

Twistor actions, Wilson loops, d logs and dilogs

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cf: review with Adamo, Bullimore & Skinner 1104.2890, &
recent work with Lipstein arxiv:1212.6228, 1307.1443.

[Also. work by Alday, Arkani-Hamed, Cachazo, Caron-Huot,
Drummond, Henn, Heslop, Korchemsky, Maldacena,
Sokatchev. (Annecy, Oxford, Perimeter and Princeton IAS).]

$\mathcal{N} = 4$ Super Yang-Mills

The analogue of the harmonic oscillator for 4-dimensional quantum field theory?

- Toy version of standard model.
- Best behaved nontrivial 4d field theory (UV finite, superconformal $SU(2, 2|4)$ symmetry, . . .).
- Particle spectrum

helicity	-1	-1/2	0	1/2	1
# of particles	1	4	6	4	1

- Susy changes helicity so that particles form irrep of ‘super’-group $SU(2, 2|4)$ like single particle.
- Contains QCD and more classically.
- ‘completely integrable’ in planar (large N) sector.
- much twistor geometry in their amplitudes:
 - ① (Ambo-)Twistor string description, (next lecture)
 - ② Grassmannian residue formula,
 - ③ polyhedra volumes \leadsto the amplituhedron,
 - ④ Focus here on the *holomorphic Wilson loop*.

Scattering amplitudes for $\mathcal{N} = 4$ super Yang-Mills

4-Momentum:

$$p = (E, p_1, p_2, p_3) = E(1, v_1, v_2, v_3),$$

massless $\Leftrightarrow |\mathbf{v}| = c = 1 \Leftrightarrow p \cdot p := E^2 - p_1^2 - p_2^2 - p_3^2 = 0.$ \Leftrightarrow

$$\Leftrightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} E + p_3 & p_1 + ip_2 \\ p_1 - ip_2 & E - p_3 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} (\tilde{\lambda}_{0'} \quad \tilde{\lambda}_{1'})$$

Supermomentum: $P = (\lambda, \tilde{\lambda}, \eta) \in \mathbb{C}^{4|0} \times \mathbb{C}^{0|4}$, where
 $\eta_i, i = 1, \dots, 4$ anti-commute \leadsto wave functions:

$$\Xi(\lambda, \tilde{\lambda}, \eta) = A_+ + \Psi^i \eta_i + \Phi^{ij} \eta_i \eta_j + \tilde{\Psi}^{ijk} \eta_i \eta_j \eta_k + A_- \eta_1 \eta_2 \eta_3 \eta_4.$$

Amplitude: for n -particle process is

$$\mathcal{A}(1, \dots, n) = \mathcal{A}(P_1, \dots, P_n)$$

MHV degree: $m = \{ \text{weight in } \eta \text{s} \} / 4 - 2.$

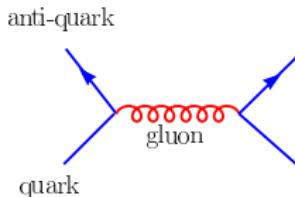
If $k \sim \# \text{ of } -\text{ve helicity particles, susy} \Rightarrow \mathcal{A} = 0 \text{ for } k = 0, 1$

For $k = 2, \mathcal{A} \neq 0$, ‘Maximal Helicity Violating’ (MHV).

$m = k - 2 := \text{the MHV degree.}$

Ordinary Feynman diagrams

Contributions



Feynman diagrams are more than pictures. They represent algebraic formulas for the propagation and interaction of particles.

$$\text{wavy line} \quad \frac{\eta_{\mu\nu}}{p^2}$$

$$g\gamma_\mu$$

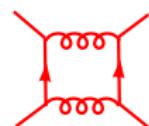
$$g[(p_1 - p_2)_\rho \eta_{\mu\nu} + (p_2 - p_3)_\mu \eta_{\nu\rho} + (p_3 - p_1)_\nu \eta_{\rho\mu}]$$



LO



NLO

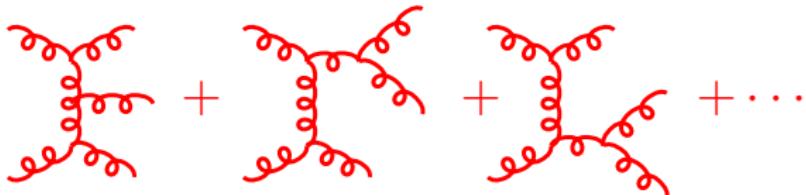


Trees \leftrightarrow classical, loops \leftrightarrow quantum.

