Combinatorial Dyson-Schwinger equations and systems I

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To a given QFT is attached a family of graphs.

### Feynman graphs

1. A finite number of possible half-edges.
2. A finite number of possible vertices.
3. A finite number of possible external half-edges (external structure).
4. The graph is connected and 1-PI.

To each external structure is associated a formal series in the Feynman graphs.
In QED

1. Half-edges: \( \rightarrow \) (electron), \( \sim \sim \) (photon).

2. Vertices: \( \sim \sim \).

3. External structures: \( \sim \circ \), \( \sim \sim \), \( \rightarrow \rightarrow \).
Examples in QED

\[\text{Feynman definition}
\text{Combinatorial structures on Feynman graphs}
\text{Hopf algebra of Feynman graphs}\]

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Combinatorial Dyson-Schwinger equations and systems I
Subgraphs and contraction

1. A subgraph of a Feynman graph $\Gamma$ is a subset $\gamma$ of the set of half-edges $\Gamma$ such that $\gamma$ and the vertices of $\Gamma$ with all half edges in $\gamma$ is itself a Feynman graph.

2. If $\Gamma$ is a Feynman graph and $\gamma_1, \ldots, \gamma_k$ are disjoint subgraphs of $\Gamma$, $\Gamma/\gamma_1 \ldots \gamma_k$ is the Feynman graph obtained by contraction of $\gamma_1, \ldots, \gamma_k$. 
Insertion

Let $\Gamma_1$ and $\Gamma_2$ be two Feynman graphs. According to the external structure of $\Gamma_1$, you can replace a vertex or an edge of $\Gamma_2$ by $\Gamma_1$ in order to obtain a new Feynman graph.

Examples in QED

\[
\begin{align*}
\text{\begin{tikzpicture}
\draw (0,0) -- (0.5,0) -- (1,0);
\end{tikzpicture}} & \quad \mapsto \quad \begin{tikzpicture}
\draw (0,0) -- (0.5,0) -- (1,0);
\draw (0.25,0.25) .. controls (0.75,0.75) and (1.25,0.25) .. (1,0);
\end{tikzpicture} & = & \begin{tikzpicture}
\draw (0,0) -- (0.5,0) -- (1,0);
\draw (0.25,0.25) .. controls (0.75,0.75) and (1.25,0.25) .. (1,0);
\draw (1,0) -- (1.5,0) -- (2,0);
\end{tikzpicture} \\
\text{\begin{tikzpicture}
\draw (0,0) -- (0.5,0) -- (1,0);
\draw (0.25,0.25) .. controls (0.75,0.75) and (1.25,0.25) .. (1,0);
\draw (1,0) -- (1.5,0) -- (2,0);
\draw (2,0) -- (2.5,0) -- (3,0);
\end{tikzpicture}} & \quad \mapsto \quad \begin{tikzpicture}
\draw (0,0) -- (0.5,0) -- (1,0);
\draw (0.25,0.25) .. controls (0.75,0.75) and (1.25,0.25) .. (1,0);
\draw (1,0) -- (1.5,0) -- (2,0);
\draw (2,0) -- (2.5,0) -- (3,0);
\draw (3,0) -- (3.5,0) -- (4,0);
\end{tikzpicture} & = & \begin{tikzpicture}
\draw (0,0) -- (0.5,0) -- (1,0);
\draw (0.25,0.25) .. controls (0.75,0.75) and (1.25,0.25) .. (1,0);
\draw (1,0) -- (1.5,0) -- (2,0);
\draw (2,0) -- (2.5,0) -- (3,0);
\draw (3,0) -- (3.5,0) -- (4,0);
\draw (4,0) -- (4.5,0) -- (5,0);
\end{tikzpicture}
\end{align*}
\]
Construction

Let $H_{FG}$ be a free commutative algebra generated by the set of Feynman graphs. It is given a coproduct: for all Feynman graph $\Gamma$,

$$\Delta(\Gamma) = \sum_{\gamma_1 \ldots \gamma_k \subseteq \Gamma} \gamma_1 \ldots \gamma_k \otimes \Gamma / \gamma_1 \ldots \gamma_k.$$
The Hopf algebra $H_{FG}$ is graded by the number of loops:

$$|\Gamma| = \#E(\Gamma) - \#V(\Gamma) + 1.$$ 

Because of the 1-PI condition, it is connected, that is to say $(H_{FG})_0 = K1_{H_{FG}}$. What is its dual?

