Twistor actions, Wilson loops, *d* logs and dilogs

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cf: review with Adamo, Bullimore & Skinner 1104.2890, & recent work with Lipstein arxiv:1212.6228, 1307.1443.

[Also. work by Alday, Arkani-Hamed, Cachazo, Caron-Huot, Drummond, Henn, Heslop, Korchemsky, Maldacena, Sokatchev. (Annecy, Oxford, Perimeter and Princeton IAS).]

$\mathcal{N} = 4$ Super Yang-Mills

The analogue of the harmonic oscillator for 4-dimensional quantum field theory?

- Toy version of standard model.
- Best behaved nontrivial 4d field theory (UV finite, superconformal $SU(2,2|4)$ symmetry, $\ldots$).
- Particle spectrum

helicity	-1	-1/2	0	1/2	1
# of particles	1	4	6	$\bar{4}$	1

- Susy changes helicity helicity so that particles form irrep of 'super'-group $\mathrm{SU}(2,2|4)$ like single particle.
- Contains QCD and more classically.
- 'completely integrable' in planar (large N) sector.
- much twistor geometry in their amplitudes:
  1. (Ambi-)Twistor string description, (next lecture)
  2. Grassmannian residue formula,
  3. polyhedra volumes $\rightsquigarrow$ the amplituhedron,
  4. Focus here on the *holomorphic Wilson loop*.

Scattering amplitudes for $\mathcal{N} = 4$ super Yang-Mills

4-Momentum:

$$p = (E, p_1, p_2, p_3) = E(1, v_1, v_2, v_3),$$

massless$\leftrightarrow |\mathbf{v}| = c = 1 \Leftrightarrow p \cdot p := E^2 - p_1^2 - p_2^2 - p_3^2 = 0. \Leftrightarrow$

$$\Leftrightarrow \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} E + p_3 & p_1 + ip_2 \\ p_1 - ip_2 & E - p_3 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} \begin{pmatrix} \tilde{\lambda}_{0'} & \tilde{\lambda}_{1'} \end{pmatrix}$$

Supermomentum: $P = (\lambda, \tilde{\lambda}, \eta) \in \mathbb{C}^{4|0} \times \mathbb{C}^{0|4}$, where $\eta_i, i = 1, \dots, 4$ anti-commute $\rightsquigarrow$ wave functions:

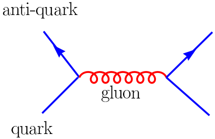
$$\Xi(\lambda, \tilde{\lambda}, \eta) = A_+ + \Psi^i \eta_i + \Phi^{ij} \eta_i \eta_j + \tilde{\Psi}^{ijk} \eta_i \eta_j \eta_k + A_- \eta_1 \eta_2 \eta_3 \eta_4 \,.$$

Amplitude: for n-particle process is

$$\mathcal{A}(1, \dots, n) = \mathcal{A}(P_1, \dots, P_n)$$

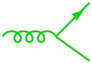
MHV degree: $m = \{$ weight in ηs $\}/4 - 2$.
If $k \sim$ # of $-$ve helicity particles, susy $\Rightarrow \mathcal{A} = 0$ for $k = 0, 1$
For $k = 2$, $\mathcal{A} \neq 0$, 'Maximal Helicity Violating' (MHV).
$m = k - 2 :=$ the *MHV degree*.

Contributions

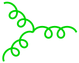


Feynman diagrams are more than pictures. They represent algebraic formulas for the propagation and interaction of particles.

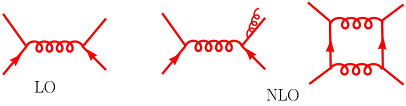
$\frac{\eta_{\mu\nu}}{p^2}$



$g\,\gamma_\mu$



$g[\,(p_1 - p_2)_\rho\,\eta_{\mu\nu} + (p_2 - p_3)_\mu\,\eta_{\nu\rho} + (p_3 - p_1)_\nu\,\eta_{\rho\mu}\,]$



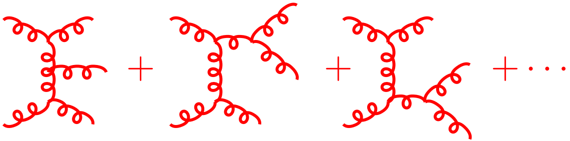
Trees $\leftrightarrow$ classical, loops $\leftrightarrow$ quantum.

