Loops and vertices in QCD

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Outline

- Will review the renormalization of a massless non-abelian gauge theory and the computer algebra algorithms used
- Definition of various renormalization schemes aside from the usual $\overline{\mathrm{MS}}$ scheme will be discussed which are derived from the structure of the vertex functions
- As an application to problems with mass scales will briefly consider the infrared structure of the gluon propagator affected by the Gribov construction
- Relation of results to special functions will be introduced en route
- Or, this is where the periods go

Quantum Chromodynamics (QCD)

- QCD is the non-abelian gauge theory of the strong interactions
- It requires a choice of gauge, which will be the linear covariant gauge here
- The Lagrangian for massless quarks is

$$L^{\mathrm{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} - \frac{1}{2\alpha} (\partial^\mu A^a_\mu)^2 - \bar{c}^a \, (\partial^\mu D_\mu c)^a + i \bar{\psi}^{iI} \not{D} \psi^{iI}$$

  where $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ and α is the linear covariant gauge parameter
- Graphically it involves 2, 3 and 4 point vertices
- So underlying structure of anomalous dimensions will be similar to that of scalar ϕ^n theory, $n = 3$ and 4
- In this gauge QCD is renormalizable and the renormalization group functions have been computed to four (and five) loops in $\overline{\mathrm{MS}}$

Renormalization

- To renormalize the theory introduce renormalized variables by rescaling all bare fields and parameters (coupling constants, gauge parameters, masses) by renormalization constants Z_i
- Conventions:

$$A_o^{a\,\mu} = \sqrt{Z_A}\, A^{a\,\mu}\,,\; c_o^a = \sqrt{Z_c}\, c^a\,,\; \psi_o = \sqrt{Z_\psi}\psi\,,\; g_o = Z_g\, g\,,\; \alpha_o = Z_\alpha^{-1} Z_A$$

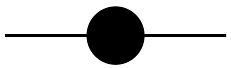
- Wave function and gauge parameter renormalization constants defined by ensuring respective 2-point functions are finite
- Coupling constant renormalization constant determined from vertex renormalization in a way which is consistent with underlying gauge symmetry (Slavnov-Taylor identities)
- The specific definition of a renormalization constant is not unique but depends on a renormalization scheme such as $\overline{\mathrm{MS}}$
- To quantify nature of divergences need to introduce a regularization which preserves symmetries of the theory
- Will use dimensional regularization in $d = 4 - 2\epsilon$ dimensions; then $g_o = \mu^\epsilon Z_g\, g$ where μ is the mass scale associated with this regularization

Renormalization schemes

- General features of a scheme are
  - (a) the momentum configuration of the external legs of the Green's function where the renormalization constants are to be defined
  - (b) the prescription defining the Z's
- Schemes can be classified as mass dependent or mass independent; physical or unphysical
- For example mass dependent schemes could be those where the subtraction point is at the physical mass of the external particle
- Or where the renormalization constants after renormalization depend on some mass scale
- Mass independent schemes could have the squared external momenta equal to μ^2 but the Z's do not depend on any mass scale
- Structure of renormalization group functions in a mass independent scheme is simpler and invariably computationally easier to determine

Specific schemes - $\overline{\text{MS}}$

- Most widely used renormalization scheme is minimal subtraction (MS) which is a mass independent scheme
- Computationally easiest to determine, especially for massless theories
- Consider ϵ expansion of formal 2-point function at one loop



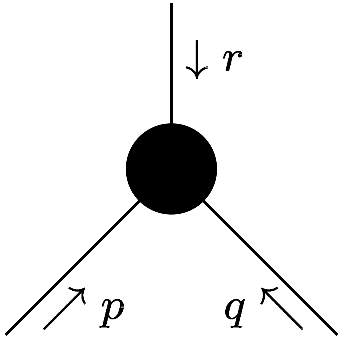
$$= \left[1 + \left[\frac{A_2}{\epsilon} + B_2 + z_{\phi\,1} + C_2\epsilon + O(\epsilon^2)\right] g^2\right] p^2$$

where A_2, B_2 and C_2 depend on all parameters except the coupling constant

- B_2 and C_2 will involve $\ln(p^2/\mu^2)$
- The formal definition of MS at one loop is to choose the counterterm $z_{\phi\,1}$ at the subtraction point ($p^2 = \mu^2$) so that only the divergences are removed
- Once the 2-point function is finite the regularization can be lifted ($\epsilon \to 0$)
- $\overline{\text{MS}}$ is MS but with a certain additional finite part also absorbed into $z_{\phi\,1}$

3-point functions

- Need notation for 3-point vertex functions
- Three external momenta, p, q and r, but only two are independent



due to energy momentum conservation

$$p + q + r = 0$$

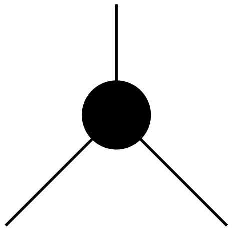
- Leads to two massless scales

$$x = \frac{p^2}{r^2} \ , \ y = \frac{q^2}{r^2}$$

- A non-exceptional momentum configuration is one where the energy momentum is satisfied but none of p, q or r are zero

Vertex function renormalization - generalities

- Consider the formal structure of the one loop 3-point vertex for a generic field theory



$$= \left[1 + \left[\frac{A_3}{\epsilon} + B_3 + C_3\epsilon + z_{g\,1} + \frac{3}{2}z_{\phi\,1} + O(\epsilon^2)\right] g^2\right] g$$

where A_3, B_3 and C_3 equally depend on all parameters except the coupling constant but are also now functions of x, y and r^2

- The finite parts will involve logarithms and dilogarithms of functions of these variables at one loop
- For certain external momenta configurations the finite parts can be simpler functions
- Procedure to renormalize is same as for 2-point functions
- The wave function counterterm $z_{\phi\,1}$ is already determined from 2-point function in a scheme
- First specify a subtraction point, then specify the scheme or method to define the renormalization point