**Cartier-Quillen-Milnor-Moore theorem**

Let $H$ be a cocommutative, graded, connected Hopf algebra over a field of characteristic zero. Then it is the enveloping algebra of its primitive elements.
This theorem can be applied to the graded dual of $H_{FG}$.

**Primitive elements of $H_{FG}^*$**

- Basis of primitive elements: for any Feynman graph $\Gamma$,

  $$f_{\Gamma}(\gamma_1 \ldots \gamma_k) = \# \text{Aut}(\Gamma) \delta_{\gamma_1 \ldots \gamma_k, \Gamma}.$$

- The Lie bracket is given by:

  $$[f_{\Gamma_1}, f_{\Gamma_2}] = \sum_{\Gamma = \Gamma_1 \rightarrow \Gamma_2} f_{\Gamma} - \sum_{\Gamma = \Gamma_2 \rightarrow \Gamma_1} f_{\Gamma}.$$
We define:

\[ f_{\Gamma_1} \circ f_{\Gamma_2} = \sum_{\Gamma = \Gamma_1 \mapsto \Gamma_2} f_{\Gamma}. \]

The product \( \circ \) is not associative, but satisfies:

\[ f_1 \circ (f_2 \circ f_3) - (f_1 \circ f_2) \circ f_3 = f_2 \circ (f_1 \circ f_3) - (f_2 \circ f_1) \circ f_3. \]

It is (left) prelie.
In the context of QFT, we shall consider some special infinite sums of Feynman graphs:

**Example in QED**

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example1}\end{array} = \sum_{n \geq 1} x^n \left( \sum_{\gamma \in \{\text{}(n)\}} \sum_{\gamma} s_{\gamma \gamma} \right). \\
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example2}\end{array} = -\sum_{n \geq 1} x^n \left( \sum_{\gamma \in \{\text{}(n)\}} \sum_{\gamma} s_{\gamma \gamma} \right).
\end{align*}
\]
Example in QED

\[
\begin{align*}
\quad &\quad = - \sum_{n \geq 1} x^n \left( \sum_{\gamma \in (n)} s_{\gamma \gamma} \right). 
\end{align*}
\]

They live in the completion of \( H_{FG} \).
How to describe these formal series?

- For any primitive Feynman graph $\gamma$, one defines the insertion operator $B_\gamma$ over $H_{FG}$. This operator associates to a graph $G$ the sum (with symmetry coefficients) of the insertions of $G$ into $\gamma$.

- The propagators then satisfy a system of equations involving the insertion operators, called systems of Dyson-Schwinger equations.
Example

In QED:

\[ B = \frac{1}{2} + \frac{1}{2} \]

\[ B = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \]
In QED:

\[
\begin{align*}
\text{Feynman graph} & \quad = \quad \sum_{\gamma} x|\gamma| B_{\gamma} \\
\text{Dyson-Schwinger equations} & \quad = \quad -xB \\
\text{Reformulation with trees} & \quad = \quad -xB \\
\text{More realistic Dyson-Schwinger equations} & \quad = \quad -xB
\end{align*}
\]
Other example (Bergbauer, Kreimer)

\[ X = \sum_{\gamma \text{ primitive}} B_\gamma \left( (1 + X)^{|\gamma|+1} \right). \]
Question

For a given system of Dyson-Schwinger equations \((S)\), is the subalgebra generated by the homogeneous components of \((S)\) a Hopf subalgebra?
Proposition

The operators $B_\gamma$ satisfy: for all $x \in H_{FG}$,

$$\Delta \circ B_\gamma(x) = B_\gamma(x) \otimes 1 + (Id \otimes B_\gamma) \circ \Delta(x).$$

This relation allows to lift any system of Dyson-Schwinger equation to the Hopf algebra of decorated rooted trees.
The Hopf algebra of rooted trees $H_R$ (or Connes-Kreimer Hopf algebra) is the free commutative algebra generated by the set of rooted trees.