Locality: only simple poles from propagators at $(\sum p_i)^2 = 0$.

Need for new ideas

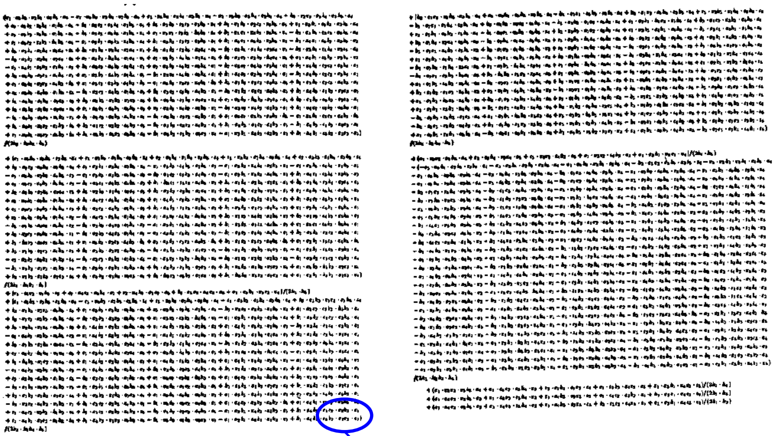
Consider the five-gluon tree-level amplitude of QCD. Enters in calculation of multi-jet production at hadron colliders.

Described by following Feynman diagrams:



If you follow the textbooks you discover a disgusting mess.

Result of a brute force calculation:



$$k_1 \cdot k_4\, \varepsilon_2 \cdot k_1\, \varepsilon_1 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5$$

However, result for helicity $(+ + - - -)$ part of the amplitude is

$$\mathcal{A}(1,2,3,4,5) = \delta\left(\sum_{a=1}^{5} p_a\right) \frac{\langle \lambda_b\, \lambda_c\rangle^4}{\langle \lambda_1\, \lambda_2\rangle\langle \lambda_2\, \lambda_3\rangle\langle \lambda_3\, \lambda_4\rangle\langle \lambda_4\, \lambda_5\rangle\langle \lambda_5\, \lambda_1\rangle}$$

where b and c are the $+$ helicity particles and

$$\langle i\, j\rangle := \langle \lambda_i\, \lambda_j\rangle := \lambda_{i0}\lambda_{j1} - \lambda_{i1}\lambda_{j0}$$

(similarly use $[i,j]$ for $\tilde{\lambda}_i$s).

More generally (Parke-Taylor 1984, Nair 1986)

$$\mathcal{A}^{tree}_{MHV}(1,\dots,n) = \frac{\delta^{4|8}\left(\sum_{a=1}^{n}(p_a, \eta_a\lambda_a)\right)}{\prod_{a=1}^{n}\langle \lambda_a\, \lambda_{a+1}\rangle}$$

Super twistor space is $\mathbb{CP}^{3|4}$ with homogeneous coords:

$$Z = (\lambda_\alpha, \mu^{\dot\alpha}, \chi_i) \in \mathbb{T} := \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^{0|4}, \qquad Z \sim \zeta Z\,, \zeta \in \mathbb{C}^*\,.$$

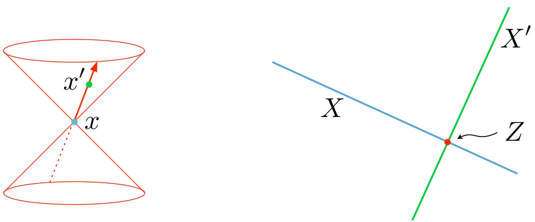
$\mathbb{T}$ = fund. repn of superconformal group $SU(2,2|4)$.