Vertex function renormalization - schemes

- Only unspecified quantity is $z_{g\,1}$
- At one loop in renormalizable field theories A_3 should be independent of x, y and r^2
- The minimal subtraction scheme is defined in such a way that at the subtraction point only the poles in ϵ are absorbed into $z_{g\,1}$
- Vertex functions allow for a large variety of scheme definitions
- One set is the physical mass dependent schemes known as MOM or momentum subtraction of Celmaster and Gonsalves
- They are defined at the completely symmetric point

$$x = y = 1$$

  with $r^2 = -\mu^2$ where μ is the scale introduced to ensure the coupling constant is dimensionless in d-dimensions
- Symmetry of subtraction point simplifies structure of the basic Feynman graphs comprising the vertex functions
- Configuration is non-exceptional and hence avoids potential infrared issues

- MOM schemes are defined in such a way that at the subtraction point there are no $O(g^2)$ corrections
- Hence $z_{g\,1}$ has a non-zero finite part in addition to the pole
- This finite part will correspond to evaluations of the logarithms and dilogarithms
- For QCD there are three distinct vertices and hence three separate MOM schemes defined relative to the triple gluon, ghost-gluon and quark-gluon vertices
- At higher loop the definition of the scheme is the same but renormalization constants are constructed iteratively
- One feature of the renormalization group functions is that they will depend on the renormalization scheme after a few low loop orders
- The leading term is always independent of the scheme
- In mass independent schemes in theories with one coupling constant the β-function is scheme independent to two loops and independent of the gauge parameter to all orders
- In mass dependent schemes the β-function is scheme dependent and depends on the gauge parameter at two loops and beyond
- Variables such as g are defined relative to a scheme

Renormalization - practicalities

- Need to be able to extract renormalization constants at high loop order
- Requires symbolic manipulation languages (such as FORM) and algorithms to evaluate integrals and handle the large amounts of algebra
- One method is to use values of subtracted diagrams; all subgraph divergences removed from a graph to leave the 'true' divergence
- Alternative method of Larin and Vermaseren is to determine n-point functions as functions of the *bare* parameters
- Then counterterms are introduced by rescalings such as

$$\phi_o = \sqrt{Z_\phi}\,\phi \quad , \quad g_o = \mu^\epsilon Z_g g$$

- Remaining overall divergence for that n-point function absorbed into the unknown counterterm for that n-point function at that loop order

Renormalization - packages

- Currently two main computer algebra approaches to renormalization
- MINCER package from 1980's by Chetyrkin et al evaluates massless 2-point functions to three loops and $O(\epsilon)$ in $d = 4 - 2\epsilon$ dimensions
- Used to renormalize 2- and 3-point functions of QCD in a variety of gauges and (mass independent) schemes
- There is a technical shortcut for 3-point functions which can be used if infrared safe
- In 4-dimensions finite integrals such as the 3-point integral

$$I(p,q) \; = \; \int_k \frac{1}{k^2(k-p)^2(k+q)^2}$$

  are infrared safe
- Tempting to evaluate the finite part by setting $q = 0$ to give

$$I(p,0) \; = \; \int_k \frac{1}{(k^2)^2(k-p)^2}$$

  but this is (infrared) divergent $I(p,0) = \frac{1}{\epsilon}$; this is an exceptional momentum configuration

- However for QCD vertex functions such nullifications of one external moment for a 3-point function are viable and allowed as such infrared sick integrals do not arise
- Structure of Feynman rules is such that quark propagator, ghost-gluon vertex and triple gluon vertex have numerator momenta which protect the nullified denominator from being infrared divergent
- This and its generalization to higher n-point functions is known as infrared rearrangement
- Wide application of MINCER to 3-point functions but not 4-point
- Limitation is that MINCER not useful for MOM type renormalization or for beyond three loops
- Current practice is to use the Laporta algorithm
- Uses integration by parts to establish relations between Feynman integrals lurking within a Green's functions
- These are solved algebraically in terms of a small set of master integrals
- Various packages such as REDUZE developed for this; builds databases or relations
- These are evaluated by direct methods such as Schwinger parameters and thence related to polylogarithms and higher functions
- MINCER type masters are known to four loops [Baikov & Chetyrkin]

Application - QCD vertex functions

- Consider the three QCD 3-point vertices

$$
\begin{aligned}
\left\langle A^a_\mu(p) A^b_\nu(q) A^c_\sigma(r) \right\rangle &= f^{abc} \Sigma^{\mathrm{ggg}}_{\mu\nu\sigma}(p,q) \\
\left\langle \psi^i(p) \bar{\psi}^j(q) A^c_\sigma(r) \right\rangle &= T^c_{ij} \Sigma^{\mathrm{qqg}}_{\sigma}(p,q) \\
\left\langle c^a(p) \bar{c}^b(q) A^c_\sigma(r) \right\rangle &= f^{abc} \Sigma^{\mathrm{ccg}}_{\sigma}(p,q)
\end{aligned}
$$

  with

$$p + q + r = 0$$

- Vertices carry Lorentz structure; so have to decompose into a basis of Lorentz tensors built from the external momenta and any other relevant tensor such as the metric
- For these 3-point functions colour group structure factors off; not always the case