The set of rooted forests is a linear basis of $H_R$:
The coproduct is given by admissible cuts:

\[
\Delta(t) = \sum_{c \text{ admissible cut}} P_c(t) \otimes R_c(t).
\]

<table>
<thead>
<tr>
<th>cut</th>
<th>( W^c(t) )</th>
<th>( R^c(t) )</th>
<th>( P^c(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Admissible?</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( W^c(t) )</td>
<td>( \cup )</td>
<td>( \vdash )</td>
<td>( \vdash )</td>
</tr>
<tr>
<td>( R^c(t) )</td>
<td>( \cup )</td>
<td>( \vdash )</td>
<td>( \vdash )</td>
</tr>
<tr>
<td>( P^c(t) )</td>
<td>1</td>
<td>( \vdash )</td>
<td>( \vdash )</td>
</tr>
</tbody>
</table>

\[
\Delta(\begin{array}{c}
\vdash
\end{array}) = \begin{array}{c}
\vdash \otimes 1 + 1 \otimes \vdash + \vdash + . \vdash + . \vdash + . \vdash + . \vdash + . \vdash + . \vdash
\end{array}.
\]
The grafting operator of $H_R$ is the map $B : H_R \rightarrow H_R$, associating to a forest $t_1 \ldots t_n$ the tree obtained by grafting $t_1, \ldots, t_n$ on a common root. For example:

$$B(\bullet.) = V.$$

**Proposition**

For all $x \in H_R$:

$$\Delta \circ B(x) = B(x) \otimes 1 + (Id \otimes B) \circ \Delta(x).$$

So $B$ is a 1-cocycle of $H_R$. 
Universal property

Let $A$ be a commutative Hopf algebra and let $L : A \to A$ be a 1-cocycle of $A$. Then there exists a unique Hopf algebra morphism $\phi : H_R \to A$ with $\phi \circ B = L \circ \phi$.

This will be generalized to the case of several 1-cocycles with the help of decorated rooted trees.
\( H_R \) is graded by the number of vertices and \( B \) is homogeneous of degree 1.

Let \( Y = B_{\gamma}(f(Y)) \) be a Dyson-Schwinger equation in a suitable Hopf algebra of Feynman graphs \( H_{FG} \), such that \( |\gamma| = 1 \).

There exists a Hopf algebra morphism \( \phi : H_R \to H_{FG} \), such that \( \phi \circ B = B_{\gamma} \circ \phi \). This morphism is homogeneous of degree 0.

Let \( X \) be the solution of \( X = B(f(X)) \). Then \( \phi(X) = Y \) and for all \( n \geq 1 \), \( \phi(X(n)) = Y(n) \).

Consequently, if the subalgebra generated by the \( X(n) \)’s is Hopf, so is the subalgebra generated by the \( Y(n) \)’s.
**Definition**

Let \( f(h) \in \mathbb{C}[[h]] \).

- The combinatorial Dyson-Schwinger equations associated to \( f(h) \) is:
  \[
  X = B(f(X)),
  \]
  where \( X \) lives in the completion of \( H_R \).
- This equation has a unique solution \( X = \sum X(n) \), with:
  \[
  \begin{align*}
  X(1) & = p_0, \\
  X(n + 1) & = \sum_{k=1}^{n} \sum_{a_1 + \ldots + a_k = n} p_k B(X(a_1) \ldots X(a_k)),
  \end{align*}
  \]
  where \( f(h) = p_0 + p_1 h + p_2 h^2 + \ldots \)
\[ X(1) = p_0 ., \]
\[ X(2) = p_0 p_1 ., \]
\[ X(3) = p_0 p_1^2 . + p_0^2 p_2 \bigtriangledown , \]
\[ X(4) = p_0 p_1^3 . + p_0^2 p_1 p_2 \bigtriangledown + 2p_0^2 p_1 p_2 \bigtriangledown + p_0^3 p_3 \bigtriangledown . \]
Examples

