a, b, c, d \in I$:



Proposition

For all $a \in I$, $x \in H_R^I$:

$$\Delta \circ B_a(x) = B_a(x) \otimes 1 + (Id \otimes B_a) \circ \Delta(x).$$

Universal property

Let A be a commutative Hopf algebra and for all $a \in I$, let $L_a : A \longrightarrow A$ such that for all $x \in A$:

$$\Delta \circ L_a(x) = L_a(x) \otimes 1 + (Id \otimes L_a) \circ \Delta(x).$$

Then there exists a unique Hopf algebra morphism $\phi : H_R^I \longrightarrow A$ with $\phi \circ B_a = L_a \circ \phi$ for all $a \in A$.

Definitions

Let I be a graded set and let $f_i(h) \in \mathbb{C}[[h]]$ for all $i \in I$.

- The combinatorial Dyson-Schwinger equations associated to $(f_i(h))_{i \in I}$ is:

$$X = \sum_{i \in I} B_i(f_i(X)),$$

  where X lives in the completion of H_R^I.
- This equation has a unique solution $X = \sum X(n)$.
- The subalgebra of H_R^I generated by the $X(n)$'s is denoted by $H_{(f)}$.
- We shall say that the equation is Hopf if $H_{(f)}$ is a Hopf subalgebra.

Lemma

Let us assume that the equation associated to (f) is Hopf. If $f_i(0) = 0$, then $f_i = 0$.

We now assume that $f_i(0) = 1$ for all $i \in I$.

Lemma

Let us assume that the equation associated to (f) is Hopf. If $i, j \in I$ have the same degree, then $f_i = f_j$.

Grouping 1-cocycles by degrees, we now assume that $I \subseteq \mathbb{N}^*$.

Let us choose $i \in I$. We restrict our solution to i, that is to say we delete any tree with a decoration which is not equal to i. The obtained element X' is solution of:

$$X' = B_i(f_i(X')),$$

and this equation is Hopf. By the study of equations with only one 1-cocycle:

Lemma

For all $i \in I$, there exists $\alpha_i, \beta_i \in \mathbb{C}$ such that :

$$f_i = \begin{cases} e^{\alpha_i h} \text{ if } \beta_i = 0, \\ (1 - \alpha_i \beta_i h)^{-1/\beta_i} \text{ if } \beta_i \neq 0. \end{cases}$$

Theorem

One of the following assertions holds:

1. there exists $\lambda, \mu \in \mathbb{C}$ such that, if we put:

$$Q(h) = \begin{cases} (1-\mu h)^{-\frac{\lambda}{\mu}} \text{ if } \mu \neq 0, \\ e^{\lambda h} \text{ if } \mu = 0, \end{cases}$$

   then:

$$(E) : x = \sum_{i \in I} B_j \left((1-\mu x)Q(x)^i\right).$$

2. There exists $m \geq 0$ and $\alpha \in \mathbb{C} - \{0\}$ such that:

$$(E) : x = \sum_{\substack{i \in I \\ m \mid i}} B_i(1+\alpha x) + \sum_{\substack{i \in I \\ m \nmid i}} B_i(1).$$

Theorem

For all $\lambda, \mu \in \mathbb{C}$, the algebra generated by the components of the solution of the Dyson-Schwinger equation of the first type is a Hopf subalgebra.

Corollary

If $\mu \neq -1$ and $\lambda = 1 + \mu$,

$$\Delta(X) = X \otimes 1 + \sum_{j=1}^{\infty} (1 + \lambda' X)^{1+\frac{j}{\lambda'}} \otimes X(j),$$

with $\lambda' = \dfrac{-1}{1+\mu}$.

Description of the prelie algebra in the second case: to simplify, we assume that $1 \in I$.

Theorem

$$X = \sum_{\substack{i \in I \\ m \mid j}} B_i(1 + \alpha X) + \sum_{\substack{i \in I \\ m \nmid i}} B_i(1),$$

with $\alpha \in \mathbb{C} - \{0\}$. The dual of $H_{(f)}$ is the enveloping algebra of a pre-Lie algebra $\mathfrak{g}$, such that:

- $\mathfrak{g}$ has a basis $(f_i)_{i \geq 1}$.
- For all $i, j \geq 1$:

$$f_i \circ f_j = \begin{cases} 0 \text{ if } m \nmid j, \\ f_{i+j} \text{ if } m \mid j. \end{cases}$$

The product $\circ$ is associative.