Method of computation

- Method to determine amplitudes, $\Sigma^{\mathrm{V}}_{(k)}(p,q)$, is to decompose into tensor basis, $\{\mathcal{P}^{\mathrm{V}}_{(k)\,\{\mu_i\}}(p,q)\}$, by projection

$$\begin{aligned}
\Sigma^{\mathrm{ccg}}_{\sigma}(p,q) &= \sum_{k=1}^{2} \mathcal{P}^{\mathrm{ccg}}_{(k)\,\sigma}(p,q)\,\Sigma^{\mathrm{ccg}}_{(k)}(p,q) \\
\Sigma^{\mathrm{qqg}}_{\sigma}(p,q) &= \sum_{k=1}^{6} \mathcal{P}^{\mathrm{qqg}}_{(k)\,\sigma}(p,q)\,\Sigma^{\mathrm{qqg}}_{(k)}(p,q) \\
\Sigma^{\mathrm{ggg}}_{\mu\nu\sigma}(p,q) &= \sum_{k=1}^{14} \mathcal{P}^{\mathrm{ggg}}_{(k)\,\mu\nu\sigma}(p,q)\,\Sigma^{\mathrm{ggg}}_{(k)}(p,q)
\end{aligned}$$

- This produces scalar integrals to compute either by MINCER or by Laporta algorithm
- For ghost-gluon vertex basis is $\{p_\sigma, q_\sigma\}$ with projection matrix

$$\mathcal{M}^{\mathrm{ccg}} = \frac{1}{\Delta_G}\begin{pmatrix} 4y & -2(1-x-y) \\ -2(1-x-y) & 4x \end{pmatrix}$$

with $\Delta_G(x,y) = x^2 - 2xy + y^2 - 2x - 2y + 1$

- For a nullified external momentum q the basis is $\{p_\sigma\}$ and the projection is p^σ/p^2
- More complicated for other vertices
- Using above method for automatic renormalization and applying MINCER to extract $\overline{\mathrm{MS}}$ renormalization constants to give

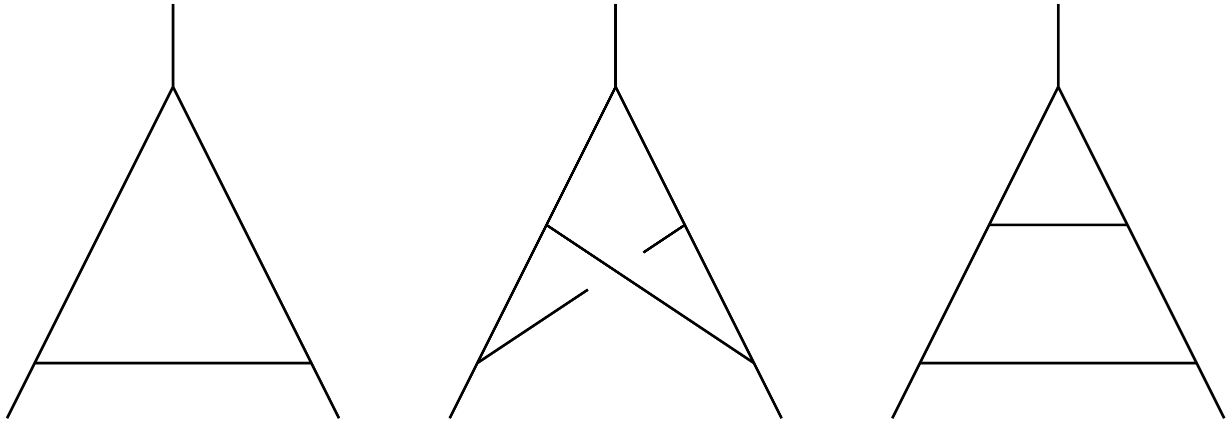
$$\Sigma_\sigma^{\mathrm{ccg}}(p,0) = \left[1 + \frac{\alpha}{2} C_A a + \left[\frac{43}{16}\alpha - \frac{9}{16}\zeta(3)\alpha + \frac{3}{16}\zeta(3)\alpha^2 + \frac{7}{16}\alpha^2\right] C_A^2 a^2 + O(a^3)\right] p_\sigma$$

where $a = g^2/(16\pi^2)$

- No $O(a)$ corrections in the Landau gauge consistent with the Slavnov-Taylor identity and the non-renormalization theorem of Taylor for the ghost-gluon vertex
- For $p \neq 0$ and $q \neq 0$ use Laporta which requires 3-point masters

Integral families

- For application of REDUZE need to define basic integral families



and two permutations of final graph

- Masters which emerge from REDUZE do not necessarily have the same topologies
- To two loops all masters for 3-point functions are of the form of the first topology
- Define

$$I(\alpha,\beta,\gamma) \;=\; \int_k \frac{1}{(k^2)^\alpha((k-p)^2)^\beta((k+q)^2)^\gamma}$$

One loop master

- In compact notation one loop master is

$$I(1,1,1) = -\frac{1}{\mu^2}\left[\Phi_1(x,y) + \Psi_1(x,y)\epsilon + \left[\frac{\zeta(2)}{2}\Phi_1(x,y) + \chi_1(x,y)\right]\epsilon^2 + O(\epsilon^3)\right]$$

where, [Ussyukina & Davydychev],

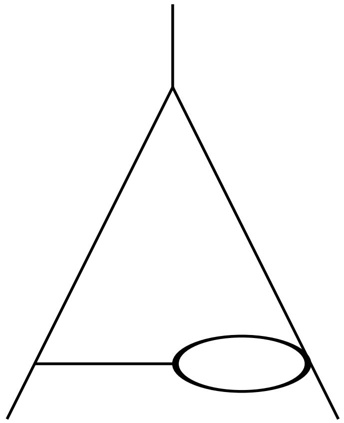
$$\Phi_1(x,y) = \frac{1}{\lambda}\left[2\mathrm{Li}_2(-\rho x) + 2\mathrm{Li}_2(-\rho y) + \ln\left(\frac{y}{x}\right)\ln\left(\frac{(1+\rho y)}{(1+\rho x)}\right) + \ln(\rho x)\ln(\rho y) + \frac{\pi^2}{3}\right]$$

$$\lambda(x,y) = \sqrt{\Delta_G}\ , \ \rho(x,y) = \frac{2}{1-x-y+\lambda(x,y)}$$

- Require $O(\epsilon^2)$ terms due to spurious poles resulting from factors of $1/(d-4)$ appearing after solution of integration by parts equations
- $\Psi_1(x,y)$ involves $\mathrm{Li}_3(z)$ and $\chi_1(x,y)$ has a harmonic polylogarithm [Birthwright et al]
- For instance arguments of $\mathrm{Li}_2(z)$ are complex at symmetric point