A point in super Minkowski space, $\mathbb{M}$, coords $(x, \theta) \leftrightarrow$ a line $X = \mathbb{CP}^1 \subset \mathbb{PT}$ via incidence relations

$$\mu^{\dot\alpha} = i x^{\alpha\dot\alpha}\lambda_\alpha\,, \qquad \chi_i = \theta_i^\alpha \lambda_\alpha\,.$$

Two points x, x' are null separated iff X and X' intersect.

Space-time Twistor Space



Linear Maxwell fields and Penrose transform

- Let $f \in \mathcal{O}(n) \Leftrightarrow f(\gamma Z) = \gamma^n f(Z)$.
- Linear Maxwell fields obtained from
$$(a, b) \in \Omega^{0,1} \times \Omega^{0,1}(-4)\,, \quad \bar\partial a = \bar\partial b = 0\,,$$
modulo gauge freedom $(a, b) \to (a + \bar\partial\alpha, b + \bar\partial\beta)$.
- Thus $(a, b) \in H^1(\mathcal{O}) \times H^1(\mathcal{O}(-4))\,.$

Action: above twistor data arises from

$$S = \int_{\mathbb{PT}} a \wedge \bar\partial b \wedge \mathrm{D}^3 Z\,, \qquad \mathrm{D}^3 Z = \varepsilon^{IJKL} Z_I \mathrm{d}Z_J \wedge \mathrm{d}Z_K \wedge \mathrm{d}Z_L\,.$$

Space-time fields: given by

$$\phi_{\dot\alpha\dot\beta}(x) = \int_X \frac{\partial^2 a}{\partial\mu^{\dot\alpha}\partial\mu^{\dot\beta}}\,\mathrm{D}\lambda\,, \quad \phi_{\alpha\beta}(x) = \int_X \lambda_\alpha\lambda_\beta\, b\,\mathrm{D}\lambda\,, \quad \mathrm{D}\lambda = \lambda_\gamma \mathrm{d}\lambda^\gamma\,.$$

Ex: Plane wave, momentum $\lambda_i\tilde\lambda_i$ helicity s:

$$a, b \text{ etc. } = \int_{\mathbb{C}} \frac{\mathrm{d}t}{t^{1-2s}} \bar\delta^2(t\lambda - \lambda_i) \mathrm{e}^{it[\mu,\tilde\lambda_i]}.$$

Theorem (Ward 1978)

Local solutions (E', D_A) to self-dual Yang Mills equs on $U \subset \mathbb{C}^4$ correspond 1:1 to holomorphic vector bundles $E \to \mathbb{PT}(U)$.

Idea: $E'_x =$ holomorphic sections of $E|_X$.

Exhibits complete integrability of self-duality equations (so, no scattering).

Applications:

- ADHM classification of instantons.
- Construction of monopoles.
- Unification of theory of integrable systems.

Data & field equs. Hol vector bundles $E \to \mathbb{PT}$ given by

$$\bar\partial_a = \bar\partial_0 + a\,, \quad a \in \Omega^{0,1} \otimes sl(n, \mathbb{C})\,, \quad \text{with} \quad F^{0,2} := \bar\partial_a^2 = 0.$$

Action. Introduce Lagrange multiplier $b \in \Omega^{0,1}(-4)$,

$$S[a,b] = \int_{\mathbb{PT}} \mathrm{tr}\,(b_\wedge F^{0,2})\; D^3 Z \qquad D^3 Z \in \Omega^{3,0}(4)\,.$$

Action also gives $\bar\partial_a b = 0$; so b gives an ASD linear field:

$$B(x) = \int_X H^{-1} b H_\wedge D^3 Z \in \Omega^{2-}\,, \qquad D_A B = 0\,.$$

On space-time. Chalmers-Siegel SD YM action:

$$S[A,B] = \int_{\mathbb{M}} \mathrm{tr}\, F_{A\wedge} B\,, \qquad (A,B) \in (\Omega^1, \Omega^{2-}) \otimes sl(N)$$

Has degrees of freedom of full YM, but only SD interactions.
Extend to full YM: add $g^2 \int \mathrm{tr}\, B_\wedge B$ term.