Locality: only simple poles from propagators at $(\sum p_i)^2 = 0$.

Consider the five-gluon tree-level amplitude of QCD. Enters in calculation of multi-jet production at hadron colliders.

Described by following Feynman diagrams:



If you follow the textbooks you discover a disgusting mess.

Result of a brute force calculation:

```
+a1 -a2 -a3 -a4 -a5 -a6 -a7 -a8 -a9 -a10 -a11 -a12 -a13 -a14 -a15 -a16 -a17 -a18 -a19 -a20 -a21 -a22 -a23 -a24 -a25 -a26 -a27 -a28 -a29 -a30 -a31 -a32 -a33 -a34 -a35 -a36 -a37 -a38 -a39 -a40 -a41 -a42 -a43 -a44 -a45 -a46 -a47 -a48 -a49 -a50 -a51 -a52 -a53 -a54 -a55 -a56 -a57 -a58 -a59 -a60 -a61 -a62 -a63 -a64 -a65 -a66 -a67 -a68 -a69 -a70 -a71 -a72 -a73 -a74 -a75 -a76 -a77 -a78 -a79 -a80 -a81 -a82 -a83 -a84 -a85 -a86 -a87 -a88 -a89 -a90 -a91 -a92 -a93 -a94 -a95 -a96 -a97 -a98 -a99 -a100 -a101 -a102 -a103 -a104 -a105 -a106 -a107 -a108 -a109 -a110 -a111 -a112 -a113 -a114 -a115 -a116 -a117 -a118 -a119 -a120 -a121 -a122 -a123 -a124 -a125 -a126 -a127 -a128 -a129 -a130 -a131 -a132 -a133 -a134 -a135 -a136 -a137 -a138 -a139 -a140 -a141 -a142 -a143 -a144 -a145 -a146 -a147 -a148 -a149 -a150 -a151 -a152 -a153 -a154 -a155 -a156 -a157 -a158 -a159 -a160 -a161 -a162 -a163 -a164 -a165 -a166 -a167 -a168 -a169 -a170 -a171 -a172 -a173 -a174 -a175 -a176 -a177 -a178 -a179 -a180 -a181 -a182 -a183 -a184 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```

The Parke-Taylor MHV amplitude

However, result for helicity (+ + - - -) part of the amplitude is

$$\mathcal{A}(1, 2, 3, 4, 5) = \delta \left(\sum_{a=1}^5 p_a \right) \frac{\langle \lambda_b \lambda_c \rangle^4}{\langle \lambda_1 \lambda_2 \rangle \langle \lambda_2 \lambda_3 \rangle \langle \lambda_3 \lambda_4 \rangle \langle \lambda_4 \lambda_5 \rangle \langle \lambda_5 \lambda_1 \rangle}$$

where b and c are the + helicity particles and

$$\langle i j \rangle := \langle \lambda_i \lambda_j \rangle := \lambda_{i0} \lambda_{j1} - \lambda_{i1} \lambda_{j0}$$

(similarly use $[i, j]$ for $\tilde{\lambda}_i$ s).

More generally (Parke-Taylor 1984, Nair 1986)

$$\mathcal{A}_{MHV}^{tree}(1, \dots, n) = \frac{\delta^{4|8} (\sum_{a=1}^n (p_a, \eta_a \lambda_a))}{\prod_{a=1}^n \langle \lambda_a \lambda_{a+1} \rangle}$$