- If \( f(h) = 1 + h \):
  \[
  X = . + : + \{ + \} + \{ + \} + \ldots
  \]

- If \( f(h) = (1 - h)^{-1} \):
  \[
  X = . + : + \{ + \} + \{ + \} + 2 \{ + \} + \{ + \} + \ldots
  \]

\[\begin{align*}
  &+\text{ graph} + 3 \{ + \} + \{ + \} + 2 \{ + \} + 2 \{ + \} + \{ + \} + \{ + \} + \ldots
  
\end{align*}\]
Let \( f(h) \in \mathbb{C}[[h]] \). The homogeneous components of the unique solution of the combinatorial Dyson-Schwinger equation associated to \( f(h) \) generate a subalgebra of \( H_R \) denoted by \( H_f \).

\[ H_f \text{ is not always a Hopf subalgebra} \]

For example, for \( f(h) = 1 + h + h^2 + 2h^3 + \cdots \), then:

\[
X = . + \overset{\bullet}{1} + \overset{\bullet}{v} + \overset{\bullet}{1} + 2 \overset{\bullet}{v} + 2 \overset{\bullet}{\overset{\bullet}{v}} + \overset{\bullet}{\overset{\bullet}{v}} + \overset{\bullet}{1} + \cdots
\]

So:

\[
\Delta(X(4)) = X(4) \otimes 1 + 1 \otimes X(4) + (10X(1)^2 + 3X(2)) \otimes X(2) + (X(1)^3 + 2X(1)X(2) + X(3)) \otimes X(1) + X(1) \otimes (8 \overset{\bullet}{v} + 5\overset{\bullet}{1}).
\]
If $f(0) = 0$, the unique solution of $X = B(f(X))$ is 0. From now, up to a normalization we shall assume that $f(0) = 1$.

**Theorem**

Let $f(h) \in \mathbb{C}[[h]]$, with $f(0) = 1$. The following assertions are equivalent:

1. $H_f$ is a Hopf subalgebra of $H_R$.
2. There exists $(\alpha, \beta) \in \mathbb{C}^2$ such that $(1 - \alpha \beta h)f'(h) = \alpha f(h)$.
3. There exists $(\alpha, \beta) \in \mathbb{C}^2$ such that $f(h) = 1$ if $\alpha = 0$ or $f(h) = e^{\alpha h}$ if $\beta = 0$ or $f(h) = (1 - \alpha \beta h)^{-\frac{1}{\beta}}$ if $\alpha \beta \neq 0$. 
1 \implies 2. We put \( f(h) = 1 + p_1 h + p_2 h^2 + \cdots \). Then \( X(1) = \ldots \)

Let us write:

\[
\Delta(X(n+1)) = X(n+1) \otimes 1 + 1 \otimes X(n+1) + X(1) \otimes Y(n) + \ldots .
\]

1. By definition of the coproduct, \( Y(n) \) is obtained by cutting a leaf in all possible ways in \( X(n+1) \). So it is a linear span of trees of degree \( n \).

2. As \( H_f \) is a Hopf subalgebra, \( Y(n) \) belongs to \( H_f \).

Hence, there exists a scalar \( \lambda_n \) such that \( Y(n) = \lambda_n X_n \).
Lemma

Let us write:

\[ X = \sum_t a_t t. \]

For any rooted tree \( t \):

\[ \lambda_{|t|} a_t = \sum_{t'} n(t, t') a_{t'}, \]

where \( n(t, t') \) is the number of leaves of \( t' \) such that the cut of this leaf gives \( t \).
We here assume that $f$ is not constant. We can prove that $p_1 \neq 0$.