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Supersymmetric Ward correspondence

Super Calabi-Yau:
$\mathbb{CP}^{3|4}$ has weightless super volume form

$$D^{3|4}Z = D^3 Z \, \mathrm{d}\chi_1 \ldots \mathrm{d}\chi_4 \in \Omega_{Ber}.$$

'Super-Ward' for $\mathcal{N} = 4$ SYM:
A dbar-op $\bar{\partial}_{\mathcal{A}} = \bar{\partial}_0 + \mathcal{A}$ on a bundle on $\mathbb{CP}^{3|4}$ has expansion

$$\mathcal{A} = a + \chi_a \psi^a + \chi_a \chi_b \phi^{ab} + \chi^{3a} \tilde{\psi}_a + \chi^4 b$$

and $\bar{\partial}^2_{\mathcal{A}} = 0 \;\leftrightarrow$ solns to SD $\mathcal{N} = 4$ SYM on space-time.
[ψ and $\tilde{\psi}$ give fermions and ϕ scalars.]

Action for fields with self-dual interactions:[Witten]
SD interactions $\leftrightarrow$ holomorphic Chern-Simons action

$$S_{sd} = \int_{\mathbb{PT}} \mathrm{tr}(\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}^3)_\wedge D^{3|4}Z \, .$$

Extension to full SYM:

$$S_{full}[\mathcal{A}] = S_{sd}[\mathcal{A}] + S_{int}[\mathcal{A}]$$

includes non-local interaction term:

$$\begin{aligned} S_{int}[\mathcal{A}] &= g^2 \int_{\mathbb{M}} \mathrm{d}^{4|8}\mathbf{x} \ \log \ \det(\bar{\partial}_{\mathcal{A}}|_X) \\ &= g^2 \sum_{n=2}^{\infty} \int_{\mathbb{M}\times(\times^n X)} \mathrm{d}^{4|8}\mathbf{x} \frac{\mathrm{tr}\left(\mathcal{A}(\lambda_1)\mathcal{A}(\lambda_2)\ldots\mathcal{A}(\lambda_n)\right) D\lambda_1 \ldots D\lambda_n}{\langle\lambda_1\lambda_2\rangle\langle\lambda_2\lambda_3\rangle\ldots\langle\lambda_n\lambda_1\rangle} \end{aligned}$$

X is $\mathbb{CP}^1$ corresponding to $\mathbf{x} \in \mathbb{M}^{4|8}$, $\lambda_i \in X_i$, i^{th} factor in $\times^n X$,

$$K_{ij} = \frac{D\lambda_j}{\langle\lambda_i \lambda_j\rangle}$$

is Cauchy kernel of $\bar{\partial}^{-1}$ on X at λ_i, λ_j.

Axial gauge, Feynman rules & MHV formalism

Choose 'reference twistor' Z_*, impose gauge $\bar{Z}_* \cdot \frac{\partial}{\partial \bar{Z}} \lrcorner \mathcal{A} = 0$.

- Cubic Chern-Simons vertex vanishes.
- Propagator = delta-function forcing Z, Z', Z_* to be collinear.

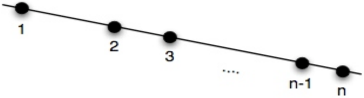
$$\Delta(Z, Z') = \frac{1}{2\pi i} \bar{\delta}^{2|4}(Z, Z_*, Z') := \frac{1}{2\pi i} \int \frac{\mathrm{d}s\mathrm{d}t}{st} \bar{\delta}^{4|4}(Z_* + sZ + tZ')$$

[Here $\bar{\delta}^1(z) = \bar{\partial} \frac{1}{2\pi i\, z}$ for $z \in \mathbb{C}$ and $\delta^{0|1}(\chi) = \chi$ for χ odd.]