Two loop masters

- One two loop master is, [Ussyukina & Davydychev],



$$
\begin{aligned}
= & - \left[\Phi_1(x,y)\right. \\
& + \left[\Psi_1(x,y) - \tfrac{1}{2}\ln(x)\Phi_1(x,y) - \tfrac{1}{2}\ln(y)\Phi_1(x,y)\right]\epsilon \\
& + \left.\left[\tfrac{\zeta(2)}{2}\Phi_1(x,y) + \chi_3(x,y)\right]\epsilon^2\right]\tfrac{1}{\mu^2} + O(\epsilon^3)
\end{aligned}
$$

- Explicit forms of $\chi_1(x,y)$ and $\chi_3(x,y)$ are known but only their *difference* appears in the final vertex function for all external momenta configurations
- This difference can be determined using symmetry of the integrals such as uniqueness (conformal integration)

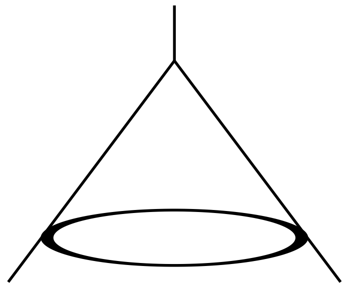
$$\chi_3(x,y) = \chi_1(x,y) + \Phi_2(x,y) - \frac{1}{2}\ln(xy)\Psi_1(x,y) + \frac{1}{4}\left[\ln^2(x) + \ln^2(y)\right]\Phi_1(x,y)$$

- Agrees with Gorbahn & Jäger for a restricted configuration
- $\Phi_2(x,y)$ involves $\mathrm{Li}_4(z)$

- For example, at the fully symmetric point

$$\chi_3(1,1) - \chi_1(1,1) = \frac{1}{36}\psi'''\left(\frac{1}{3}\right) - \frac{2\pi^4}{27}$$

- Other main master is



which has more involved ϵ expansion

- For example $\zeta(3)$ and $\mathrm{Li}_3(z)$ appear at $O(\epsilon^2)$
- Symmetric point masters related to cyclotomic harmonic polylogarithms

$$\int_0^1 dx\, \frac{\ln x}{1 - x + x^2}$$

- Now assemble all contributions; use QGRAF to generate graphs

Results

- Landau gauge ghost-gluon vertex

$$
\begin{aligned}
&\Sigma^{\mathrm{ccg},\alpha=0}_{(1)}(p,q)\\
&= -1\\
&+\Big[-\frac{9}{4}\Phi_1(x,y)y^2\Delta_G^{-1} - \frac{15}{16}\Phi_1(x,y)y - \frac{3}{4}\Phi_1(x,y)xy\Delta_G^{-1} - \frac{1}{2} - \frac{1}{4}y\\
&\quad -\frac{1}{4}\ln(x)y\Delta_G^{-1} - \frac{1}{4}\ln(x)x\Delta_G^{-1} - \frac{1}{8}\ln(x)y - \frac{1}{8}\ln(x)x + \frac{1}{16}\Phi_1(x,y)x\\
&\quad +\frac{1}{8}\ln(y) + \frac{1}{8}\ln(y)y + \frac{1}{8}\ln(y)x + \frac{3}{16}\Phi_1(x,y) + \frac{1}{4}x + \frac{1}{4}\ln(x)\\
&\quad +\frac{1}{4}\ln(x)\Delta_G^{-1} + \frac{5}{4}\Phi_1(x,y)y\Delta_G^{-1} + \frac{3}{2}\ln(y)y\Delta_G^{-1} - \ln(y)y^2\Delta_G^{-1}\\
&\quad -\ln(y)xy\Delta_G^{-1} - \Phi_1(x,y)xy^2\Delta_G^{-1} + \Phi_1(x,y)y^3\Delta_G^{-1}\\
&\quad +2\ln(x)xy\Delta_G^{-1}\Big] C_A a
\end{aligned}
$$