For $t$ the ladder $(B)^n(1)$, we obtain:

$$p_1^{n-1} \lambda_n = 2(n - 1)p_1^{n-2}p_2 + p_1^n.$$ 

Hence:

$$\lambda_n = 2 \frac{p_2}{p_1} (n - 1) + p_1.$$ 

We put $\alpha = p_1$ and $\beta = 2 \frac{p_2}{p_1} - 1$, then:

$$\lambda_n = \alpha(1 + (n - 1)(1 + \beta)).$$
For $t$ the corolla $B(\cdot^{n-1})$, we obtain:

$$\lambda_n p_{n-1} = np_n + (n - 1)p_{n-1}p_1.$$ 

Hence:

$$\alpha(1 + (n - 1)\beta)p_{n-1} = np_n.$$ 

Summing:

$$(1 - \alpha\beta h)f'(h) = \alpha f(h).$$
\[ X(1) = \cdot, \]
\[ X(2) = \alpha \cdot, \]
\[ X(3) = \alpha^2 \left( \frac{(1 + \beta)}{2} \cdot V + \mathbb{1} \right), \]
\[ X(4) = \alpha^3 \left( \frac{(1 + 2\beta)(1 + \beta)}{6} \cdot V + (1 + \beta) \cdot V + \frac{(1 + \beta)}{2} \cdot V + \mathbb{1} \right), \]
\[ X(5) = \alpha^4 \left( \frac{(1 + 3\beta)(1 + 2\beta)(1 + \beta)}{24} \cdot V + \frac{(1 + 2\beta)(1 + \beta)}{2} \cdot V \right) \]
\[ + \frac{(1 + \beta)^2}{2} \cdot V + (1 + \beta) \cdot V + \frac{(1 + 2\beta)(1 + \beta)}{6} \cdot V \]
\[ + \frac{(1 + \beta)}{2} \cdot V + (1 + \beta) \cdot V + \frac{(1 + \beta)}{2} \cdot V + \mathbb{1} \right) \]
Particular cases

- If $(\alpha, \beta) = (1, -1)$, $f = 1 + h$ and $X(n) = (B)^n(1)$ for all $n$.

- If $(\alpha, \beta) = (1, 1)$, $f = (1 - h)^{-1}$ and:

  $$X(n) = \sum_{|t|=n} \#\{\text{embeddings of } t \text{ in the plane}\} t.$$ 

- If $(\alpha, \beta) = (1, 0)$, $f = e^h$ and:

  $$X(n) = \sum_{|t|=n} \frac{1}{\#\{\text{symmetries of } t\}} t.$$
(Left) prelie algebra

A prelie algebra $g$ is a vector space with a linear product $\circ$ such that for all $x, y, z \in g$:

$$x \circ (y \circ z) - (x \circ y) \circ z = y \circ (x \circ z) - (y \circ x) \circ z.$$  

Associated Lie bracket

If $\circ$ is a prelie product on $g$, its antisymmetrization is a Lie bracket.
Primitive elements of the dual of $H_R$

For any rooted tree $t$ let us define:

$$f_t : \begin{cases} H_R & \longrightarrow & \mathbb{C} \\ F & \longrightarrow & S_t \delta_{F,t} \end{cases}$$

The family $(f_t)$ is a basis of the primitive elements of $H_R^\ast$. The Lie bracket is given by:

$$[f_{t_1}, f_{t_2}] = \sum_{t'_1 = t_1 \rightarrow t_2} f_{t'} - \sum_{t'_2 = t_2 \rightarrow t_1} f_{t'}.$$

$$[\cdot, V] = \begin{array}{c} \cdot \end{array} + \begin{array}{c} \cdot \end{array} + \begin{array}{c} \cdot \end{array} - \begin{array}{c} \cdot \end{array} = \begin{array}{c} \cdot \end{array} + 2 \begin{array}{c} \cdot \end{array} - \begin{array}{c} \cdot \end{array}.$$
We define:

\[ f_{t_1} \circ f_{t_2} = \sum_{t' = t_1 \rightarrow t_2} f_{t'} . \]

This product is prelie.

