Ambitwistor strings, the scattering equations, tree formulae and beyond

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With David Skinner. arxiv:1311.2564 and Yvonne Geyer & Arthur Lipstein 1404.6219.

[Cf. also Cachazo, He, Yuan arxiv:1306.2962, 1306.6575, 1307.2199, 1309.0885]

String formulae for tree-level scattering

Remarkable compact formulae:

- Twistor-string for $N = 4$ Yang-Mills [Witten, Roiban, Spradlin, Volovich, 2003/4].
- Some (conformal-) supergravity amplitudes [Adamo, Mason 2012]
- $N = 8$ supergravity [Cachazo-Geyer, Cachazo-Skinner, CMS, 2012],
- ABJM [Huang,Lee 20012].
- Pfaffian/current-correlator formulae for spins 2 , 1, 0 in all dimensions [Cachazo, He, Yuan, 2013].
- CS N=8 formula arises from $N = 8$ twistor-string [Skinner, 2013]

This talk: Such formulae $\leftrightarrow$ *'ambitwistor string theories'*:

- Ambitwistors $\mathbb{A} =$ complexified space of null geodesics.
- $\mathbb{A} =$ reduction of complex cotangent bdle of space-time.
- Theory is critical in 10d.
- In 4d, $\mathbb{A} =$ cotangent bdle of twistor space or dual twistors *or* new ambidextrous formulation.

New family of infinite tension ($\alpha' = 0$) chiral analogues of standard strings that extend original twistor-string.

Take n null momenta $k_i \in \mathbb{R}^d$, $i = 1, \dots, n$, $k_i^2 = 0$, $\sum_i k_i = 0$,

- define $P : \mathbb{CP}^1 \to \mathbb{C}^d$

$$P(\sigma) := \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i}, \qquad \sigma, \sigma_i \in \mathbb{CP}^1 .$$

- Solve for $\sigma_i \in \mathbb{CP}^1$ with the *scattering equations*

$$\mathrm{Res}_{\sigma_i} P(\sigma) \cdot P(\sigma) = k_i \cdot P(\sigma_i) = \sum_{j=1}^{n} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0 .$$

- Then $P(\sigma) \cdot P(\sigma) = 0 \ \forall \sigma$.
- For Mobius invariance $\Rightarrow P \in \mathbb{C}^d \otimes K$, $K = \Omega^{1,0}\mathbb{CP}^1$
- only $n - 3$ scattering equations are independent.
- There are $(n - 3)!$ solutions.

First arose for high energy string scattering [Gross-Mende 1988].
Underpin twistor-string formulae also [Witten 2004].

Scatter n spin s massless particles, momenta k_i, $k_i^2 = 0$,

- polarizations ϵ_{1i} for spin 1, $\epsilon_{1i} \otimes \epsilon_{2i}$ for spin-2

$$k_i \cdot \epsilon_{ri} = 0\,, \qquad \epsilon_{ri} \sim \epsilon_{ri} + \alpha_r k_i\,, \quad r = 1, 2.$$

- Introduce skew $2n \times 2n$ matrices

$$M_r = \begin{pmatrix} A & C_r \\ -C_r^t & B_r \end{pmatrix}\,,$$

$$A_{ij} = \frac{k_i \cdot k_j}{\sigma_i - \sigma_j},, \quad B_{rij} = \frac{\epsilon_{ri} \cdot \epsilon_{rj}}{\sigma_i - \sigma_j}\,, \quad C_{rij} = \frac{k_i \cdot \epsilon_{rj}}{\sigma_i - \sigma_j}\,, \qquad \text{for } i \neq j$$

and $A_{ii} = B_{ii} = 0$, but $C_{rii} = \epsilon_{ri} \cdot P(\sigma_i)$.

- Tree-level gravity amplitude in d-dims is sum

$$\mathcal{M}(1, \ldots, n) = \delta^d \left(\sum_i k_i \right) \int_{\mathbb{CP}^{1\,n}} \frac{Pf'(M_1)Pf'(M_2)}{\mathrm{Vol\ SL}(2, \mathbb{C})} \prod_i{}' \bar{\delta}(k_i \cdot P(\sigma_i)) d\sigma_i$$

[For YM, replace $Pf'(M_2)$ by Parke-Taylor $(\prod_i(\sigma_i - \sigma_{i-1}))^{-1}$.]