Twistor space (Penrose 1967)

Super twistor space is $\mathbb{CP}^{3|4}$ with homogeneous coords:

$$Z = (\lambda_\alpha, \mu^{\dot{\alpha}}, \chi_i) \in \mathbb{T} := \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^{0|4}, \quad Z \sim \zeta Z, \zeta \in \mathbb{C}^*.$$

\mathbb{T} = fund. repn of superconformal group $SU(2, 2|4)$.

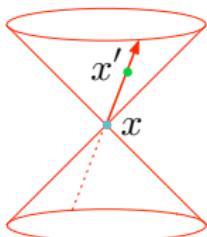
A point in super Minkowski space, \mathbb{M} , coords $(x, \theta) \leftrightarrow$ a line

$X = \mathbb{CP}^1 \subset \mathbb{PT}$ via incidence relations

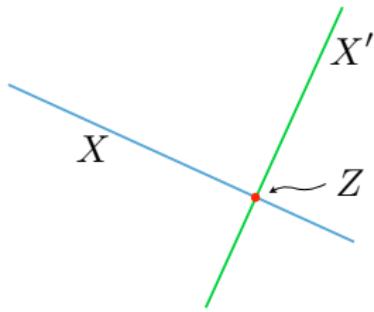
$$\mu^{\dot{\alpha}} = ix^{\alpha\dot{\alpha}}\lambda_\alpha, \quad \chi_i = \theta_i^\alpha\lambda_\alpha.$$

Two points x, x' are null separated iff X and X' intersect.

Space-time



Twistor Space



Linear Maxwell fields and Penrose transform

- Let $f \in \mathcal{O}(n) \Leftrightarrow f(\gamma Z) = \gamma^n f(Z)$.
- Linear Maxwell fields obtained from

$$(a, b) \in \Omega^{0,1} \times \Omega^{0,1}(-4), \quad \bar{\partial}a = \bar{\partial}b = 0,$$

modulo gauge freedom $(a, b) \rightarrow (a + \bar{\partial}\alpha, b + \bar{\partial}\beta)$.

- Thus $(a, b) \in H^1(\mathcal{O}) \times H^1(\mathcal{O}(-4))$.

Action: above twistor data arises from

$$S = \int_{\mathbb{PT}} a \wedge \bar{\partial}b \wedge D^3Z, \quad D^3Z = \varepsilon^{IJKL} Z_I dZ_J \wedge dZ_K \wedge dZ_L.$$

Space-time fields: given by

$$\phi_{\dot{\alpha}\dot{\beta}}(x) = \int_X \frac{\partial^2 a}{\partial \mu^{\dot{\alpha}} \partial \mu^{\dot{\beta}}} D\lambda, \quad \phi_{\alpha\beta}(x) = \int_X \lambda_\alpha \lambda_\beta b D\lambda, \quad D\lambda = \lambda_\gamma d\lambda^\gamma.$$

Ex: Plane wave, momentum $\lambda_i \tilde{\lambda}_i$ helicity s :

$$a, b \text{ etc. } = \int_{\mathbb{C}} \frac{dt}{t^{1-2s}} \bar{\delta}^2(t\lambda - \lambda_i) e^{it[\mu, \tilde{\lambda}_i]}.$$

The Nonlinear Ward Construction

Theorem (Ward 1978)

Local solutions (E', D_A) to self-dual Yang Mills equs on $U \subset \mathbb{C}^4$ correspond 1:1 to holomorphic vector bundles $E \rightarrow \mathbb{P}\mathbb{T}(U)$.

Idea: $E'_x =$ holomorphic sections of $E|_x$.

Exhibits complete integrability of self-duality equations
(so, no scattering).

Applications:

- ADHM classification of instantons.
- Construction of monopoles.
- Unification of theory of integrable systems.

Data & field equs. Hol vector bundles $E \rightarrow \mathbb{PT}$ given by

$$\bar{\partial}_a = \bar{\partial}_0 + a, \quad a \in \Omega^{0,1} \otimes \mathfrak{sl}(n, \mathbb{C}), \quad \text{with} \quad F^{0,2} := \bar{\partial}_a^2 = 0.$$

Action. Introduce Lagrange multiplier $b \in \Omega^{0,1}(-4)$,

$$S[a, b] = \int_{\mathbb{PT}} \text{tr} (b \wedge F^{0,2}) D^3 Z \quad D^3 Z \in \Omega^{3,0}(4).$$

Action also gives $\bar{\partial}_a b = 0$; so b gives an ASD linear field:

$$B(x) = \int_X H^{-1} b H \wedge D^3 Z \in \Omega^{2-}, \quad D_A B = 0.$$

On space-time. Chalmers-Siegel SD YM action:

$$S[A, B] = \int_M \text{tr } F_{A \wedge B}, \quad (A, B) \in (\Omega^1, \Omega^{2-}) \otimes \mathfrak{sl}(N)$$

Has degrees of freedom of full YM, but only SD interactions.

Extend to full YM: add $g^2 \int \text{tr } B \wedge B$ term.

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Has degrees of freedom of full YM, but only SD interactions.

Extend to full YM: add $g^2 \int \text{tr } B \wedge B$ term.

Supersymmetric Ward correspondence

Super Calabi-Yau:

$\mathbb{C}\mathbb{P}^{3|4}$ has weightless super volume form

$$D^{3|4}Z = D^3 Z \, d\chi_1 \dots d\chi_4 \in \Omega_{Ber}.$$

'Super-Ward' for $\mathcal{N} = 4$ SYM:

A dbar-op $\bar{\partial}_{\mathcal{A}} = \bar{\partial}_0 + \mathcal{A}$ on a bundle on $\mathbb{C}\mathbb{P}^{3|4}$ has expansion

$$\mathcal{A} = a + \chi_a \psi^a + \chi_a \chi_b \phi^{ab} + \chi^{3a} \tilde{\psi}_a + \chi^4 b$$

and $\bar{\partial}_{\mathcal{A}}^2 = 0 \leftrightarrow$ solns to SD $\mathcal{N} = 4$ SYM on space-time.
[ψ and $\tilde{\psi}$ give fermions and ϕ scalars.]

Action for fields with self-dual interactions: [Witten]

SD interactions \leftrightarrow holomorphic Chern-Simons action

$$S_{sd} = \int_{\mathbb{PT}} \text{tr}(\mathcal{A} \wedge \bar{\partial} \mathcal{A} + \frac{2}{3} \mathcal{A}^3) \wedge D^{3|4}Z.$$

Incorporating interactions of full $\mathcal{N} = 4$ SYM

Nair, M., Boels, Skinner, 2006

Extension to full SYM:

$$S_{full}[\mathcal{A}] = S_{sd}[\mathcal{A}] + S_{int}[\mathcal{A}]$$

includes non-local interaction term:

$$\begin{aligned} S_{int}[\mathcal{A}] &= g^2 \int_{\mathbb{M}} d^{4|8}\mathbf{x} \log \det(\bar{\partial}_{\mathcal{A}}|_X) \\ &= g^2 \sum_{n=2}^{\infty} \int_{\mathbb{M} \times (\times^n X)} d^{4|8}\mathbf{x} \frac{\text{tr}(\mathcal{A}(\lambda_1)\mathcal{A}(\lambda_2)\dots\mathcal{A}(\lambda_n)) D\lambda_1\dots D\lambda_n}{\langle \lambda_1\lambda_2 \rangle \langle \lambda_2\lambda_3 \rangle \dots \langle \lambda_n\lambda_1 \rangle} \end{aligned}$$

X is \mathbb{CP}^1 corresponding to $\mathbf{x} \in \mathbb{M}^{4|8}$, $\lambda_i \in X_i$, i^{th} factor in $\times^n X$,

$$K_{ij} = \frac{D\lambda_j}{\langle \lambda_i \lambda_j \rangle}$$

is Cauchy kernel of $\bar{\partial}^{-1}$ on X at λ_i, λ_j .

Axial gauge, Feynman rules & MHV formalism

Choose ‘reference twistor’ Z_* , impose gauge $\bar{Z}_* \cdot \frac{\partial}{\partial \bar{Z}} \lrcorner \mathcal{A} = 0$.

- Cubic Chern-Simons vertex vanishes.
- Propagator = delta-function forcing Z, Z', Z_* to be collinear.

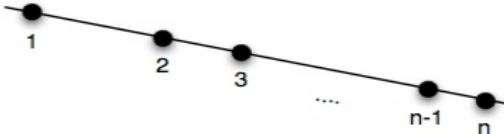
$$\Delta(Z, Z') = \frac{1}{2\pi i} \bar{\delta}^{2|4}(Z, Z_*, Z') := \frac{1}{2\pi i} \int \frac{ds dt}{st} \bar{\delta}^{4|4}(Z_* + sZ + tZ')$$

[Here $\bar{\delta}^1(z) = \bar{\partial} \frac{1}{2\pi i z}$ for $z \in \mathbb{C}$ and $\delta^{0|1}(\chi) = \chi$ for χ odd.]

- log-det term expands to give ‘MHV vertices’:

$$V(Z_1, \dots, Z_n) = \int_{\mathbb{M} \times L_X^n} \frac{d^{4|4}Z_A d^{4|4}Z_B}{Vol GL(2)} \prod_{r=1}^n \frac{\bar{\delta}^{3|4}(Z_r, Z_A + \sigma_r Z_B)}{(\sigma_{r-1} - \sigma_r)} d\sigma_r.$$

- Vertices force Z_1, \dots, Z_n to lie on line Z_A to Z_B :



- On momentum space gives ‘MHV rules’ for amplitudes with vertices = off-shell MHV amplitudes, non-local,

Large N and region momentum space

- For $SU(N)$, $N \rightarrow \infty$, only single trace terms survive
- gives ordering of particles: if particle i has colour $c_i \in su(N)$, coeff of $\text{tr}(c_{i_1} c_{i_2} \dots c_{i_n})$ gives ordering i_1, \dots, i_n .

Supermomentum conservation \leadsto null polygon $\{X_i\} = \{(x_i, \theta_i)\}$
 \in ‘region momentum space’ = \mathbb{M} , super Minkowski space:

$$(p_i^{AA'}, \eta_i^a \lambda^A) = (x_i^{AA'} - x_{i+1}^{AA'}, \theta_i^A - \theta_{i+1}^A).$$