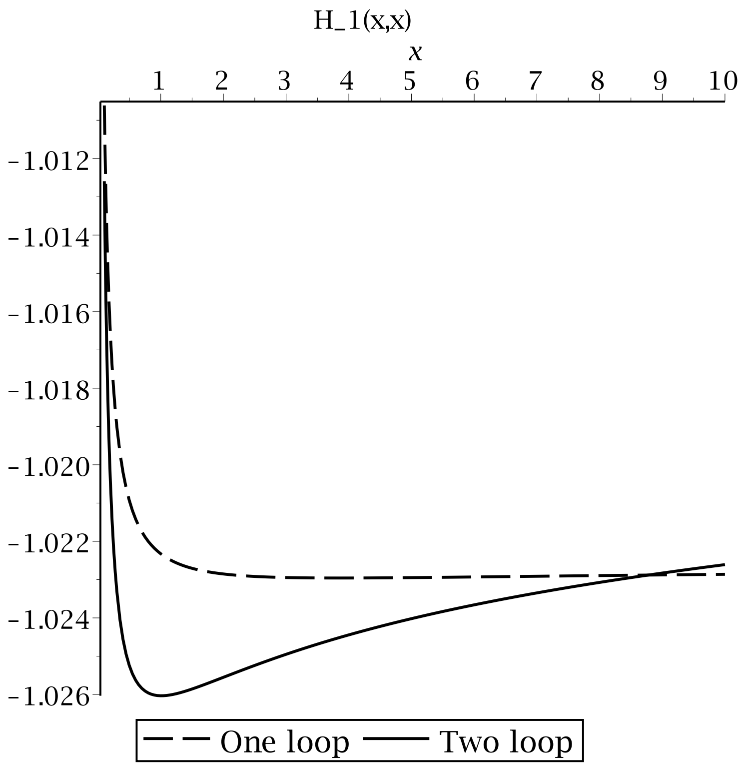
$$
\begin{aligned}
+ \Big[& -\frac{149}{18}\ln(x)xy\Delta_G^{-1} - \frac{59}{12}\ln(y)y\Delta_G^{-1} - \frac{163}{36}\Phi_1(x,y)y\Delta_G^{-1} \\
& -\frac{149}{36}\Phi_1(x,y)y^3\Delta_G^{-1} - \frac{19}{12}\ln(y)\Phi_1(x,y)y^2\Delta_G^{-1} - \frac{3}{2}\ln(x)\Phi_1(x,y)y^2\Delta_G^{-1} \\
& -\frac{47}{36}\ln(x) - \frac{25}{24}\Phi_1(x,y) - \frac{11}{12}\ln^2(y)y^2\Delta_G^{-1} - \frac{11}{12}\ln(y)\Phi_1(x,y)xy^2\Delta_G^{-1} \\
& -\frac{19}{24}\ln(x)\Phi_1(x,y)y - \frac{2}{3}\ln(x)\Phi_1(x,y)xy\Delta_G^{-1} - \frac{2}{3}\ln^2(y)xy\Delta_G^{-1} \\
& -\frac{11}{18}\ln(y) - \frac{1}{2}\ln(x)\Phi_1(x,y)xy^2\Delta_G^{-1} - \frac{1}{2}\ln(y)\Phi_1(x,y)xy\Delta_G^{-1} \\
& -\frac{11}{24}\ln(y)\Phi_1(x,y)y - \frac{7}{18}\ln(x)\Delta_G^{-1} - \frac{11}{36}x - \frac{1}{4}\ln^2(x)y^2\Delta_G^{-1} \\
& -\frac{1}{4}\ln(x)\ln(y)y^2\Delta_G^{-1} - \frac{5}{24}\ln(x)\Phi_1(x,y)x - \frac{1}{6}\ln^2(x)y \\
& -\frac{1}{6}\ln(x)\ln(y)y^3\Delta_G^{-1} - \frac{1}{6}\ln(x)\ln(y)x\Delta_G^{-1} - \frac{1}{6}\Omega_2\left(\frac{y}{x},\frac{1}{x}\right)x \\
& -\frac{1}{6}\Omega_2\left(\frac{x}{y},\frac{1}{y}\right)x - \frac{1}{6}\Phi_1(x,y)y^2 - \frac{1}{12}\ln^2(x)x\Delta_G^{-1} \\
& -\frac{1}{12}\ln^2(x)xy^2\Delta_G^{-1} - \frac{1}{12}\ln(x)\ln(y)x - \frac{1}{12}\ln(x)\Phi_1(x,y)\Delta_G
\end{aligned}
$$

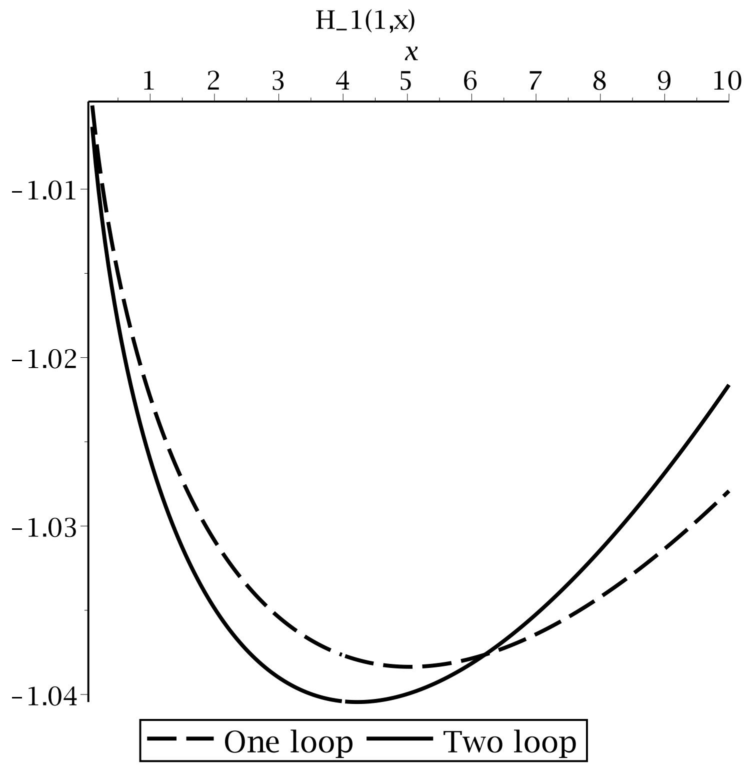
$$-\frac{1}{12}\ln^2(y)y-\frac{1}{12}\ln^2(y)xy^2\Delta_G^{-1}-\frac{1}{36}\ln(y)y-\frac{1}{36}\ln(y)x+\frac{1}{36}\ln(x)y$$
$$+\frac{1}{36}\ln(x)x+\frac{1}{24}\ln(y)\Phi_1(x,y)x+\frac{1}{12}\ln^2(x)\Delta_G^{-1}+\frac{1}{12}\ln^2(x)y\Delta_G^{-1}$$
$$+\frac{1}{12}\ln^2(x)y^3\Delta_G^{-1}+\frac{1}{12}\ln^2(y)+\frac{1}{12}\ln^2(y)y^3\Delta_G^{-1}+\frac{1}{12}\ln^2(y)x$$
$$+\frac{1}{12}\ln(y)\Phi_1(x,y)\Delta_G+\frac{1}{12}\Phi_1(x,y)\Delta_G+\frac{1}{8}\ln(y)\Phi_1(x,y)$$
$$+\frac{1}{6}\ln(x)\ln(y)\Delta_G^{-1}+\frac{1}{6}\ln(x)\ln(y)xy^2\Delta_G^{-1}+\frac{1}{6}\Omega_2\left(\frac{y}{x},\frac{1}{x}\right)$$
$$+\frac{1}{6}\Omega_2\left(\frac{y}{x},\frac{1}{x}\right)y+\frac{1}{6}\Omega_2\left(\frac{x}{y},\frac{1}{y}\right)+\frac{1}{6}\Omega_2\left(\frac{x}{y},\frac{1}{y}\right)y+\frac{1}{6}\Phi_1(x,y)xy$$
$$+\frac{1}{4}\ln^2(x)+\frac{1}{4}\ln(x)\ln(y)y+\frac{7}{24}\ln(x)\Phi_1(x,y)+\frac{11}{36}y+\frac{3}{8}\ln(x)\ln(y)$$
$$+\frac{7}{18}\ln(x)y\Delta_G^{-1}+\frac{7}{18}\ln(x)x\Delta_G^{-1}+\frac{41}{72}\Phi_1(x,y)x+\frac{7}{12}\ln(x)\ln(y)y\Delta_G^{-1}$$
$$+\frac{2}{3}\ln(x)\Phi_1(x,y)y^3\Delta_G^{-1}+\frac{3}{4}\ln(x)\ln(y)xy\Delta_G^{-1}+\frac{3}{4}\ln(y)\Phi_1(x,y)y^3\Delta_G^{-1}$$
$$+\frac{5}{6}\ln(x)\Phi_1(x,y)y\Delta_G^{-1}+\frac{5}{6}\ln(y)\Phi_1(x,y)y\Delta_G^{-1}+\frac{31}{36}+\frac{4}{3}\ln^2(x)xy\Delta_G^{-1}$$