- log-det term expands to give 'MHV vertices':

$$V(Z_1, \ldots, Z_n) = \int_{\mathbb{M} \times L_X^n} \frac{\mathrm{d}^{4|4} Z_A \mathrm{d}^{4|4} Z_B}{Vol\, GL(2)} \prod_{r=1}^{n} \frac{\bar{\delta}^{3|4}(Z_r, Z_A + \sigma_r Z_B)}{(\sigma_{r-1} - \sigma_r)} d\sigma_r \,.$$

- Vertices force $Z_1, \ldots Z_n$ to lie on line Z_A to Z_B:



- On momentum space gives 'MHV rules' for amplitudes with vertices $=$ off-shell MHV amplitudes, non-local, [CSW].

Large N and region momentum space

- For $\mathrm{SU}(N)$, $N \to \infty$, only single trace terms survive
- gives ordering of particles: if particle i has colour $c_i \in su(N)$, coeff of $\mathrm{tr}(c_{i_1} c_{i_2} \dots c_{i_n})$ gives ordering $i_1, \dots, i_n$.

Supermomentum conservation $\rightsquigarrow$ null polygon $\{X_i\} = \{(x_i, \theta_i)\}$ $\in$ 'region momentum space'$= \mathbb{M}$, super Minkowski space:

$$(p_i^{AA'}, \eta_i^a \lambda^A) = (x_i^{AA'} - x_{i+1}^{AA'}, \theta_i^A - \theta_{i+1}^A)\,.$$

Instead of amplitude, we consider 'ratio' $\widetilde{R}$:

$$\mathcal{A} = \mathcal{A}_{MHV}^{\mathrm{tree}} \widetilde{R}(X_1, \dots, X_n)$$

Conjecture (Alday, Maldacena)

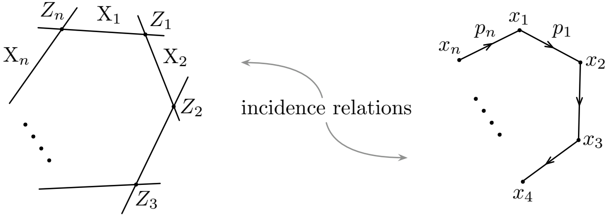
At MHV, arbitrary loop order, $\widetilde{R} =$ Yang-Mills correlation function of a Wilson-loop around momentum polygon in $\mathbb{M}$.

Corollary: $\widetilde{R}$ dual conformal invariant (up to known anomaly).

Much evidence: [Drummond, Henn, Korchemsky & Sokatchev; Brandhuber, Heslop, & Travaglini].

Momentum polygons in twistor space

A null polygon in space-time $\leftrightarrow$ generic polygon in $\mathbb{PT}$ [Hodges].



Change variables so that $\widetilde{R}(X_1, \ldots, X_n) = R(Z_1, \ldots, Z_n)$.

Important simplification: $Z_i \in \mathbb{PT}$ are unconstrained.

What is analogue of Alday-Maldacena conjecture on $\mathbb{PT}$?

For Wilson-loop, need holonomy around polygon in $\mathbb{PT}$.

- Polygon vertices Z_i, edges $X_i = \langle Z_{i-1}, Z_i \rangle$.
- Parametrize X_i by σ with $Z(\sigma) = \sigma Z_{i-1} + Z_i$.
- On X_i find global frame $F_i(\sigma)$ of $E|_{X_i}$ with $\bar\partial_{\mathcal{A}}|_{X_i} F_i = 0$, boundary condition $F_i(\infty) = F_i|_{Z_{i-1}} = 1$.
- Define
$$W = \operatorname{tr} \prod_{i=1}^{n} F_i|_{Z_i} = \operatorname{tr} \prod_{i=1}^{n} F_i(0)\,.$$
- Agrees with space-time Wilson loop on-shell.