**Theorem (Chapoton-Livernet)**

As a prelie algebra, \( Prim(H_R^*) \) is freely generated by \( f \).
Faà di Bruno prelie algebra

$g_{FdB}$ has a basis $(e_i)_{i \geq 1}$, and the prelie product is defined by:

$$e_i \circ e_j = (j + \lambda)e_{i+j}.$$  

For all $i, j, k \geq 1$:

$$e_i \circ (e_j \circ e_k) - (e_i \circ e_j) \circ e_k = k(k + \lambda)e_{i+j+k}.$$  

Theorem

- If $β \neq -1$ and $α = 1$,

$$\Delta(X) = X \otimes 1 + \sum_{j=1}^{\infty} (1 + \lambda X)^{1+\frac{j}{\lambda}} \otimes X(j),$$

with $\lambda = \frac{-1}{1 + β}$.

- If $β = -1$ and $α = 1$,

$$\Delta(X) = 1 \otimes X + X \otimes 1 + X \otimes X.$$
Corollary

If $\alpha \neq 0$, the prelie algebra of the primitive elements of the dual of the Hopf algebra generated by the $X(i)$’s has a basis $(e_i)_{i \geq 1}$.

- If $\beta \neq -1$, $e_i \circ e_j = (\lambda + j)e_{i+j}$ (Faà di Bruno case).
- If $\beta = -1$, $e_i \circ e_j = e_{i+j}$ (symmetric case).
In QFT, generally Dyson-Schwinger equations involve several 1-cocycles, for example [Bergbauer-Kreimer]:

\[ X = \sum_{n=1}^{\infty} B_n((1 + X)^{n+1}), \]

where \( B_n \) is the insertion operator into a primitive Feynman graph with \( n \) loops.
Let $I$ be a set. Set of rooted trees decorated by $I$:

\[ a, a \in I; \quad a, (a, b) \in I^2; \quad b \mapsto c = c \mapsto a, b, c \in I^3; \]

\[ \begin{array}{cccc}
\mapsto a & \mapsto b & \mapsto c & \mapsto d \\
\mapsto a & \mapsto b & \mapsto c & \mapsto d \\
\mapsto a & \mapsto b & \mapsto c & \mapsto d \\
\mapsto a & \mapsto b & \mapsto c & \mapsto d \\
\end{array} = \ldots \begin{array}{cccc}
\mapsto a & \mapsto b & \mapsto c & \mapsto d \\
\mapsto a & \mapsto b & \mapsto c & \mapsto d \\
\mapsto a & \mapsto b & \mapsto c & \mapsto d \\
\mapsto a & \mapsto b & \mapsto c & \mapsto d \\
\end{array} = \ldots = \begin{array}{cccc}
\mapsto a & \mapsto b & \mapsto c & \mapsto d \\
\mapsto a & \mapsto b & \mapsto c & \mapsto d \\
\mapsto a & \mapsto b & \mapsto c & \mapsto d \\
\mapsto a & \mapsto b & \mapsto c & \mapsto d \\
\end{array}.

The Connes-Kreimer construction is extended to obtain the Hopf algebra $H^I_R$.

\[
\Delta (\begin{array}{ccc}a & b & c \\
b & c & d \end{array}) = \begin{array}{ccc}a & b & c \\
b & c & d \end{array} \otimes \mathbf{1} + \mathbf{1} \otimes \begin{array}{ccc}a & b & c \\
b & c & d \end{array} + \begin{array}{ccc}a & b & c \\
b & c & d \end{array} + \begin{array}{ccc}a & b & c \\
b & c & d \end{array} + \begin{array}{ccc}a & b & c \\
b & c & d \end{array}.
\]
For all $d \in l$, there is a grafting operator $B_d : H^l_R \to H^l_R$. For example, if $a, b, c, d \in l$:

$$B_a(b \cdot c \cdot d) = b \cdot c \cdot d.$$

**Proposition**

For all $a \in l$, $x \in H^l_R$:

$$\Delta \circ B_a(x) = B_a(x) \otimes 1 + (\text{id} \otimes B_a) \circ \Delta(x).$$
Universal property

Let $A$ be a commutative Hopf algebra and for all $a \in I$, let $L_a : A \rightarrow A$ such that for all $x \in A$:

$$\Delta \circ L_a(x) = L_a(x) \otimes 1 + (Id \otimes L_a) \circ \Delta(x).$$

Then there exists a unique Hopf algebra morphism $\phi : H^l_R \rightarrow A$ with $\phi \circ B_a = L_a \circ \phi$ for all $a \in A$. 
Let $I$ be a graded set and let $f_i(h) \in \mathbb{C}[[h]]$ for all $i \in I$.