$$
\begin{aligned}
&+\frac{83}{24}\Phi_1(x,y)y+\frac{15}{4}\Phi_1(x,y)xy\Delta_G^{-1}+\frac{149}{36}\ln(y)y^2\Delta_G^{-1}\\
&+\frac{149}{36}\ln(y)xy\Delta_G^{-1}+\frac{149}{36}\Phi_1(x,y)xy^2\Delta_G^{-1}+\frac{26}{3}\Phi_1(x,y)y^2\Delta_G^{-1}\\
&+\ln^2(y)y\Delta_G^{-1}\Big]\,C_A T_F N_f a^2\ +\ \ldots
\end{aligned}
$$

- Other amplitudes similar

Graphical illustration





- Comparison of one and two loop functions in several directions for projection 1 in Landau gauge with $\alpha_s = 0.125$ where $H_k(x,y) = \Sigma^{\text{ccg}}_{(k)}(p,q)$
- Two loop corrections not significant

MOM renormalization

- Can now examine vertices in MOM schemes by following earlier prescription
- Based on symmetric point; $x = y = 1$
- Restriction of masters to this point produces the one loop basis $\{\mathbb{Q}, \pi^2, \psi'\left(\frac{1}{3}\right)\}$ for the renormalization group functions and vertices
- With $s_n(z) = \frac{1}{\sqrt{3}} \Im \left[\mathrm{Li}_n \left(\frac{e^{iz}}{\sqrt{3}} \right) \right]$, the MOM basis at two loops is

$$\left\{ \mathbb{Q}, \pi^2, \zeta(3), \zeta(4), \psi'\left(\frac{1}{3}\right), \psi'''\left(\frac{1}{3}\right), s_2\left(\frac{\pi}{2}\right), s_2\left(\frac{\pi}{6}\right), \right.$$
$$\left. s_3\left(\frac{\pi}{2}\right), s_3\left(\frac{\pi}{6}\right), \frac{\ln^2(3)\pi}{\sqrt{3}}, \frac{\ln(3)\pi}{\sqrt{3}}, \frac{\pi^3}{\sqrt{3}} \right\}$$

- In $\overline{\mathrm{MS}}$ to three loops the basis will involve $\zeta(3)$
- Illustrate the relation by considering the coupling constant in MOMh and $\overline{\mathrm{MS}}$
- After renormalization in each scheme can define the relation between parameters via

$$a_{\mathrm{MOMh}} = \frac{a}{\left(C_g(a,\alpha)\right)^2}$$

where $C_g(a,\alpha) = Z_g^{\mathrm{MOMh}} / Z_g$

MOMh coupling constant map

- Explicit relation for Landau gauge is

$$
\begin{aligned}
a' \quad = \quad & a + \left[\left[15\psi'(\frac{1}{3}) - 10\pi^2 + 615 \right] C_A - 240 T_F N_f \right] \frac{a^2}{108} \\
& + \left[\left[450(\psi'(\frac{1}{3}))^2 - 600\pi^2\psi'(\frac{1}{3}) - 458928\psi'(\frac{1}{3}) - 3213\psi'''(\frac{1}{3}) \right.\right. \\
& -3825792 s_2(\frac{\pi}{6}) + 7651584 s_2(\frac{\pi}{2}) + 6376320 s_3(\frac{\pi}{6}) - 5101056 s_3(\frac{\pi}{2}) \\
& +8768\pi^4 + 305952\pi^2 + 7776\Sigma + 153576\zeta(3) + 6521760 \\
& \left. -26568 \frac{\ln^2(3)\pi}{\sqrt{3}} + 318816 \frac{\ln(3)\pi}{\sqrt{3}} + 28536 \frac{\pi^3}{\sqrt{3}} \right] C_A^2 + 460800 T_F^2 N_f^2 \\
& + \left[206784\psi'(\frac{1}{3}) + 1492992 s_2(\frac{\pi}{6}) - 2985984 s_2(\frac{\pi}{2}) - 2488320 s_3(\frac{\pi}{6}) \right. \\
& +1990656 s_3(\frac{\pi}{2}) - 137856\pi^2 - 995328\zeta(3) - 4015296 \\
& \left. +10368 \frac{\ln^2(3)\pi}{\sqrt{3}} - 124416 \frac{\ln(3)\pi}{\sqrt{3}} - 11136 \frac{\pi^3}{\sqrt{3}} \right] C_A T_F N_f \\
& \left. + \left[1492992\zeta(3) - 1710720 \right] C_F T_F N_f \right] \frac{a^3}{93312}
\end{aligned}
$$