Instead of amplitude, we consider ‘ratio’ \tilde{R} :

$$\mathcal{A} = \mathcal{A}_{MHV}^{\text{tree}} \tilde{R}(X_1, \dots, X_n)$$

Conjecture (Alday, Maldacena)

At MHV, arbitrary loop order, \tilde{R} = Yang-Mills correlation function of a Wilson-loop around momentum polygon in \mathbb{M} .

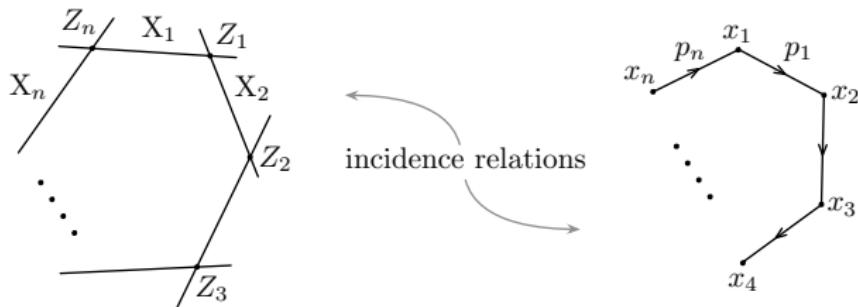
Corollary: \tilde{R} dual conformal invariant (up to known anomaly).

Much evidence: [Drummond, Henn, Korchemsky & Sokatchev; Brandhuber, Heslop, & Travaglini].



Momentum polygons in twistor space

A null polygon in space-time \leftrightarrow generic polygon in \mathbb{PT} [Hodges].



Change variables so that $\tilde{R}(X_1, \dots, X_n) = R(Z_1, \dots, Z_n)$.

Important simplification: $Z_i \in \mathbb{PT}$ are unconstrained.

What is analogue of Alday-Maldacena conjecture on \mathbb{PT} ?

Holomorphic Wilson loops

For Wilson-loop, need holonomy around polygon in \mathbb{PT} .

- Polygon vertices Z_i , edges $X_i = \langle Z_{i-1}, Z_i \rangle$.
- Parametrize X_i by σ with $Z(\sigma) = \sigma Z_{i-1} + Z_i$.
- On X_i find global frame $F_i(\sigma)$ of $E|_{X_i}$ with $\bar{\partial}_{\mathcal{A}}|_{X_i} F_i = 0$, boundary condition $F_i(\infty) = F_i|_{Z_{i-1}} = 1$.
- Define

$$W = \text{tr} \prod_{i=1}^n F_i|_{Z_i} = \text{tr} \prod_{i=1}^n F_i(0).$$

- Agrees with space-time Wilson loop on-shell.

Perturbatively iterate $F_i = 1 + \bar{\partial}^{-1}(\mathcal{A}F_i)$ to get

$$F_i = 1 + \sum_{r=1}^{\infty} \prod_{s=1}^r \bar{\partial}_{s-1 s}^{-1} \mathcal{A}(\sigma_s), \quad (\bar{\partial}_{rs}^{-1} f)(\sigma_r) = \int_{L_{X_i}} \frac{f(\sigma_s) d\sigma_s}{\sigma_r - \sigma_s}$$

The S-matrix as a holomorphic Wilson loop

Theorem (Bullimore, M., Skinner, 2010-11)

The all-loop integrand for the planar n-particle amplitude for super Yang-Mills is identical to that for the holomorphic Wilson loop in twistor space:

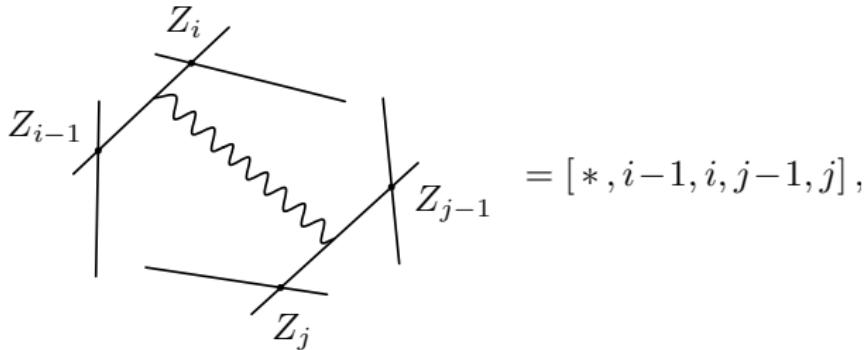
$$\mathcal{A}(1, \dots, n) = \langle W(Z_1, \dots, Z_n) \rangle \mathcal{A}_{\text{MHV}}^{\text{tree}}.$$

- Tree amplitudes arise at $g = 0$ when the Wilson-loop is calculated in the self-dual sector.
- The loop expansion for $\mathcal{A} =$ expansion for W in g .