Geometry of ambitwistor space

Complexify real space-time $M_{\mathbb{R}} \rightsquigarrow M$, and null covectors P.

$\mathbb{A} :=$ space of complex null geodesics with scale of P.

- $\mathbb{A} = T^*M|_{P^2=0}/\{D_0\}$ where $D_0 := P \cdot \nabla =$ geodesic spray.
- D_0 has Hamiltonian P^2 wrt symplectic form $\omega = dP_\mu \wedge dx^\mu$.
- Symplectic potential $\theta = P_\mu dx^\mu$, $\omega = d\theta$, descend to $\mathbb{A}$.

Projectivise: $P\mathbb{A} :=$ space of *unscaled* complex light rays.

- On $P\mathbb{A}$, $\theta \in \Omega^1_{P\mathbb{A}} \otimes L$ is a holomorphic contact structure.

Theorem (LeBrun 1983)

The complex structure on $P\mathbb{A}$ determines M and conformal metric g. The correspondence is stable under arbitrary deformations of the complex structure of $P\mathbb{A}$ that preserve θ.

Linearized LeBrun correspondence Baston & M. 1986

θ determines complex structure on $P\mathbb{A}$ via $\theta \wedge d\theta^{d-2}$. So:

$$\text{Deformations of complex structure } \leftrightarrow [\delta\theta] \in H^1_{\bar\partial}(P\mathbb{A}, L)\,.$$

Analyze with double fibration

$$\begin{array}{ccc} & PT^*M|_{P^2=0} & \\ \pi_1 \swarrow D_0 & & \searrow \pi_2 \\ P\mathbb{A}_S & & M\,. \end{array}$$

For $\delta g_{\mu\nu} = \mathrm{e}^{ik\cdot x}\epsilon_{\mu\nu}$ on flat space-time

$$\delta\theta = \bar\delta(k\cdot P)\mathrm{e}^{ik\cdot X}\epsilon_{\mu\nu}P^\mu P^\nu\,.$$

Delta-function support on $k\cdot P = 0 \Rightarrow$ the scattering equations.

Proof: The correspondence between δg and $\delta\theta$ is

- $\pi_1^*\delta\theta = \bar\partial j$ on $PT^*M|_{P^2=0}$ for some j modulo $P\cdot V(x)$.
- $\bar\partial D_0 j = D_0\pi_1^*\delta\theta = 0$ so $D_0 j$ is holomorphic in (P, X), thus
- $D_0 j = \delta g_{\mu\nu}(X)P^\mu P^\nu$ for some variation in the metric δg.

For momentum eigenstate, $j = \mathrm{e}^{ik\cdot X}\frac{\epsilon_{\mu\nu}P^\mu P^\nu}{k\cdot P}$. $\square$

From null geodesics to chiral strings

For real space-time $(M_{\mathbb{R}}, g_{\mathbb{R}})$ dimension d,

- **Phase space action:** null geodesic γ, $(X, P) : \mathbb{R} \to T^*M_{\mathbb{R}}$

$$S = \int_\gamma (P \cdot dX - eP^2/2) \,,$$

- $e \in \Omega^1(\gamma)$ is 'einbein' and Lagrange multiplier for $P^2 = 0$.
- Gauge freedom $\delta(X, P, e) = (\alpha P, 0, 2d\alpha)$.

Phase space of real null geodesics: $\mathbb{A}_{\mathbb{R}} := T^*M_{\mathbb{R}}|_{P^2=0}/\{\text{gauge}\}$

Complexify: $\gamma \rightsquigarrow \Sigma$, Riemann surface, and $(M_{\mathbb{R}}, g_{\mathbb{R}}) \rightsquigarrow (M, g)$.