Renormalization group

- Can transform between schemes using the other conversion functions defined from the renormalization constants

$$C_\phi(a,\alpha) \;=\; \frac{Z_\phi^{\mathrm{MOMh}}}{Z_\phi}$$

  where $\phi \in \{A, c, \psi\}$

- Then renormalization group functions in different schemes are related by

$$\begin{aligned}\gamma_\phi^{\mathrm{MOMi}}\left(a_{\mathrm{MOMi}}, \alpha_{\mathrm{MOMi}}\right) \;&=\; \left[\gamma_\phi(a) + \beta(a)\frac{\partial}{\partial a}\ln C_\phi(a,\alpha)\right. \\ &\left.\qquad + \,\alpha\gamma_\alpha(a,\alpha)\frac{\partial}{\partial \alpha}\ln C_\phi(a,\alpha)\right]_{\overline{\mathrm{MS}}\to\mathrm{MOMi}}\end{aligned}$$

  where mapping indicates that $\overline{\mathrm{MS}}$ variables are mapped back to MOMi ones

- Knowledge of conversion functions at L loops in one scheme and $\overline{\mathrm{MS}}$ renormalization group functions at $(L+1)$ loops means the $(L+1)$ loop renormalization group functions can be deduced in the first scheme at $(L+1)$ loops *without* an explicit $(L+1)$ loop computation in that scheme

Extension to problems with masses

- Current interest in the infrared behaviour of the QCD propagators and vertices in the low energy region
- In intermediate energy range lattice gauge theory analysis suggests there are power corrections to that predicted from high energy
- These are either dimension two or dimension four and from operator product expansion would suggest existence of underlying dimension two or four operators
- Important for understanding running coupling constant definition
- Effects can be modeled by non-zero gluon mass or Gribov mass
- Basic idea is to examine such corrections at the symmetric point vertices at one loop
- Care required in naively expanding massive integrals to avoid spurious infrared infinities
- Method developed by Smirnov, Tausk, Davydychev and Behrends for various limits
- Define master vertex integral

$$I(\alpha,\beta,\gamma;m_1^2,m_2^2,m_3^2) ~=~ \int_k \frac{1}{[k^2+m_1^2]^\alpha[(k-p)^2+m_2^2]^\beta[(k+q)^2+m_3^2]^\gamma} \Bigg|$$

- If $m_1 = 0$ and/or $m_2 = 0$ then potential infrared poles but integral is finite

Expansion

- Expand asymptotically using a method which corrects for the appearance of these spurious singularities
- Graphically
$$I_\Gamma \sim \sum_\lambda I_{\Gamma/\lambda} \circ \mathcal{T}_{\{m_i\};\{q_i\}} I_\lambda$$
where Γ is the original graph and λ are certain subgraphs in the asymptotic expansion
- First term is always the naive expansion
- Subgraphs λ here for $O(m_i^2)$ corrections are given by all possible routings of the (two) external momenta around the graph
- In each of these subgraphs the identified subgraph λ is expanded in the masses and the momenta q_i which are external to λ itself
- This process is denoted by $\mathcal{T}_{\{m_i\};\{q_i\}} I_\lambda$ and this is substituted into the *reduced* graph $I_{\Gamma/\lambda}$ and then the loop momenta integrated
- Requires an additional integration by parts database to complete the integrals in the expansion

- For example, if $m_i \neq 0$

$$
\begin{aligned}
I(1,1,1;m_1^2,m_2^2,m_3^2) \quad &= \quad \big[I(1,1,1;0,0,0) - m_1^2 I(2,1,1;0,0,0) \\
&- m_2^2 I(1,2,1;0,0,0) \\
&- m_3^2 I(1,1,2;0,0,0)\big] \\
&+ \left[\frac{1}{p^2 q^2} I(1,0,0;m_1,0,0)\right. \\
&- \frac{1}{(p^2)^2 q^2} \int \frac{k^2 + m_2^2 - 2kp}{(k^2+m_1^2)} \\
&- \left.\frac{1}{p^2 (q^2)^2} \int \frac{k^2 + m_3^2 + 2kq}{(k^2+m_1^2)}\right] \\
&+ \ldots
\end{aligned}
$$

for general external momentum configuration

- Divergences in the naive expansion are cancelled from the extra terms in the graphical expansion

Application - Gribov problem

- Gribov problem arises from the inability to globally fix a covariant gauge in a non-abelian gauge theory
- Different gauge configurations can satisfy the same gauge condition leading to an overcounting in path integral construction
- Locally gauge is fixed uniquely and no issues in ultraviolet analyses of QCD
- Infrared structure is affected and (Gribov) copies have to be factored out of path integral
- Gribov effected this by restricting the path integral to the first Gribov region
- Leads to a new action with an additional non-local term in Landau gauge [Gribov; Zwanziger]

$$\frac{\gamma^4}{2} f^{acp} f^{bdp} A^a_\mu \left(\frac{1}{\partial^\nu D_\nu} \right)^{cd} A^{b\,\mu} \; - \; \frac{d N_A \gamma^4}{2g^2}$$

where γ is the Gribov mass

- Non-locality can be localized to produce a renormalizable local Lagrangian but with extra ghost fields [Zwanziger]