- The combinatorial Dyson-Schwinger equations associated to $(f_i(h))_{i \in I}$ is:

$$X = \sum_{i \in I} B_i(f_i(X)),$$

where $X$ lives in the completion of $H^I_R$.

- This equation has a unique solution $X = \sum X(n)$.

- The subalgebra of $H^I_R$ generated by the $X(n)$'s is denoted by $H_{(f)}$.

- We shall say that the equation is Hopf if $H_{(f)}$ is a Hopf subalgebra.
Lemma

Let us assume that the equation associated to \((f)\) is Hopf. If \(f_i(0) = 0\), then \(f_i = 0\).

We now assume that \(f_i(0) = 1\) for all \(i \in I\).

Lemma

Let us assume that the equation associated to \((f)\) is Hopf. If \(i, j \in I\) have the same degree, then \(f_i = f_j\).

Grouping 1-cocycles by degrees, we now assume that \(I \subseteq \mathbb{N}^*\).
Let us choose $i \in I$. We restrict our solution to $i$, that is to say we delete any tree with a decoration which is not equal to $i$. The obtained element $X'$ is solution of:

$$X' = B_i(f_i(X')),$$

and this equation is Hopf. By the study of equations with only one 1-cocycle:

**Lemma**

For all $i \in I$, there exists $\alpha_i, \beta_i \in \mathbb{C}$ such that:

$$f_i = \begin{cases} 
  e^{\alpha_i h} & \text{if } \beta_i = 0, \\
  (1 - \alpha_i \beta_i h)^{-1/\beta_i} & \text{if } \beta_i \neq 0.
\end{cases}$$
Theorem

One of the following assertions holds:

1. there exists $\lambda, \mu \in \mathbb{C}$ such that, if we put:

   \[ Q(h) = \begin{cases} 
   (1 - \mu h)^{-\frac{\lambda}{\mu}} & \text{if } \mu \neq 0, \\
   e^{\lambda h} & \text{if } \mu = 0,
   \end{cases} \]

   then:

   \[ (E) : x = \sum_{i \in I} B_j \left( (1 - \mu x) Q(x)^i \right). \]

2. There exists $m \geq 0$ and $\alpha \in \mathbb{C} - \{0\}$ such that:

   \[ (E) : x = \sum_{i \in I} B_i (1 + \alpha x) + \sum_{i \in I} B_i (1). \]
Theorem
For all $\lambda, \mu \in \mathbb{C}$, the algebra generated by the components of the solution of the Dyson-Schwinger equation of the first type is a Hopf subalgebra.

Corollary
If $\mu \neq -1$ and $\lambda = 1 + \mu$,

$$
\Delta(X) = X \otimes 1 + \sum_{j=1}^{\infty} (1 + \lambda' X)^{1+\frac{j}{\lambda'}} \otimes X(j),
$$

with $\lambda' = \frac{-1}{1 + \mu}$.
Description of the prelie algebra in the second case: to simplify, we assume that $1 \in I$.

**Theorem**

\[
X = \sum_{i \in I} B_i(1 + \alpha X) + \sum_{i \in I} \frac{m}{m \parallel i} B_i(1),
\]

with $\alpha \in \mathbb{C} - \{0\}$. The dual of $H_f$ is the enveloping algebra of a pre-Lie algebra $g$, such that:

- $g$ has a basis $(f_i)_{i \geq 1}$.
- For all $i, j \geq 1$:

\[
f_i \circ f_j = \begin{cases} 
0 & \text{if } m \parallel j, \\
 f_{i+j} & \text{if } m \mid j.
\end{cases}
\]

The product $\circ$ is associative.