- **Ambitwistor string action:** $X : \Sigma \to M$, $P \in K \otimes X^*T^*M$

$$S = \int (P \cdot \bar\partial X - e\, P^2/2) \,.$$

with $e \in \Omega^{0,1} \otimes T$, where $K = \Omega^{1,0}_\Sigma$ and $T = T^{1,0}\Sigma$.

- e again enforces $P^2 = 0$,
- flat space gauge freedom: $\delta(X, P, e) = (\alpha P, 0, 2\bar\partial\alpha)$.

Ambitwistor space: $\mathbb{A} = T^*M|_{P^2=0}/\{\text{gauge}\}$.

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Ambitwistor space: $\mathbb{A} = T^*M|_{P^2=0}/\{\text{gauge}\}$.

Quantizing spin 0 ambitwistor string

To quantize, gauge fix

$$S = \int (P \cdot \bar{\partial} X - e\, P^2/2)\,.$$

with $e = 0$ and ghosts $(\tilde{b}, \tilde{c}) \in (K^2, T)$ plus usual $(b, c) \in (K^2, T)$ for diffeos

$$S_{\text{ghost}} = \int b\bar{\partial}c + \tilde{b}\bar{\partial}\tilde{c}\,.$$

This gives BRST operator

$$Q = \int cT + \tilde{c}P^2\,.$$

We have central charge

$$C = 2d - 26 - 26$$

so to quantize consistently $Q^2 = 0 \Rightarrow d = 26$.

Vertex operators and amplitudes

- Integrated vertex ops = perturbations of action $\leftrightarrow \delta g$.
- Action is $\int \theta = \int P \cdot \bar{\partial} X$ so integrated vertex operator is

$$\mathcal{V}_i = \int_\Sigma \delta\theta(\sigma_i) = \int_\Sigma \bar{\delta}(k_i \cdot P(\sigma_i))\, \mathrm{e}^{ik\cdot X(\sigma_i)} \epsilon_{i\mu\nu} P^\mu(\sigma_i) P^\nu(\sigma_i)\,.$$

- Quantum consistency implies field equations:

$$\{Q, \mathcal{V}_i\} = 0 \quad \Leftrightarrow \quad k^2 = 0, \quad k^\mu \epsilon_{\mu\nu} = 0\,.$$

- Fixed vertex operators provide Fadeev Popov determinants for fixing remaining gauge symmetries $G = SL(2,\mathbb{C}) \times \mathbb{C}^3$ for Mobius on $\mathbb{CP}^1$ and translations along D_0.

Replace fixed vertex ops by quotient by G to give amplitude as path-integral

$$\mathcal{M}(1,\ldots,n) = \int \frac{D[X,P,\ldots]}{\text{Vol } G}\, \mathrm{e}^{iS} \prod_{i=1}^{n} \mathcal{V}_i\,.$$

- Take $e^{ik_i \cdot X(\sigma_i)}$ factors into action to give

$$S = \frac{1}{2\pi}\int_\Sigma P\cdot\bar\partial X + 2\pi\sum_i ik\cdot X(\sigma_i)\,.$$

- Gives field equations $\bar\partial X = 0\,, \quad \bar\partial P = 2\pi\sum_i ik\delta^2(\sigma-\sigma_i)\,.$
- Solutions $X(\sigma) = X = \text{const.}\,, \quad P(\sigma) = \sum_i \frac{k_i}{\sigma-\sigma_i}d\sigma\,.$

Thus path-integral reduces to

$$\mathcal{M}(1,\ldots,n) = \int_{\mathbb{R}^d\times(\mathbb{CP}^1)^{n-3}} d^dX\, e^{i\sum_j k_j\cdot X}\,\frac{\prod_{i=1}^n \bar\delta(k_i\cdot P)\,\epsilon_{i\mu\nu}P^\mu(\sigma_i)P^\nu(\sigma_i)}{\text{Vol } G}$$

$$= \delta^d(\sum_i k_i)\int_{(\mathbb{CP}^1)^{n-3}}\prod_i{}'\bar\delta(k_i\cdot P)\,\epsilon_{i\mu\nu}P^\mu(\sigma_i)P^\nu(\sigma_i)$$

We see $P(\sigma)$ appearing and scattering equations.