- Consequence is that the gluon propagator is modified by the Gribov mass γ

$$\langle A^a_\mu(p) A^b_\nu(-p) \rangle \;=\; -\,\frac{\delta^{ab} D_A(p^2)}{p^2} P_{\mu\nu}(p)$$

where

$$D_A(p^2) \;=\; \frac{(p^2)^2}{[(p^2)^2 + C_A \gamma^4]}$$

which vanishes at zero momentum and has no pole

- γ is not an independent parameter and satisfies a gap equation

$$1 \;=\; C_A \left[\frac{5}{8} - \frac{3}{8} \ln \left(\frac{C_A \gamma^4}{\mu^4} \right) \right] a \;+\; O(a^2)$$

- Two loop correction known
- Only when γ satisfies the gap equation is one in the gauge theory
- Faddeev-Popov ghost propagator enhances at zero momentum
- Examine expansion of vertex functions at symmetric point in powers of γ^2/μ^2

Triple gluon vertex

- For symmetric point use compact tensor basis

$$
\begin{aligned}
\mathcal{P}^{ggg}_{(1)\mu\nu\sigma}(p,q) &= \eta_{\mu\nu}p_\sigma - \eta_{\mu\nu}q_\sigma - 2\eta_{\mu\sigma}p_\nu - \eta_{\sigma\mu}q_\nu + \eta_{\nu\sigma}p_\mu + 2\eta_{\nu\sigma}q_\mu \\
\mathcal{P}^{ggg}_{(2)\mu\nu\sigma}(p,q) &= [2p_\mu p_\nu p_\sigma + p_\mu q_\nu p_\sigma - p_\mu q_\nu q_\sigma \\
&\quad + 2q_\mu p_\nu p_\sigma - 2q_\mu p_\nu q_\sigma - 2q_\mu q_\nu q_\sigma] \frac{1}{2\mu^2} \\
\mathcal{P}^{ggg}_{(3)\mu\nu\sigma}(p,q) &= [p_\mu p_\nu q_\sigma - q_\mu p_\nu p_\sigma + q_\mu p_\nu q_\sigma - q_\mu q_\nu p_\sigma] \frac{1}{\mu^2}
\end{aligned}
$$

- Extra fields in Zwanziger construction mean that there are 30 graphs at one loop
- Use same techniques as before
- Laporta algorithm used to reduce to masters which are then expanded as above
- Amplitudes, $\Sigma^{ggg}_{(i)}(p,q,\gamma^2)$, depend on γ

Results

- For triple gluon vertex at one loop

$$\begin{aligned}\Sigma^{ggg}_{(1)}(p,q,\gamma^2) &= \Sigma^{ggg}_{(1)}(p,q,0) \\ &+ \left[\frac{13}{6}+\frac{7\pi^2}{36}-\frac{7}{24}\psi'\left(\frac{1}{3}\right)-\frac{1}{2}\ln\left[\frac{C_A\gamma^4}{\mu^4}\right]\right]\frac{C_A^2\gamma^4}{\mu^4}a \\ \Sigma^{ggg}_{(2)}(p,q,\gamma^2) &= \Sigma^{ggg}_{(2)}(p,q,0)+\frac{3\pi}{32}\frac{C_A^{3/2}\gamma^2}{\mu^2}a \\ \Sigma^{ggg}_{(3)}(p,q,\gamma^2) &= \Sigma^{ggg}_{(3)}(p,q,0)+\frac{3\pi}{32}\frac{C_A^{3/2}\gamma^2}{\mu^2}a\end{aligned}$$

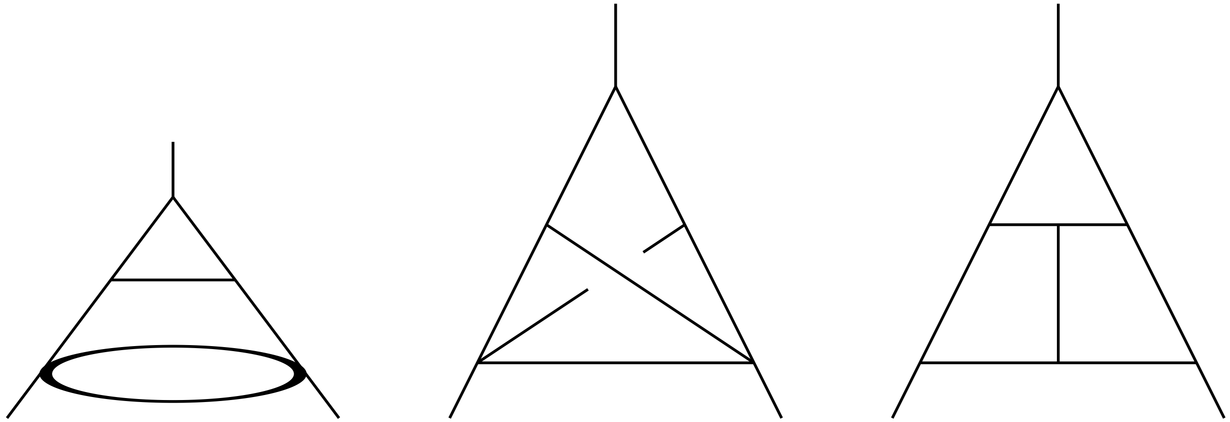
- Power corrections for channel 1 are dimension four; others are dimension two
- At asymmetric subtraction point all channels have dimension two corrections
- For ghost-gluon and quark-gluon corrections in all channels at symmetric and asymmetric points all first corrections are dimension two

Conclusions

- Have reviewed the renormalization of vertex functions in QCD in various renormalization schemes
- Discussed relation to current developments in evaluation of master integrals
- Algorithms are *in principle* now in place to systematically analyse higher n-point functions to next loop orders
- Next obvious computations are two loop quartic vertices and three loop 3-point vertices both at the symmetric point

Homework

- For three loop extensions probably will need the symmetric point evaluations of ‘masters’ such as probably



plus others

- Also will need higher orders in ϵ for one and two loop masters