Perturbatively iterate $F_i = 1 + \bar\partial^{-1}(\mathcal{A} F_i)$ to get

$$F_i = 1 + \sum_{r=1}^{\infty} \prod_{s=1}^{r} \bar\partial^{-1}_{s-1\,s} \mathcal{A}(\sigma_s)\,, \qquad (\bar\partial^{-1}_{r\,s} f)(\sigma_r) = \int_{L_{X_i}} \frac{f(\sigma_s)\,\mathrm{d}\sigma_s}{\sigma_r - \sigma_s}$$

The S-matrix as a holomorphic Wilson loop

Theorem (Bullimore, M., Skinner, 2010-11)

The all-loop integrand for the planar n-particle amplitude for super Yang-Mills is identical to that for the holomorphic Wilson loop in twistor space:

$$\mathcal{A}(1,\ldots,n) = \langle W(Z_1,\ldots,Z_n)\rangle \mathcal{A}^{tree}_{MHV}\,.$$

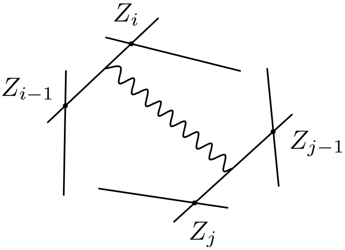
- *Tree amplitudes arise at $g=0$ when the Wilson-loop is calculated in the self-dual sector.*
- *The loop expansion for $\mathcal{A} =$ expansion for W in g.*
- *The MHV diagrams for amplitude are planar duals of those for Wilson-loop correlator in axial gauge on twistor space.*

Proof: *direct calculation of Feynman diagrams in axial gauge.*

The all-loop integrand *[Arkani-Hamed, et. al.]* *is canonical for planar gauge theories & delays confrontating infrared divergences.*

Examples

NMHV case: for $\mathcal{A}^2$ part of $\langle W \rangle$, $\langle \mathcal{A}(Z)\mathcal{A}(Z') \rangle = \Delta(Z, Z')$: obtain $\langle W \rangle = \sum_{i<j} \Delta_{ij}$ where $\Delta_{ij} =$



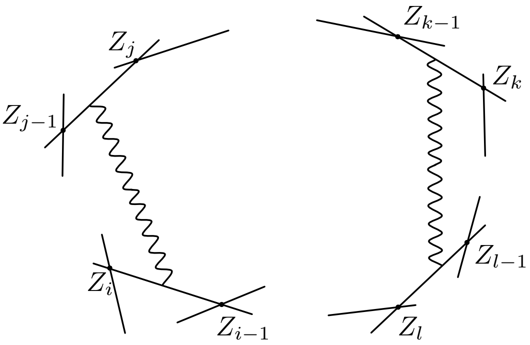
$$= [\,*\,, i-1, i, j-1, j\,]\,,$$

$$[1,2,3,4,5] \;:=\; \int \frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3\mathrm{d}s_4}{s_1 s_2 s_3 s_4}\; \bar{\delta}^{4|4}(Z_5 + \sum_{i=1}^{4} s_i Z_i)\,,$$

$$= \frac{\prod_{a=1}^{4}\big((1234)\chi_5^a + \text{cyclic}\big)}{(1234)(2345)(3451)(4512)(5123)}$$

is the ‘R-invariant’.

N^{2}MHV: quartic terms in $\mathcal{A}$ in W give Wick contractions

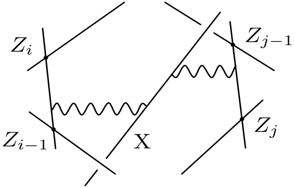


No crossed propagators for planarity.

N^{k}MHV tree amplitudes:
k propagators $\rightsquigarrow$ product of k R-invariants.

Loops

At MHV with one MHV vertex obtain $\sum_{i,j} K_{ij}$ with $K_{ij} =$



$$= \int_\Gamma \mathrm{D}^{3|4} Z_A \wedge \mathrm{D}^{3|4} Z_B \, [\,*\,, i{-}1, i, A, B'][\,*\,, j{-}1, j, A, B'']$$

Loop momenta $\leftrightarrow$ location of line $X = \langle Z_A \, Z_B \rangle$. Recall:

$$[*, i{-}1, i, A, B] := \int \frac{\mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}s_4}{s_1 s_2 s_3 s_4} \, \bar{\delta}^{4|4}(Z_* + s_1 Z_A + s_2 Z_B + s_3 Z_{i-1} + s_4 Z_i)$$

can integrate $D^{3|4} Z_A \wedge D^{3|4} Z_B$ against delta functions

$$K_{ij} = \frac{1}{(2\pi i)^2} \int \frac{\mathrm{d}s_0 \mathrm{d}s \mathrm{d}t_0 \mathrm{d}t_1}{s_0 \, s \, t_0 \, t} \, .$$

External data encoded in integration contour (see later).