Unfortunately: no good interpretation of these as amplitudes.

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Spinning light rays and super ambitwistor space

To get Pfaffians include RNS spin vectors Ψ_r^μ, fermions:

$$S[X,P,\Psi] = \int P_\mu \mathrm{d}X^\mu - \frac{e}{2}P_\mu P^\mu + \sum_{r=1}^{2} g_{\mu\nu}\Psi_r^\mu \mathrm{d}\Psi_r^\nu - \chi_r P_\mu \Psi_r^\mu$$

$\chi_r \rightsquigarrow$ constraints $P \cdot \Psi_r = 0$ that generate worldline $N = 2$ susy

$$D_r = \Psi_r \cdot \frac{\partial}{\partial X} + P \cdot \frac{\partial}{\partial \Psi_r}\,, \qquad \{D_r, D_s\} = \delta_{rs} D_0\,.$$

Super ambitwistor space:
$\mathbb{A}_s =$ symplectic quotient of (X, P, Ψ_r)-space by $P^2, P \cdot \Psi_r$.

Symplectic potential: $\theta = P \cdot dX + \Psi_r \cdot d\Psi_r$

Super LeBrun correspondence holds with perturbations

$$\delta\theta = \mathrm{e}^{ik\cdot X}\,\bar{\delta}(k\cdot P)\prod_{r=1}^{2}\epsilon_{r\mu}(P^\mu + \Psi_r^\mu k\cdot\Psi_r)\,.$$

Note: polarization states $\epsilon_{1\mu}\epsilon_{2\nu} \rightsquigarrow$ NS sector of type II sugra.

Use chiral RNS-like action

$$S[X,P,\Psi] = \int_\Sigma P\cdot\bar\partial X - \frac{e}{2}P_\mu P^\mu + \sum_{r=1}^{2}\Psi_r\cdot\bar\partial\Psi_r + \chi_r P\cdot\Psi_r$$

with $N=2$ susy (degenerate).

- To quantize, gauge fix $\chi_r = 0 \rightsquigarrow$ bosonic ghosts (β_r,γ_r) in $(K^{3/2}, T^{1/2})$ for fermionic symmetry (and (b,c), $(\tilde b,\tilde c)$).
- We obtain BRST operator

$$Q = \int cT + \tilde c P^2 + \gamma_r P\cdot\Psi_r\,.$$

- For $Q^2 = 0$ central charge C must vanish

$$C = 2d + \frac{d}{2} + \frac{d}{2} - 26 + 11 - 26 + 11 = 3(D-10)$$

- So critical in $d = 10$ dimensions.

- Integrated vertex operator

$$\mathcal{V}_i = \int_\Sigma e^{ik\cdot X(\sigma_i)}\,\bar\delta(k\cdot P(\sigma_i))\prod_{r=1}^{2}\epsilon_{r\mu}(P^\mu(\sigma_i)+\Psi_r^\mu(\sigma_i)k\cdot\Psi_r(\sigma_i))$$

- need two fixed operators for γ_r zero modes (fixing susy)

$$U_i = e^{k_i\cdot X(\sigma_i)}\prod_r \epsilon_r\cdot\Psi_r(\sigma_i)$$

- and an extra fixed one to fix 3rd c and $\tilde c$ zero modes

$$V_i = \prod_{r=1}^{2}\epsilon_{r\mu}(P^\mu(\sigma_i)+\Psi_r^\mu(\sigma_i)k\cdot\Psi_r(\sigma_i))$$

So amplitudes are given by

$$\mathcal{M}(1,\ldots,n) = \left\langle c_1\tilde c_1\prod_r\gamma_{r1}U_1\,c_2\tilde c_2\prod_r\gamma_{r2}U_2 c_3\tilde c_3 V_3\mathcal{V}_4\ldots\mathcal{V}_n\right\rangle .$$

Much works as before giving $\delta^d(\sum_i k_i)\prod_i{}'\bar\delta(k_i\cdot P)$ etc..