- The MHV diagrams for amplitude are planar duals of those for Wilson-loop correlator in axial gauge on twistor space.

Proof: direct calculation of Feynman diagrams in axial gauge.

The all-loop integrand [Arkani-Hamed, et. al.] is canonical for planar gauge theories & delays confronting infrared divergences.

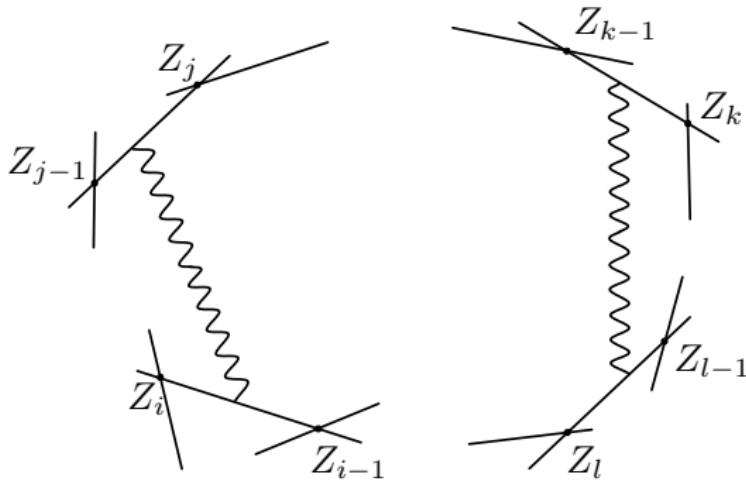
NMHV case: for \mathcal{A}^2 part of $\langle W \rangle$, $\langle \mathcal{A}(Z)\mathcal{A}(Z') \rangle = \Delta(Z, Z')$:
 obtain $\langle W \rangle = \sum_{i < j} \Delta_{ij}$ where $\Delta_{ij} =$



$$\begin{aligned} [1, 2, 3, 4, 5] &:= \int \frac{ds_1 ds_2 ds_3 ds_4}{s_1 s_2 s_3 s_4} \bar{\delta}^{4|4}(Z_5 + \sum_{i=1}^4 s_i Z_i), \\ &= \frac{\prod_{a=1}^4 ((1234)\chi_5^a + \text{cyclic})}{(1234)(2345)(3451)(4512)(5123)} \end{aligned}$$

is the ‘R-invariant’.

N^2 MHV: quartic terms in \mathcal{A} in W give Wick contractions

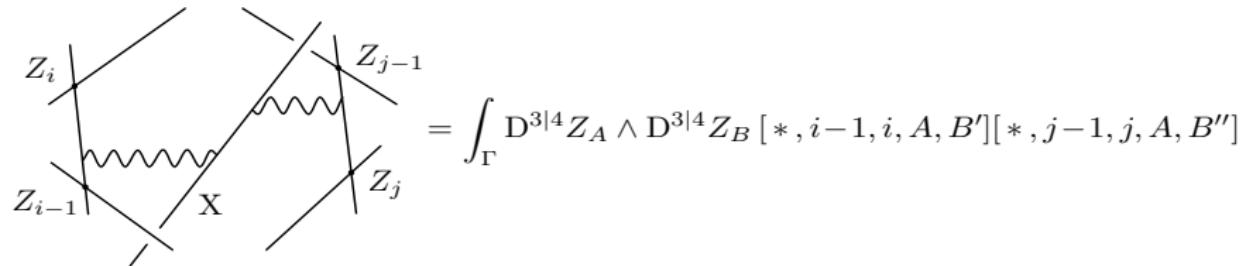


No crossed propagators for planarity.

N^k MHV tree amplitudes:

k propagators \leadsto product of k R-invariants.

At MHV with one MHV vertex obtain $\sum_{i,j} K_{ij}$ with $K_{ij} =$



Loop momenta \leftrightarrow location of line $X = \langle Z_A Z_B \rangle$. Recall:

$$[*, i-1, i, A, B] := \int \frac{ds_1 ds_2 ds_3 ds_4}{s_1 s_2 s_3 s_4} \bar{\delta}^{4|4}(Z_* + s_1 Z_A + s_2 Z_B + s_3 Z_{i-1} + s_4 Z_i)$$

can integrate $D^{3|4} Z_A \wedge D^{3|4} Z_B$ against delta functions

$$K_{ij} = \frac{1}{(2\pi i)^2} \int \frac{ds_0 ds dt_0 dt_1}{s_0 s t_0 t}.$$

External data encoded in integration contour (see later).

Applications to loop integrals

In general: loop order = # MHV vertices.

Loop integrand: At l -loops, integral is over l loop 4-momenta L_r , $r = 1, \dots, l$.

$$\langle W \rangle = \int_{(\mathbb{R}^4)^l} F(L_r, Z_i) \prod_{r=1}^l d^4 L_r .$$

Expect $\langle W \rangle =$ polylogs of degree $2l$ of invariants a_j of Z_i .

Definition

A polylog of degree $2l$ is an iterated integral

$$Plog(a_1, a_2, \dots) = \int_{[0,1]^{2l}} \prod_{m=1}^{2l} \frac{dR_m}{R_m}$$

where $R_m =$ rational fns of a_i and integration parameters s_m .

Conjecture: At least at MHV, can express integrated amplitude as polylog of pure transcendentality degree $2l$.