In general: loop order $=$ # MHV vertices.

Loop integrand: At l-loops, integral is over l loop 4-momenta L_r, $r = 1, \ldots, l$.

$$\langle W \rangle = \int_{(\mathbb{R}^4)^l} F(L_r, Z_i) \prod_{r=1}^{l} d^4 L_r \,.$$

Expect $\langle W \rangle =$ polylogs of degree $2l$ of invariants a_j of Z_i.

Definition

A polylog of degree $2l$ is an iterated integral

$$Plog(a_1, a_2, \ldots) = \int_{[0,1]^{2l}} \prod_{m=1}^{2l} \frac{dR_m}{R_m}$$

where $R_m =$ rational fns of a_i and integration parameters s_m.

Conjecture: At least at MHV, can express integrated amplitude as polylog of pure transcendentality degree $2l$.

In our formulation

- L_r are encoded in (Z_{A_r}, Z_{B_r}).
- At MHV, have two propagators per vertex
- can integrate Z_{A_r}, Z_{B_r} against δ-functions in propagators

Theorem (Lipstein,M.)

The MHV loop integrand can be explicitly expressed as

$$\langle W \rangle = \frac{1}{(2\pi)^{2l}} \int \prod_{m=1}^{4l} \frac{ds_m}{s_m}$$

- *The invariants of Z_i are encoded in the compact contour.*
- *At higher MHV degree we have $4(l+k)$-d logs integrated against k delta functions $\delta^{4|4}(Z+\ldots)$s.*

Manifests pure transcendentality, but have $4l$ $d\log$ integrals on compact contour *not* $2l$ product of intervals (must cancel $(2\pi)^l$).

Integration without Feynman parameters

Lipstein & M arxiv:1307.1443

$$\text{1-loop MHV } = \sum_{i<j} K_{ij}\,, \qquad K_{ij} = \int \frac{ds_0\, ds\, dt_0\, dt}{s_0 t_0 s\, t}\,.$$

For contour: normalize $Z_i \cdot \bar{Z}_* = 1$ and set $a_{ij} = Z_i \cdot \bar{Z}_j$, then

$$s = -\frac{(a_{i-1\,j} - v)\bar{t} + a_{i-1\,j-1} - v}{(a_{ij} - v)\bar{t} + a_{ij-1} - v}\,, \quad s_0 = \bar{s}_0, \quad t_0 = \bar{t}_0\,, \quad v = s_0 - t_0\,.$$

- s_0, t_0 integrals need $i\epsilon$,
- $s_o + t_0$ integral decouples and incorporates $i\epsilon$.
- Obtain definite integral in $v \in [v_*, \infty]$, $v_* = x_{ij}^2 / 2\langle\bar{\eta}|x_{ij}|\eta]$.
- Stokes reduces compact complex integrals $\rightsquigarrow$ logs.

$$K_{ij} = \mathrm{Li}_2\left(\frac{a_{i\,j}}{v_*}\right) + \mathrm{Li}_2\left(\frac{a_{i-1\,j-1}}{v_*}\right) - \mathrm{Li}_2\left(\frac{a_{i-1\,j}}{v_*}\right) - \mathrm{Li}_2\left(\frac{a_{i\,j-1}}{v_*}\right) + c.c.$$

Summary:

- Amplitude/Wilson-loop duality is planar duality for Feynman diagrams on twistor space.
- The susy amplitude/Wilson-loop duality exists on space-time, [M & Skinner, Caron-Huot] but harder and controversial.
- Naturally obtain $d\log$ form for loop integrand.
- Integration is direct without Feynman parameters. Can always do l real integrals.
- Can similarly compute more general correlation functions.

Wilson loops for real Chern-Simons $\rightsquigarrow$ link invariants.

Amplitudes are 'holomorphic self-linking' of polygons in $\mathbb{PT}$.

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Thank You!