Amplitude formulae with Pfaffians

- New ingredient is the correlation function of the Ψ's.
- Ψ_1, Ψ_2 independent so contractions computed separately.
- Ψ's appear twice in $\mathcal{V}_i$, as $k_i \cdot \Psi_i$ or $\epsilon_i \cdot \Psi_i$.
- Contractions give for example

$$A_{ij} := \left\langle k_i \cdot \Psi_i k_j \cdot \Psi_j \right\rangle = \frac{k_i \cdot k_j}{\sigma_i - \sigma_j}\,, \quad B_{ij} := \left\langle \epsilon_i \cdot \Psi_i\, \epsilon_j \Psi_j \right\rangle = \frac{\epsilon_i \cdot \epsilon_j}{\sigma_i - \sigma_j}\,.$$

- For $C_{ij} = \langle k_i \cdot \psi_i\, \epsilon_j \cdot \psi_j \rangle$, $P(\sigma_i) \cdot \epsilon_i$ gives diagonal entry.
- Two $k \cdot \Psi$s are missing in $U_i +$ ghost contribution $\rightsquigarrow Pf'(M)$.

Theorem

We obtain CHY formula

$$\mathcal{M}(1,\ldots,n) = \delta^d\left(\sum_i k_i\right) \int_{\mathbb{CP}^{1\,n}} \frac{Pf'(M_1)Pf'(M_2)}{\mathrm{Vol}\ \mathrm{SL}(2,\mathbb{C})} \prod_i{}' \bar{\delta}(k_i \cdot P(\sigma_i)) d\sigma_i$$

We can start with other formulations of null superparticles

- **Heterotic model:** as above but $r = 1$ and current algebra (SO(32) or $E_8 \times E_8$ for $Q^2 = 0$) $\rightsquigarrow$ CHY Yang-Mills formula.
- Bosonic case $+2$ current algebras $\rightsquigarrow$ CHY scalar formula.
- Green-Schwarz version:

$$S = \int P \cdot \bar{\partial} X + P_\mu \gamma^\mu_{\alpha\beta} \theta^\alpha \bar{\partial} \theta^\beta \,.$$

- Berkovits has pure spinor version $S = \int P \cdot \bar{\partial} X + p_\alpha \bar{\partial} \theta^\alpha + \ldots$.

We have chiral $\alpha' = 0$ ambitwistor strings based on LeBrun's correspondence that gives theory underlying CHY formulae

- NS sector of type II sugra extended to Ramond as in RNS string via spin-operator from bosonizing Ψs.
- Incorporates KLT/BCJ ideas.

A new representation for the loop integrand?[Adamo, Casali, Skinner]

- At genus g, P is a 1-form and acquires dg zero-modes.
- These are the loop momenta for g-loops.
- E.g., at 1-loop get sum over spin structures of

$$\mathcal{M}_n^{(1)}(\alpha;\beta) = \delta^{10}\left(\sum_i k_i\right) \int \mathrm{d}^{10}p \wedge \mathrm{d}\tau \wedge \bar{\delta}\left(P^2(\sigma_1;\tau)\right) \prod_{j=2}^{n} d\sigma_j\, \bar{\delta}(k_j \cdot P(\sigma_j)))\, \frac{\vartheta_\alpha(\tau)^4 \vartheta_\beta(\tau)^4}{\eta(\tau)^{24}}\, \mathrm{Pf}(M_\alpha)\,\mathrm{Pf}(\widetilde{M}_\beta)$$

But how to get rid of ϑ-functions?