$d \log$ form of loop integrand

Lipstein & M. arxiv:1212.6228.

In our formulation

- L_r are encoded in (Z_{A_r}, Z_{B_r}) .
- At MHV, have two propagators per vertex
- can integrate Z_{A_r}, Z_{B_r} against δ -functions in propagators

Theorem (Lipstein,M.)

The MHV loop integrand can be explicitly expressed as

$$\langle W \rangle = \frac{1}{(2\pi)^{2l}} \int \prod_{m=1}^{4l} \frac{ds_m}{s_m}$$

- The invariants of Z_i are encoded in the compact contour.
- At higher MHV degree we have $4(l+k)$ - d logs integrated against k delta functions $\delta^{4|4}(Z + \dots)s$.

Manifests pure transcendentality, but have $4l$ $d \log$ integrals on compact contour *not* $2l$ product of intervals (must cancel $(2\pi)^l$). 

Integration without Feynman parameters

Lipstein & M arxiv:1307.1443

$$\text{1-loop MHV} = \sum_{i < j} K_{ij}, \quad K_{ij} = \int \frac{ds_0 ds dt_0 dt}{s_0 t_0 s t}.$$

For contour: normalize $Z_i \cdot \bar{Z}_* = 1$ and set $a_{ij} = Z_i \cdot \bar{Z}_j$, then

$$s = -\frac{(a_{i-1j} - v)\bar{t} + a_{i-1j-1} - v}{(a_{ij} - v)\bar{t} + a_{ij-1} - v}, \quad s_0 = \bar{s}_0, \quad t_0 = \bar{t}_0, \quad v = s_0 - t_0.$$

- s_0, t_0 integrals need $i\epsilon$,
- $s_0 + t_0$ integral decouples and incorporates $i\epsilon$.
- Obtain definite integral in $v \in [v_*, \infty]$, $v_* = x_{ij}^2/2\langle\bar{\eta}|x_{ij}|\eta\rangle$.
- Stokes reduces compact complex integrals \leadsto logs.

$$K_{ij} = \text{Li}_2\left(\frac{a_{ij}}{v_*}\right) + \text{Li}_2\left(\frac{a_{i-1j-1}}{v_*}\right) - \text{Li}_2\left(\frac{a_{i-1j}}{v_*}\right) - \text{Li}_2\left(\frac{a_{ij-1}}{v_*}\right) + c.c.$$

Summary:

- Amplitude/Wilson-loop duality is planar duality for Feynman diagrams on twistor space.
- The susy amplitude/Wilson-loop duality exists on space-time, [M & Skinner, Caron-Huot] but harder and controversial.
- Naturally obtain $d \log$ form for loop integrand.
- Integration is direct without Feynman parameters.
Can always do / real integrals.
- Can similarly compute more general correlation functions.

Wilson loops for real Chern-Simons \leadsto link invariants.

Amplitudes are ‘holomorphic self-linking’ of polygons in PT.

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The end

Thank You!