Original 4d ambitwistor space

$$\mathbb{A} = \{(Z, W) \in \mathbb{PT} \times \mathbb{PT}^* | Z \cdot W = 0\}/\{Z \cdot \partial_Z - W \cdot \partial_W\}\,.$$

- Here $\mathbb{T} = \mathbb{C}^{4|\mathcal{N}}$, $\mathcal{N} \leq 3$ for Yang-MIls, $\mathcal{N} \leq 7$ for gravity.
- $Z = (\lambda_\alpha, \mu^{\dot\alpha}, \chi^a)$, $W = (\tilde\lambda_{\dot\alpha}, \tilde\mu^\alpha)$, $\alpha = 0, 1$, $\dot\alpha = \dot 0, \dot 1$, $a = 1, \ldots \mathcal{N}$.
- Relates to previous version by

$$P_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}, \quad \mu^{\dot\alpha} = ix^{\alpha\dot\alpha}\lambda_\alpha, \quad \tilde\mu^\alpha = -ix^{\alpha\dot\alpha}\tilde\lambda_{\dot\alpha}\,.$$

- Symplectic potential

$$\Theta = \frac{i}{2}(W \cdot \mathrm{d}Z - Z \cdot \mathrm{d}W) = P \cdot \mathrm{d}x$$

- The symplectic potential Θ leads to action

$$S = \int_\Sigma W \cdot \bar{\partial} Z - Z \cdot \bar{\partial} W + aZ \cdot W \,.$$

(cf., Berkovits twistor-string, RSV etc.).

- **But:** now ambidextrous so take $Z, W \in K^{1/2}$ not arbitrary.
- Use wave fns from both $\mathbb{PT}$ and $\mathbb{PT}^* \rightsquigarrow$ YM vertex ops

$$\begin{aligned} V_i &= \int \frac{\mathrm{d}s_i}{s_i} \bar{\delta}^2(\lambda_i - s_i\lambda(\sigma_i)) \mathrm{e}^{is_i[\mu\,\tilde{\lambda}_i]} J \cdot t_i \\ \widetilde{V}_i &= \int \frac{\mathrm{d}s_i}{s_i} \bar{\delta}^2(\tilde{\lambda}_i - s_i\tilde{\lambda}(\sigma_i)) \mathrm{e}^{is_i\langle\tilde{\mu}\,\lambda_i\rangle} J \cdot t_i \end{aligned}$$

- N^{k}MHV amplitude $\mathscr{M}(1,\ldots,n) = \langle \tilde{V}_1 \ldots \tilde{V}_k V_{k+1} \ldots V_n \rangle$.
- Same strategy as before with $\sigma_\alpha = \frac{1}{s}(1,\sigma)$ gives

$$\lambda(\sigma) = \sum_{i=1}^{k} \frac{\lambda_i}{(\sigma\, \sigma_i)}\,, \qquad \tilde{\lambda}(\sigma) = \sum_{i=k+1}^{n} \frac{\tilde{\lambda}_i}{(\sigma\, \sigma_i)}\,.$$

- For Yang-Mills obtain amplitude

$$\mathcal{A}(1,\ldots,n) = \int \prod_{i=1}^{n} \frac{\mathrm{d}^2\sigma_i}{(\sigma_i\, \sigma_{i+1})} \prod_{i=1}^{k} \bar{\delta}^2(\tilde{\lambda}_i - \tilde{\lambda}(\sigma)) \prod_{i=k+1}^{n} \bar{\delta}^2(\lambda_i - \lambda(\sigma_i))$$

- **Gravity:** adapt Skinner model, replace $\prod \frac{1}{(i\, i+1)}$ by $\det' \mathcal{H}$

$$\mathbb{H}_{ij} = \frac{\langle i\, j\rangle}{(i\, j)}, \quad i,j \leq k\,, \qquad \widetilde{\mathbb{H}}_{ij} = \frac{[i\, j]}{(i\, j)}\,, \quad i,j > k\,,$$

$$\mathcal{H}_{ij} = \begin{pmatrix} \mathbb{H} & 0 \\ 0 & \widetilde{\mathbb{H}} \end{pmatrix}, \qquad \mathcal{H}_{ii} = -\sum_{l \neq i} \mathcal{H}_{il}\,.$$

Summary for 4d models

- Formulae proved by comparison to original twistor strings.
- Comparison to CHY: $\det' \mathcal{H} = \mathrm{Pf}'(M)$.
- Valid for any degree of SUSY unlike original twistor-strings.
- Noncritical/anomalous as they stand, but perhaps not with additional matter and then unobstructed at loops.
- String field theory action should encode colour-kinematics....

Thank You